Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Wintersemester 2017/2018
Dr. Horn

Modern Linear Algebra: Geometric Algebra with GAALOP

Worksheet 1, supplement of worksheet 1, first problems of worksheet 3, worksheets 8 and 9, and last problems of worksheet 21

of the module „Mathematics for Business and Economics“
of joint first-year bachelor lessons at Berlin School of Economics and Law/Hochschule für Wirtschaft und Recht Berlin (BSEL/HWR Berlin)

LV-Nr. 200 691.01 & 400 691.01

Stand: 07. Jan. 2018

HWR Berlin, Wintersemester 2017/2018

Mathematics for Business and Economics

Berlin School of Economics and Law

Worksheet 1 – Exercises

Problem 1:

What is mathematics? Why are mathematical methods effective? Why does it make sense to apply mathematical methods if confronted with problems?

Please read the first part of the paper of physics Nobel prizewinner Eugene P. Wigner "The Unreasonable Effectiveness of Mathematics in the Natural Sciences – What is Mathematics?" and try to find and to identify your own epistemological position.

Problem 2:

What is your answer to the following question of mathematician and philosopher Morris Kline: „Is mathematics a collection of diamonds hidden in the depths of the universe and gradually unearthed, or is it a collection of synthetic stones manufactured by man, yet so brilliant nevertheless that they bedazzle those mathematicians who are already partially blinded by pride in their own creations?" (Quotation from Hal Hellman: Great Feuds in Mathematics. Ten of the Liveliest Disputes Ever. John Wiley & Sons, Hoboken, New Jersey 2006, p. 203.)

Is mathematics a discovery or a human invention?
Is mathematics natural and an inherent part of nature existing always and for ever,
or is mathematics artificial and a construction of the human mind?

Problem 3:

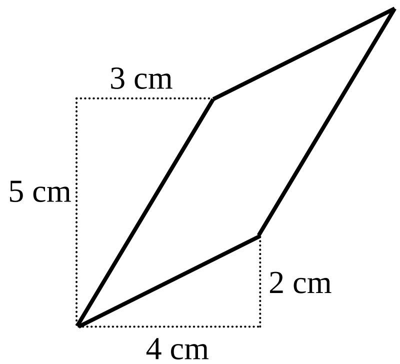
British physics Nobel prizewinner P. A. M. Dirac describes his view with the following words: "One may describe the situation by saying that the mathematician plays a game in which he himself invents the rules…" (Quotation from Paul A. M. Dirac: The Relation Between Mathematics and Physics, James Scott Prize Lecture, Proceedings of the Royal Society (Edinburgh), Vol. 59, 1938 – 1939, part II, pp. 122 – 129.

And Mathilde Marcolli, who has won the Sofja Kovalevskaya award in 2001, says: "If intelligent extraterrestrial life exists, they will most probably invent completely different mathematics," simply because "... mathematics can be invented freely, ..." (German quotation „Wenn es außerirdische Lebewesen gäbe, dann würden sie höchstwahrscheinlich auch eine vollkommen andere Mathematik erfinden," weil eben „... Mathematik frei erfunden werden kann, ..." from Antonia Rötger: Zur Person – Matilde Marcolli, MaxPlanckForschung, Das Wissenschaftsmagazin der Max-Planck-Gesellschaft, Issue 1/2005, pp. 76 – 80).

As simple example for such a free invention of mathematics, we will compare different solution strategies for the following problem:

What is the size of the area of the parallelogram on the right?

Find the area of the given parallelogram using mathematical methods you have learned at school.



Problem 4:

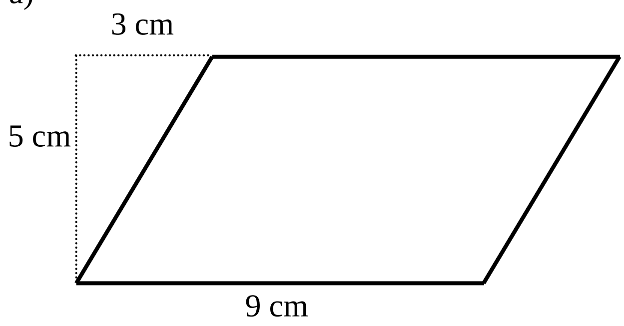
A mathematician, who invented completely different mathematics, has been the Stettin senior high school teacher (German: „Oberlehrer am Gymnasium“) Hermann Grassmann. He was able to find the extensions – or in other words: the areas – of simple geometric objects using his theory of extensions (which he called „Ausdehnungslehre“).

Please compare the solution strategy of Grassmann, which you have discussed at the first lesson, with the solution strategy you have learned at school and applied at solving problem 3.

Problem 5:

Find the areas of the following parallelograms by using the solution strategy of Grassmann.

a)



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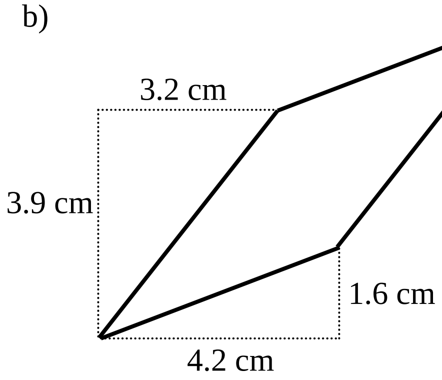

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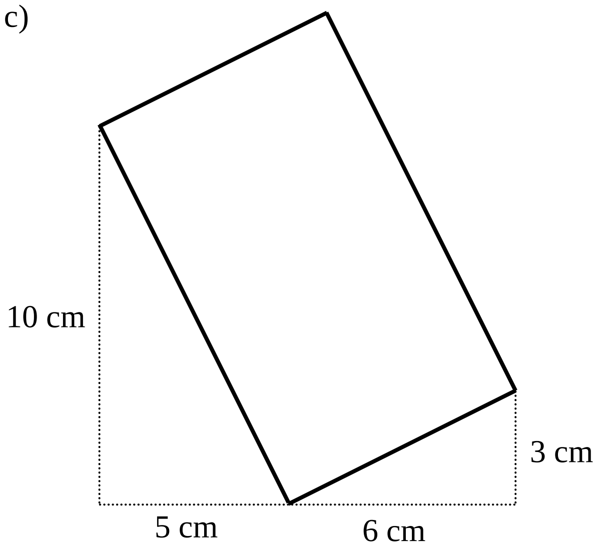

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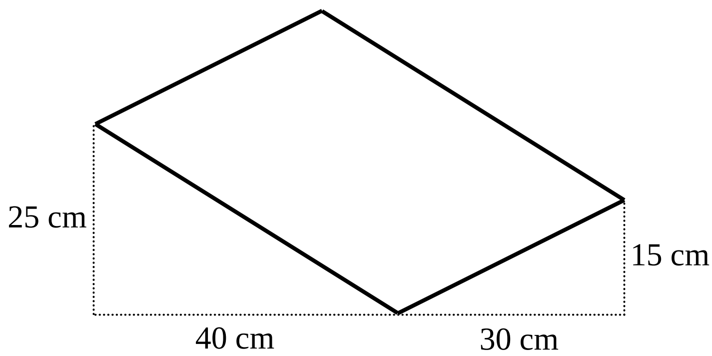

Problem 6:

To take part at this course “Mathematics for Business and Economics” successfully, mathematical basics from primary and secondary school are required. If you do not have a good knowledge of school mathematics, it will be expected that you refurbish your knowledge of school mathematics and that you remove possible deficiencies of school mathematics at your own initiative.

Lengthy repetitions of school mathematics will not be part of this business math course. You are personally responsible for your own mathematical future. Therefore please have a look into your old schoolbooks of mathematics. Please find out whether you have severe mathematical deficiencies (e.g. when calculating fractions or transforming simple equations, etc…). And please develop strategies to close possible gaps in your knowledge of school mathematics autonomously.

Problem 7:

Please get academic textbooks about mathematics for business, economics, and finance in your library, by searching through second-hand internet shops for used books, or at a bookshop.

It is of no great importance, which math books you use and read. Most introductory academic textbooks discuss the topics of the modular description. But it is of very great importance that you solve autonomously as many business math problems as possible. The books should help at and give support to your individual learning process.

As learning processes differ from person to person, you will find different books readable (or unreadable). So please compare different math books to find out which books are a good fit for you.

Problem 8:

Different textbooks use different mathematical notations. Our math course will mainly use the mathematical notations which are common in Germany, e.g. the notation used in the textbooks of Tietze (see modular description of the equivalent German math course “Wirtschaftsmathematik”, LV-Nr. 200 601.01 – Jürgen Tietze: Einführung in die angewandte Wirtschaftsmathematik, 17. Auflage, Springer Spektrum, Berlin, Heidelberg 2013).

It will be expected from you however, that you will gain the capability to deal with different mathematical notations and mathematical representations within your academic studies. Especially you should then be able to identify identical mathematical descriptions even if they are written in a mathematically totally different style.

Therefore please get several different math books and compare the mathematical styles and notations they are written in. The use of different letters or symbols for the same mathematical variable should not shatter you neither mathematically nor mentally.

Problem 9:

Please always bring your electronic pocket calculator to the lessons. And please practice to do complicated calculations with your pocket calculator. The results of the following calculations should be found by you within less than 20 seconds.

Please round all results to the nearer ten-thousandth (leave four decimal places).

a) $$400\left[\frac{1.085^{12}-1}{0.085}\right]$$

b) $$400\left[\frac{1-\frac{1}{1.085^{12}}}{0.085}\right]$$

c) $$\frac{400\cdot 1.085}{1-\frac{1}{(1+0.085)^{12}}}$$

d) $$\sqrt[5]{\frac{1.56\cdot 10^{8}}{34\cdot(5+\ln 300)}}$$

e) $$\frac{e^{\left(320\pi-10^{3}\right)}}{\ln\frac{1}{440}}$$

f) $$-2.95+\log_{10}\left[\frac{1-\frac{3}{5}}{\frac{27}{6}-3}\right]$$

Short note: At the written exam it is not allowed to use programmable electronic pocket calculators. Therefore please solve all problems with pocket calculators which are not programmable.

HWR Berlin, Wintersemester 2017/2018

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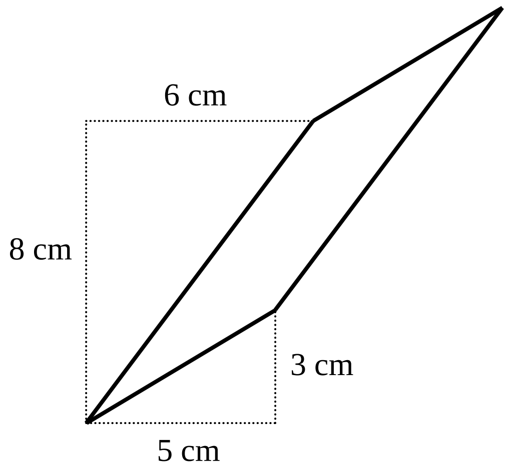
Berlin School of Economics and Law

Supplement of Worksheet 1 – Exercises

Problem 1:

The following parallelogram with the coordinates given by Nini had been discussed at the first IBMAN math lesson at Oct 12, 2017:



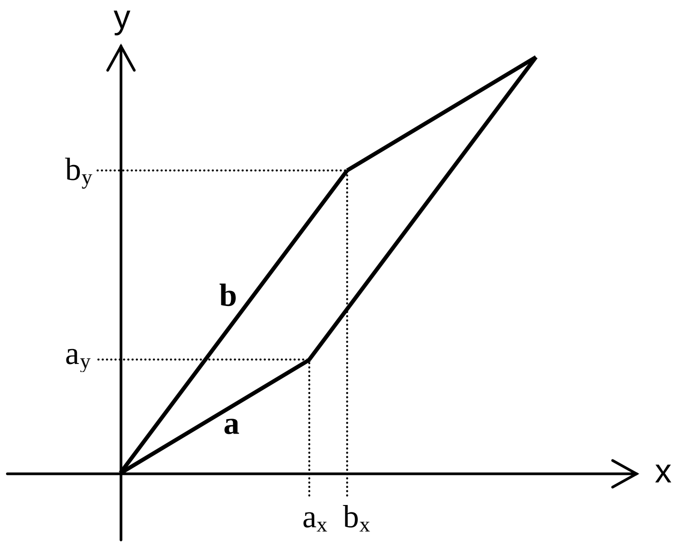
a) Find the area of this parallelogram with the strategy Hayate presented at the lesson.

b) Find the area of this parallelogram with the strategy invented by Grassmann.

Problem 2:

Now we generalize problem 1: The first side vector **a** will point a_x units of length into the direction of the x-axis and a_y units of length into the direction of the y-axis.

The second side vector **b** will point b_x units of length into the direction of the x-axis and b_y units of length into the direction of the y-axis.



Show that the equation of the formula of the area of this generalized parallelogram will be

$A_{parallelogram} = a_x\, b_y - a_y\, b_x$

by generalizing the solution strategy of Hayate.

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Berlin School of Economics and Law

Worksheet 3 – Exercises

Problem 1:

The following supply and demand functions of a market are given:

Supply: $p = x + 15$ Demand: $p = -2x + 60$

a) Find the equilibrium price p_e and equilibrium quantity x_e algebraically.

b) Compare your result with the graphical solution.

c) Transform the supply and demand functions into a system of two linear equations. Find the two coefficient vectors **a** and **b** and the resulting vector of constant terms **r**. Find the outer products (oriented areas) of the three different parallelograms which can be constructed by Pauli vectors **a**, **b**, and **r**. Find the equilibrium price p_e and equilibrium quantity x_e by the strategy of Grassmann, who simply divided the outer products.

Problem 2:

Find the equilibrium price p_e and equilibrium quantity x_e for each of the following markets algebraically. Compare your results with the graphical solutions.

a) Supply: $p = 3x + 40$
Demand: $p = -x + 120$

b) Supply: $p = 3x + 20$
Demand: $p = -\frac{1}{2}x + 90$

c) Supply: $p = \frac{1}{4}x + 400$
Demand: $p = -\frac{1}{2}x + 1000$

d) Supply: $p = 750$
Demand: $p = -\frac{5}{8}x + 1270$

Problem 3:

Solve problem 2 with Geometric Algebra.

Problem 4:

Find the equilibrium price and the equilibrium quantity for each of the following markets:

a) Supply: $x = \frac{1}{3}p - 4$
Demand: $x = -2p + 206$

b) Supply: $x = \frac{2}{5}(p - 80)$
Demand: $x = 319 - \frac{5}{7}p$

Problem 5:

Solve problem 4 with Geometric Algebra.

Problems 6 – 14 have no reference to Geometric Algebra.

HWR Berlin, Wintersemester 2017/2018

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Worksheet 8 – Exercises

Problem 1:

Find the areas of the parallelograms if the two different sides are given by the following vectors and if the two base vectors σ_x and σ_y have lengths $|\sigma_x|$ and $|\sigma_y|$ of 1 cm. Please also draw a sketch of the parallelograms.

a) $\mathbf{a} = 5\,\sigma_x + 2\,\sigma_y$, $\mathbf{b} = 2\,\sigma_x + 6\,\sigma_y$

b) $\mathbf{a} = 8\,\sigma_x + 7\,\sigma_y$, $\mathbf{b} = 2\,\sigma_x + 20\,\sigma_y$

c) $\mathbf{a} = 5\,\sigma_x - 5\,\sigma_y$, $\mathbf{b} = 3\,\sigma_x + 7\,\sigma_y$

d) $\mathbf{a} = 4\,\sigma_x + 16\,\sigma_y$, $\mathbf{b} = 9\,\sigma_x + 2\,\sigma_y$

Problem 2:

Find the areas of the parallelograms if the two different sides are given by the following vectors and if the two base vectors σ_x and σ_y have lengths $|\sigma_x|$ and $|\sigma_y|$ of 1 cm. Please also draw a sketch of the parallelograms if possible and find the precise names of the given parallelograms.

a) $\mathbf{a} = 6\,\sigma_x + 4\,\sigma_y$, $\mathbf{b} = -4\,\sigma_x + 6\,\sigma_y$

b) $\mathbf{a} = -4{,}8\,\sigma_x - 3{,}4\,\sigma_y$, $\mathbf{b} = -5.1\,\sigma_x + 7.2\,\sigma_y$

c) $\mathbf{a} = 4\,\sigma_x + 3\,\sigma_y$, $\mathbf{b} = 12\,\sigma_x + 9\,\sigma_y$

d) $\mathbf{a} = 5\,\sigma_x + 20\,\sigma_y$, $\mathbf{b} = -\sigma_x - 4\,\sigma_y$

Problem 3:

Solve the following systems of linear equations and check your results.

a) $3x + 8y = 28$, $6x + 2y = 28$

b) $4x + 9y = 29$, $5x + 6y = 31$

c) $6x + 4y = 6$, $2x + y = 3$

d) $5x - 2y = 6$, $-2x - 3y = 28$

Problem 4:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

3 units of R_1	and	6 units of R_2	to produce	1 unit of P_1
8 units of R_1	and	2 units of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 28 units of the first raw material R_1 and 28 units of the second raw material R_2 are consumed in the production process. (Hint: Results of problem 3 can be used.)

Problem 5:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

2 units of R_1	and	5 units of R_2	to produce	1 unit of P_1
7 units of R_1	and	1 unit of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 2050 units of the first raw material R_1 and 1000 units of the second raw material R_2 are consumed in the production process.

Problem 6:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

4 units of R_1	and	1 unit of R_2	to produce	1 unit of P_1
3 units of R_1	and	5 units of R_2	to produce	1 unit of P_2

In the first quarter of a year exactly 33000 units of the first raw material R_1 and 38000 units of the second raw material R_2 are consumed in the production process. In the second quarter exactly 32000 units of the first raw material R_1 and 25000 units of the second raw material R_2 are consumed in the production process.

Find the quantities of final products P_1 and P_2 which will be produced in the first quarter, and find the quantities of final products P_1 and P_2 which will be produced in the second quarter.

Problem 7:

A firm manufactures two different final products P_1 and P_2. To produce these final products two intermediate goods G_1 and G_2 are required. The production of the intermediate goods requires two different raw materials R_1 and R_2. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2
R_1	8	2
R_2	4	3

	P_1	P_2
R_1	42	28
R_2	23	26

Find the demand matrix of the second production step which shows the demand of intermediate goods to produce one unit of each final product.

Problem 8:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products two intermediate goods G_1 and G_2 are required. The production of the intermediate goods requires two different raw materials R_1 and R_2. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2
R_1	9	3
R_2	2	2

	P_1	P_2	P_3
R_1	48	21	84
R_2	12	14	32

Find the demand matrix of the second production step which shows the demand of intermediate goods to produce one unit of each final product.

Problem 9:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

7 units of R_1	and	4 units of R_2	to produce	1 unit of P_1
5 units of R_1	and	3 units of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the first raw material R_1 had been consumed in the production process.
And find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the second raw material R_2 had been consumed in the production process.

How can these results be understood?
Give an economic interpretation of the results.

Problem 10:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

10 units of R_1	and	4 units of R_2	to produce	1 unit of P_1
12 units of R_1	and	5 units of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the first raw material R_1 had been consumed in the production process.
And find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the second raw material R_2 had been consumed in the production process.

Find the inverse of the demand matrix and check your result.

Problem 11:

Find the inverses of the following matrices and check your results.

a) $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$ b) $\mathbf{B} = \begin{bmatrix} 10 & 4 \\ 19 & 8 \end{bmatrix}$ c) $\mathbf{C} = \begin{bmatrix} 10 & 6 \\ 20 & 13 \end{bmatrix}$ d) $\mathbf{D} = \begin{bmatrix} 0 & -2.5 \\ 0.2 & 3.4 \end{bmatrix}$

HWR Berlin, Wintersemester 2017/2018

Mathematics for Business and Economics

Berlin School of Economics and Law

Worksheet 9 – Exercises

Problem 1:

a) Download the "**Geometric Algebra Algorithms Optimizer**" (GAALOP), which is a software freely available at the internet. The GAALOP homepage can be found at the URL

www.gaalop.de

Activate first the "**Download**" link to reach the download page at the URL

www.gaalop.de/download

GAALOP can then be downloaded by activating the blue link "**download Gaalop**".

b) After installing GAALOP can be started by activating the "**Start**" icon or by starting the java file "**java -jar starter-1.0.0.jar**" directly, which depends on the configuration and the security adjustments of your computer.

Problem 2:

Please get acquainted with GAALOP by thinking up and then solving simple Geometric Algebra calculations.

Chose Pauli Algebra of three dimensional, Euclidean space as

Algebra to use: „**3d – vectors in 3d**"

on the basis of CluCalc.

For instance enter the three vectors

$$\mathbf{a} = 4\,\sigma_x + 8\,\sigma_y$$
$$\mathbf{b} = 10\,\sigma_x + 3\,\sigma_y$$
$$\mathbf{c} = 5\,\sigma_x - 5\,\sigma_y$$

into a GAALOP program and determine the following sums or differences:

a) $\mathbf{p} = \mathbf{a} + \mathbf{b}$ b) $\mathbf{q} = 4\,\mathbf{a} + 2\,\mathbf{b}$ c) $\mathbf{r} = \mathbf{b} - 2\,\mathbf{c}$ d) $\mathbf{s} = 65\,\mathbf{a} - 60\,\mathbf{b} + 68\,\mathbf{c}$

Problem 3:

Solve all problems of previous worksheet 8 with the help of GAALOP programs.

The problems of worksheet 8 have been problems about systems of two linear equations with two unknown variables. Such problems can be solved with Geometric Algebra by vectors, which point into only two directions and which thus are situated in the xy-plane.

At the following pages of this worksheet 9 you will now find problems about systems of three linear equations. To solve these problems the mathematics of vectors, which point into three directions and which are situated in three-dimensional space, is required. Thus vectors will now have three components, representing x, y, and z directions.

Problem 4:

a) A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

5 units of R_1,	4 units of R_2,	and	3 units of R_3	to produce	1 unit of P_1
			2 units of R_3	to produce	1 unit of P_2

Find (with the help of GAALOP) the quantities of final products P_1 and P_2 which will be produced, if exactly 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process.

b) A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

5 units of R_1,	4 units of R_2,	and	3 units of R_3	to produce	1 unit of P_1
6 units of R_1,	7 units of R_2,	and	8 units of R_3	to produce	1 unit of P_2

Find (with the help of GAALOP) the quantities of final products P_1 and P_2 which will be produced, if exactly 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process.

Please find not only a GAALOP program, but give also detailed calculations of intermediate steps using Geometric Algebra and compare with conventional solution strategies. Finally check your results.

Problem 5:

Find the volume of the parallelepipeds with the help of GAALOP if the three different sides of the parallelepipeds are given by the following vectors and if the base vectors σ_x, σ_y, and σ_z have lengths $|\sigma_x|$, $|\sigma_y|$, and $|\sigma_z|$ of 1 cm.

a) $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$
$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y$
$\mathbf{c} = 3\,\sigma_z$

b) $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$
$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y$
$\mathbf{c} = 5\,\sigma_y + 5\,\sigma_z$

c) $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$
$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y$
$\mathbf{c} = 7\,\sigma_x + 7\,\sigma_y + 7\,\sigma_z$

d) $\mathbf{a} = 2\,\sigma_x + 5\,\sigma_y + 5\,\sigma_z$
$\mathbf{b} = 3\,\sigma_x + 3\,\sigma_y + 6\,\sigma_z$
$\mathbf{c} = 4\,\sigma_x + 4\,\sigma_y + 4\,\sigma_z$

e) $\mathbf{a} = 2\,\sigma_x + 6\,\sigma_y + 10\,\sigma_z$
$\mathbf{b} = 8\,\sigma_x + 3\,\sigma_y + 12\,\sigma_z$
$\mathbf{c} = 7\,\sigma_x + 9\,\sigma_y + 4\,\sigma_z$

f) $\mathbf{a} = 4\,\sigma_x + 8\,\sigma_y - 5\,\sigma_z$
$\mathbf{b} = 3\,\sigma_x - 7\,\sigma_y + 6\,\sigma_z$
$\mathbf{c} = -2\,\sigma_x + 9\,\sigma_y - \sigma_z$

Please also draw a sketch of the parallelepipeds of the first three exercises a), b), and c) and compare all GAALOP results with the results you get by applying the rule of Sarrus to find the determinants of the coefficient matrices.

Problem 6:

Solve the following systems of linear equations with the help of GAALOP either directly or by programming intermediate steps and check your results.

a)
$3x + 8y = 28$
$6x + 2y = 28$
$2x + 4y + 2z = 28$

b)
$8x + 5y + 10z = 396$
$3x + 7y + 12z = 375$
$2x + 6y + 14z = 386$

c)
$3x - 5y + 6z = 41$
$-2x + 5y + 8z = 111$
$7x + y + 9z = 185$

d) $\frac{2}{5}x + \frac{7}{5}y + \frac{9}{5}z = 210$
$\frac{8}{5}x + \frac{1}{5}y + \frac{3}{5}z = 138$
$\frac{4}{5}x + \frac{12}{5}y + \frac{6}{5}z = 282$

Problem 7:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

7 units of R_1,	3 units of R_2,	and	4 units of R_3	to produce	1 unit of P_1
2 units of R_1,	9 units of R_2,	and	6 units of R_3	to produce	1 unit of P_2
5 units of R_1,		and	8 units of R_3	to produce	1 unit of P_3

Exactly 500 units of the first raw material R_1, 780 units of the second raw material R_2, and 880 units of the third raw material R_3 are consumed in the production process.

Find the output of final products P_1, P_2, and P_3 with the help of GAALOP.

Problem 8:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

12 units of R_1,	20 units of R_2,	and	16 units of R_3	to produce	1 unit of P_1
30 units of R_1,	15 units of R_2,	and	28 units of R_3	to produce	1 unit of P_2
10 units of R_1,	8 units of R_2,	and	25 units of R_3	to produce	1 unit of P_3

Exactly 12000 units of the first raw material R_1, 13900 units of the second raw material R_2, and 18300 units of the third raw material R_3 are consumed in the production process.

Find the output of final products P_1, P_2, and P_3 with the help of GAALOP.

Problem 9:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

9 units of R_1,	2 units of R_2,	and	7 units of R_3	to produce	1 unit of P_1
3 units of R_1,	2 units of R_2,	and	5 units of R_3	to produce	1 unit of P_2
4 units of R_1,	3 units of R_2,	and	2 units of R_3	to produce	1 unit of P_3

In the first quarter of a year exactly 98 units of the first raw material R_1, 35 units of the second raw material R_2, and 76 units of the third raw material R_3 are consumed in the production process.
In the second quarter exactly 61 units of the first raw material R_1, 30 units of the second raw material R_2, and 59 units of the third raw material R_3 are consumed in the production process.

Find the quantities of final products P_1, P_2, and P_3, which will be produced in the first quarter, and find the quantities of final products P_1, P_2, and P_3, which will be produced in the second quarter, with the help of GAALOP.

Problem 10:

A firm manufactures two different final products P_1 and P_2. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three different raw materials R_1, R_2, and R_3. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2	G_3
R_1	10	15	11
R_2	17	20	16
R_3	12	14	25

	P_1	P_2
R_1	964	814
R_2	1409	1184
R_3	1320	1093

Find the demand matrix of the second production step, which shows the demand of intermediate goods to produce one unit of each final product, with the help of GAALOP and check your result.

Problem 11:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three different raw materials R_1, R_2, and R_3. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2	G_3
R_1	8	6	6
R_2	7	5	7
R_3	5	4	0

	P_1	P_2	P_3
R_1	228	186	308
R_2	214	166	282
R_3	108	107	160

Find the demand matrix of the second production step, which shows the demand of intermediate goods to produce one unit of each final product, with the help of GAALOP and check your result.

Problem 12:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three different raw materials R_1, R_2, and R_3. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2	G_3
R_1	82	63	20
R_2	44	19	37
R_3	10	52	92

	P_1	P_2	P_3
R_1	4496	5462	4815
R_2	2530	3482	2801
R_3	3224	4062	4646

Find the demand matrix of the second production step, which shows the demand of intermediate goods to produce one unit of each final product, with the help of GAALOP and check your result.

Problem 13:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

3 units of R_1,	2 units of R_2,	and	8 units of R_3	to produce	1 unit of P_1
5 units of R_1,	6 units of R_2,	and	7 units of R_3	to produce	1 unit of P_2
4 units of R_1,	3 units of R_2,	and	10 units of R_3	to produce	1 unit of P_3

Find with the help of GAALOP the quantities of final products P_1, P_2, and P_3 which would have been produced in theory, if exactly one unit of the first raw material R_1 had been consumed in the production process.

Find with the help of GAALOP the quantities of final products P_1, P_2, and P_3 which would have been produced in theory, if exactly one unit of the second raw material R_2 had been consumed in the production process.

And find with the help of GAALOP the quantities of final products P_1, P_2, and P_3 which would have been produced in theory, if exactly one unit of the third raw material R_3 had been consumed in the production process.

Use the values just found to construct the inverse of the demand matrix and check your result.

Problem 14:

Find the inverses of the following matrices (if they exist) with the help of GAALOP and check your results.

a) $$\mathbf{A} = \begin{bmatrix} 1 & 4 & 9 \\ 7 & 2 & 6 \\ 6 & 3 & 8 \end{bmatrix}$$

b) $$\mathbf{B} = \begin{bmatrix} 0 & 4 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

c) $$\mathbf{C} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

d) $$\mathbf{D} = \begin{bmatrix} 3 & 4 & 8 \\ 10 & 5 & 10 \\ 10 & 20 & 15 \end{bmatrix}$$

HWR Berlin, Wintersemester 2017/2018

Mathematics for Business and Economics

Berlin School of Economics and Law

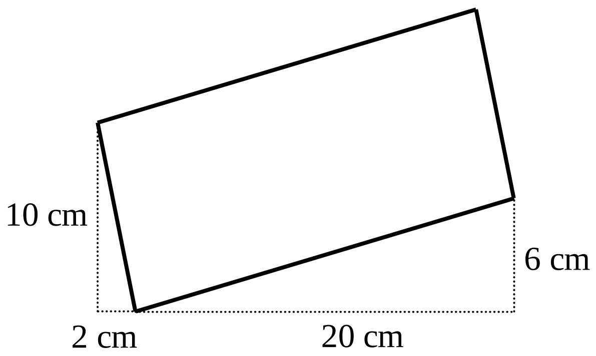
Worksheet 21 – Repetition of Linear Algebra / Exercises

Problem 1:
Problem 2:
Problem 3:
Problem 4:
Problem 5:
Problem 6:
Problem 7:

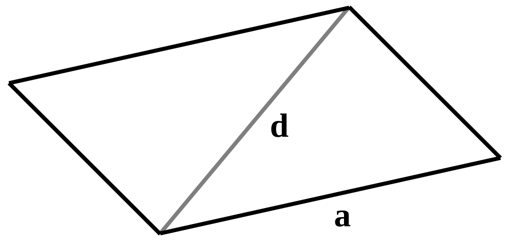
Conventional problems about Linear Algebra, which should be solved without Geometric Algebra.

Problem 8:

a) Find the area of the parallelogram.



b) Find the area of the parallelograms if the two base vectors σ_x and σ_y have lengths $|\sigma_x|$ and $|\sigma_y|$ of 1 cm.



$\mathbf{a} = 18\ \sigma_x + \ \ 4\ \sigma_y$

$\mathbf{d} = 10\ \sigma_x + 12\ \sigma_y$

Problem 9:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

8 units of R_1	and	5 units of R_2	to produce	1 unit of P_1
10 units of R_1	and	15 units of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 280 units of the first raw material R_1 and 280 units of the second raw material R_2 are consumed in the production process.

Problem 10:

A firm manufactures two different final products P_1 and P_2. To produce these final products two intermediate goods G_1 and G_2 are required. The production of the intermediate goods requires two raw materials R_1 and R_2.

The following quantities of raw materials are required in the production process:

7 units of R_1	and	8 units of R_2	to produce	1 unit of G_1
3 units of R_1	and	9 units of R_2	to produce	1 unit of G_2

Total demand of raw materials:

94 units of R_1	and	152 units of R_2	to produce	1 unit of P_1
80 units of R_1	and	175 units of R_2	to produce	1 unit of P_2

Find matrix **B** of the second production step which shows the demand of intermediate goods to produce one unit of each final product and check your result.

Problem 11:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three raw materials R_1, R_2, and R_3.

The following quantities of raw materials are required in the production process:

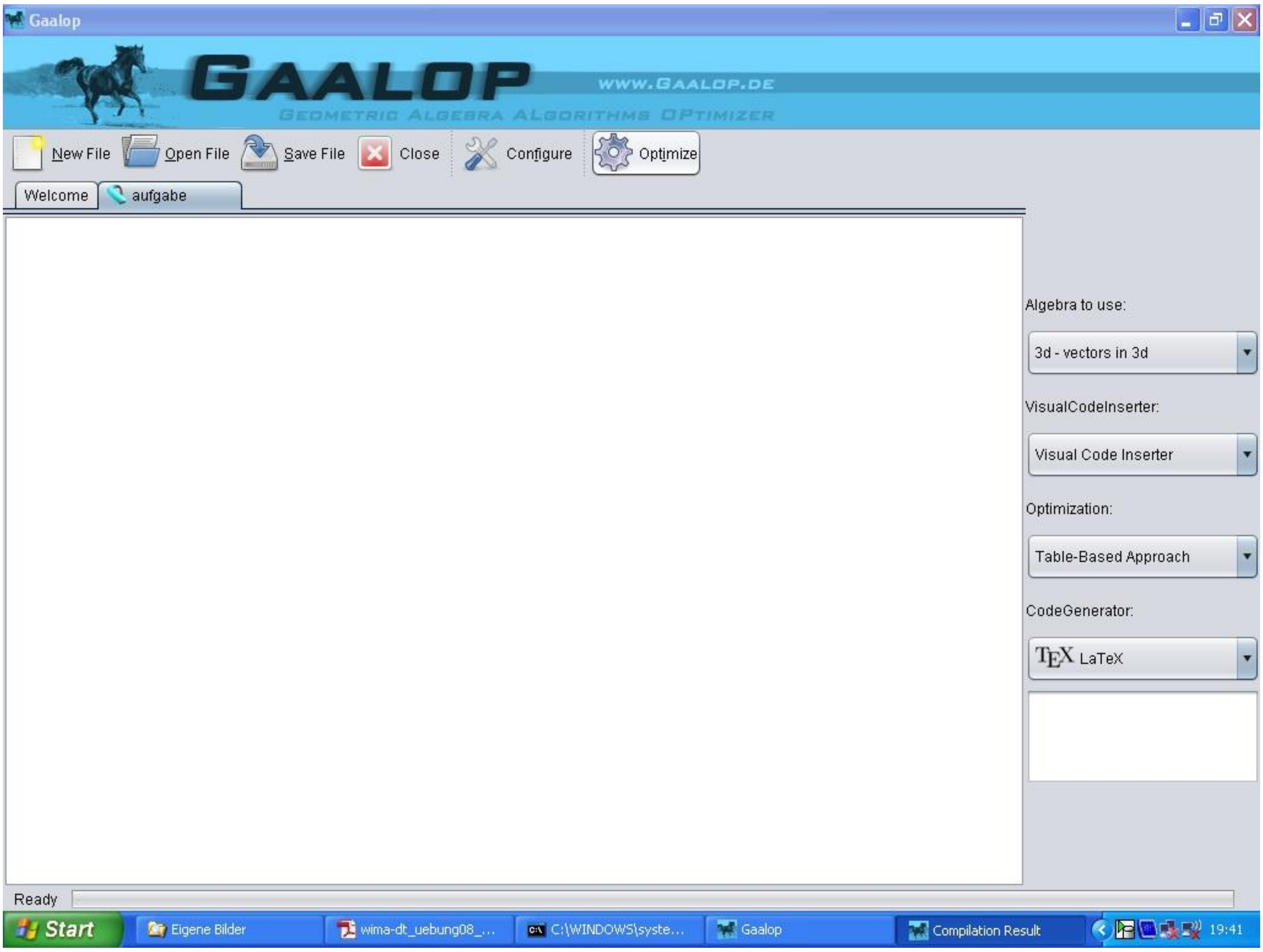
7 units of R_1 ,	6 units of R_2 ,	and	5 units of R_3	to produce	1 unit of P_1
9 units of R_1 ,	8 units of R_2 ,	and	7 units of R_3	to produce	1 unit of P_2
5 units of R_1 ,	4 units of R_2 ,	and	3 units of R_3	to produce	1 unit of P_3

Exactly 359 units of the first raw material R_1, 308 units of the second raw material R_2, and 257 units of the third raw material R_3 will be consumed in the production process.

a) Find the system of simultaneous linear equations which should be solved to find the unknown quantities of final products produced.

b) Give a GAALOP program which solves this system of linear equations at the GAALOP user interface of the next page.

c) The following solution of the system of linear equations will be expected to appear, if the ***Optimize*** button is activated: $x = 18$
$y = 17$
$z = 16$. Please check this solution.



d) Now the ***Optimize*** button is activated. Unfortunately the compiler result of the GAALOP program states that the solution values x, y, and z are undefined. What is wrong with the GAALOP program?

Give a mathematical reason why it is not possible to find the expected solution values of problem part (c) with the GAALOP program.

Problem 12:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three raw materials R_1, R_2, and R_3.

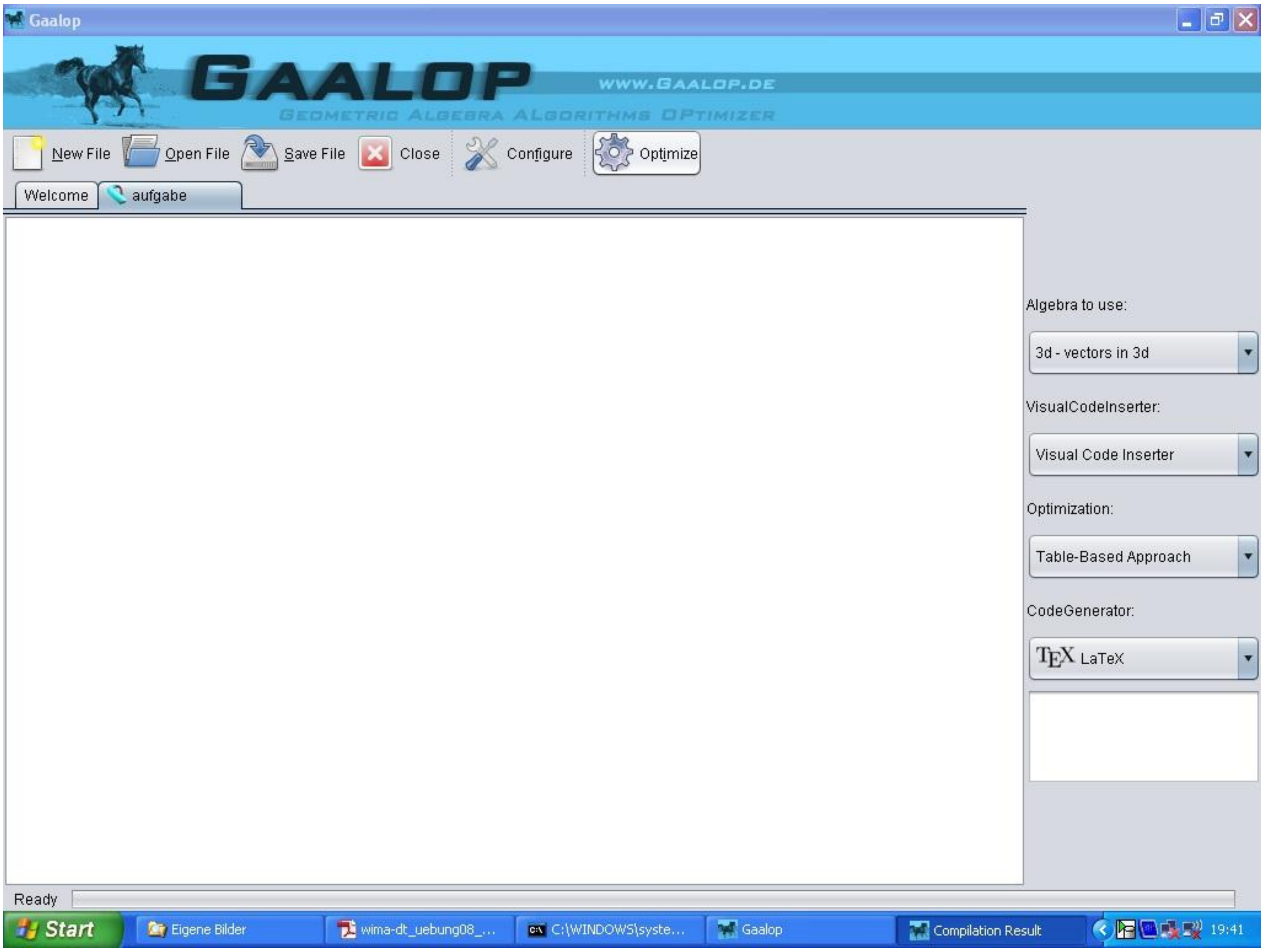
The following quantities of raw materials are required in the production process:

7 units of R_1 ,	6 units of R_2 ,	and	5 units of R_3	to produce	1 unit of P_1
9 units of R_1 ,	8 units of R_2 ,	and	7 units of R_3	to produce	1 unit of P_2
5 units of R_1 ,	4 units of R_2 ,	and	2 units of R_3	to produce	1 unit of P_3

Exactly 422 units of the first raw material R_1, 362 units of the second raw material R_2, and 283 units of the third raw material R_3 will be consumed in the production process.

a) Find the system of simultaneous linear equations which should be solved to find the unknown quantities of final products produced.

b) Give a GAALOP program which solves this system of linear equations at the following GAALOP user interface.



c) Now the ***Optimize*** button is activated. The compiler field shows the following results:

$x = 21$

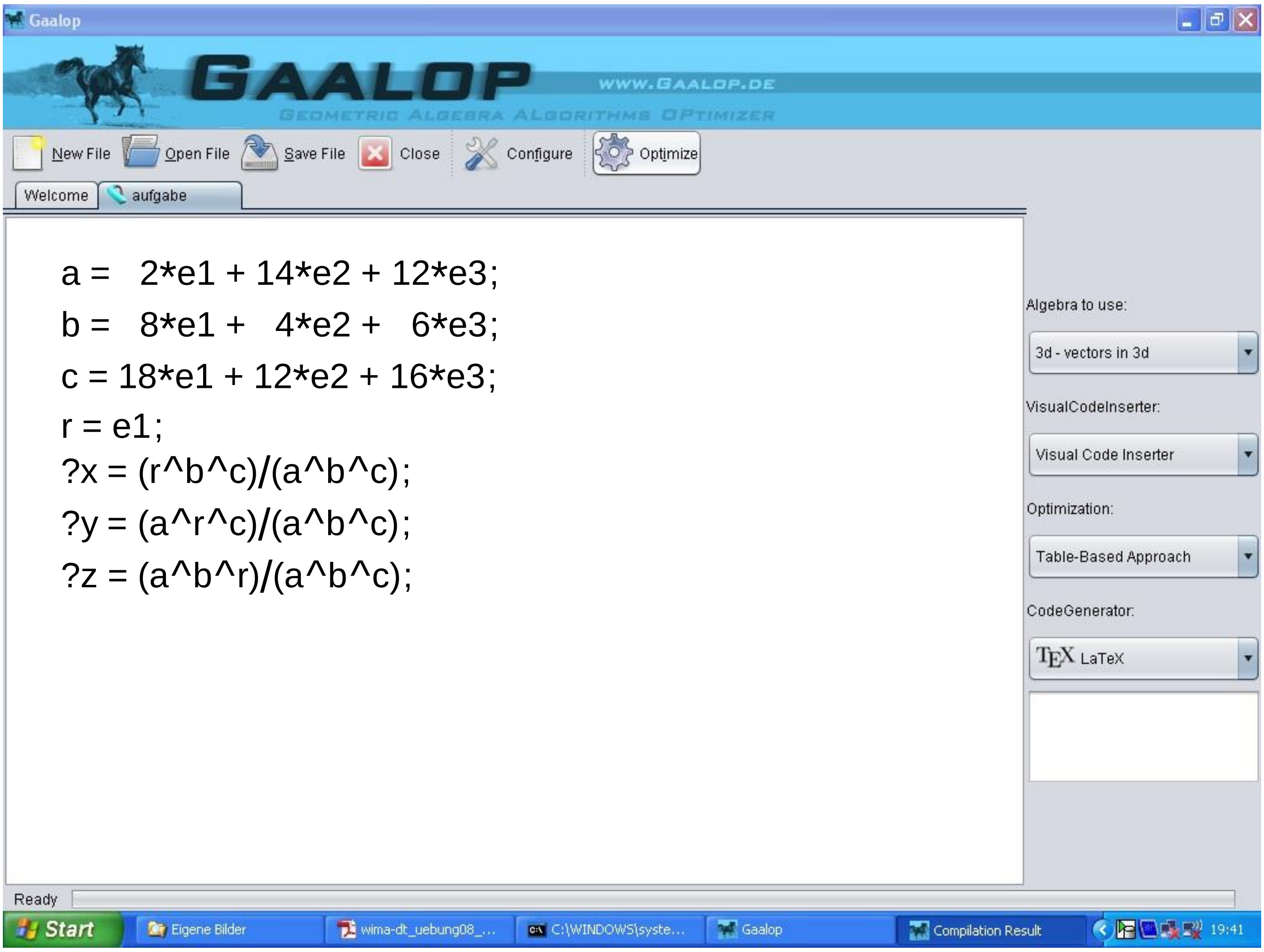
$y = 20$

$z = 19$

Please check this solution.

Problem 13:

The following GAALOP program is given:



a) Please state, which mathematical object will be calculated with the given GAALOP program.

b) Are the following results correct results of the GAALOP calculation?

$x = 1$

$y = 10$

$z = -4.5$

Please check these results.

HWR Berlin, Wintersemester 2017/2018

Mathematics for Business and Economics

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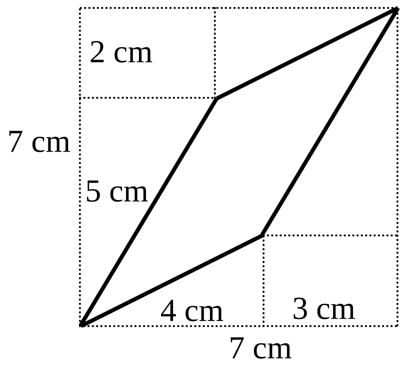
Worksheet 1 – Answers

Problem 3:

A possible starting point, which you already should have seen in school, can be to place the parallelogram into a great rectangle. This great rectangle will indeed be a great square at this problem 3, because the coordinate sums of both sides are identical: 5 + 2 = 4 + 3 = 7 Thus both sides of the rectangle have the same length.



$$A_{\text{parallelogram}} = A_{\text{square}} - 2 \cdot A_{\text{rectangle}} - 2 \cdot A_{\text{large triangle}} - 2 \cdot A_{\text{small triangle}}$$

$$= 7\,\text{cm} \cdot 7\,\text{cm} - 2 \cdot 3\,\text{cm} \cdot 2\,\text{cm} - 2 \cdot \frac{1}{2} \cdot 5\,\text{cm} \cdot 3\,\text{cm} - 2 \cdot \frac{1}{2} \cdot 4\,\text{cm} \cdot 2\,\text{cm}$$

$$= 49\,\text{cm}^2 - 2 \cdot 6\,\text{cm}^2 - 2 \cdot 7.5\,\text{cm}^2 - 2 \cdot 4\,\text{cm}^2$$

$$= 49\,\text{cm}^2 - 12\,\text{cm}^2 - 15\,\text{cm}^2 - 8\,\text{cm}^2$$

$$= 14\,\text{cm}^2$$

Problem 4:

The solution strategy of Grassmann is based on the commutation relations of the base vectors defined (and thus “invented”) by him. Because of historical reasons we will call theses base vectors Pauli vectors, symbolized by the Greek letter “sigma”: σ_x, σ_y, and σ_z.

σ_x = one step into the direction of the x-axis = base vector of x-direction

σ_y = one step into the direction of the y-axis = base vector of y-direction

σ_z = one step into the direction of the z-axis = base vector of z-direction

Every base vector should have the length of exactly one length unit. Thus they are unit vectors. Therefore Hermann Grassmann (and Wolfgang Pauli later) decided, that the square of their base vectors must be exactly one as important part of the definition (or of the “invention”):

$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$ $\Rightarrow$ This rule is called ***normalization***.

Besides that, Hermann Grassmann (and Wolfgang Pauli later) decided to calculate in a totally different way compared to the way we are used to do calculations with real numbers.

The result of a multiplication of two real numbers will not change if the order of the two factors is changed. For example the result of 3 times 7 is perfectly identical to the result of 7 times 3 (which will be 21 in both cases):

$3 \cdot 7 = 7 \cdot 3$ $\Rightarrow$ This mathematical behaviour is called ***commutativity***.

But if two Pauli vectors are multiplied, the result will change if the order of two base vectors of a multiplication is changed. An additional minus sign has then to be taken into account:

$$\sigma_x \sigma_y = -\sigma_y \sigma_x$$

$$\sigma_y \sigma_z = -\sigma_z \sigma_y$$

$$\sigma_z \sigma_x = -\sigma_x \sigma_z$$

$\Rightarrow$ This mathematical behaviour is called ***anti-commutativity***.

Hermann Grassmann (and Wolfgang Pauli later) haven chosen (or "invented") these algebraic rules, because they describe the successive walk into different directions mathematically in a pretty good way.

If we first go one step into the direction of the x-axis and if we then go another step into the direction of the y-axis (in a mathematical order of $\sigma_x \sigma_y$), we will move in an anti-clockwise orientation (like car drivers in a traffic circle on the Continent of Europe, see figure on the left).

But if we first go one step into the direction of the y-axis and if we then go another step into the direction of the x-axis (in a mathematical order of $\sigma_y \sigma_x$), we will move in a clockwise orientation (like car drivers in Britain or ghost-drivers on the Continent in a traffic circle, see figure on the right).

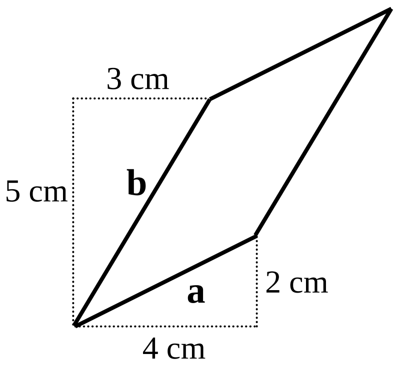
mathematically positive orientation (Continental car driver in a traffic circle)

mathematically negative orientation (Continental ghost-driver in a traffic circle)

Thus: $\sigma_x \sigma_y = -\sigma_y \sigma_x$ $\Rightarrow$ This is the basic rule of ***Pauli Algebra***. (first invented by Hermann Grassmann)

An additional minus sign thus indicates ***geometrically*** a reversal of the direction of rotation (or of the orientation), while an additional minus sign indicates **algebraically** a change of the order of two neighboring anti-commuting factors.

Now the area of a given parallelogram can be found in a very simple way simply by multiplying both side vectors of the parallelogram:



First vector **a** points simultaneously 4 units of length into x-direction and 2 units of length into y-direction:

$\Rightarrow$ $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$

Second vector **b** points simultaneously 3 units of length into x-direction and 5 units of length into y-direction:

$\Rightarrow$ $\mathbf{b} = 3\,\sigma_x + 5\,\sigma_y$

Product of the two side vectors of the parallelogram:

$$\mathbf{a}\,\mathbf{b} = (4\,\sigma_x + 2\,\sigma_y)\,(3\,\sigma_x + 5\,\sigma_y)$$

$$= 4 \cdot 3\,\sigma_x^2 + 4 \cdot 5\,\sigma_x\sigma_y + 2 \cdot 3\,\sigma_y\sigma_x + 2 \cdot 5\,\sigma_y^2$$

$$= 12\,\sigma_x^2 + 20\,\sigma_x\sigma_y + 6\,\sigma_y\sigma_x + 10\,\sigma_y^2$$

$$\sigma_x^2 \to 1, \quad \sigma_y\sigma_x \to -\,\sigma_x\sigma_y, \quad \sigma_y^2 \to 1$$

$$= 12 \cdot 1 + 20\,\sigma_x\sigma_y + 6\,(-\,\sigma_x\sigma_y) + 10 \cdot 1$$

$$= 12 + 20\,\sigma_x\sigma_y - 6\,\sigma_x\sigma_y + 10$$

$$= 22 + 14\,\sigma_x\sigma_y$$

This second part of the product, which contains two base vectors $\sigma_x\sigma_y$, is called outer product of the two vectors **a** and **b**. Mathematicians symbolize outer products by a ***wedge*** $\wedge$. This outer product is identical to the oriented area **A** of the parallelogram.
The magnitude of the outer product $|\mathbf{a} \wedge \mathbf{b}|$ therefore is identical to the area $|\mathbf{A}|$ of the parallelogram.

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 14\,\sigma_x\sigma_y$

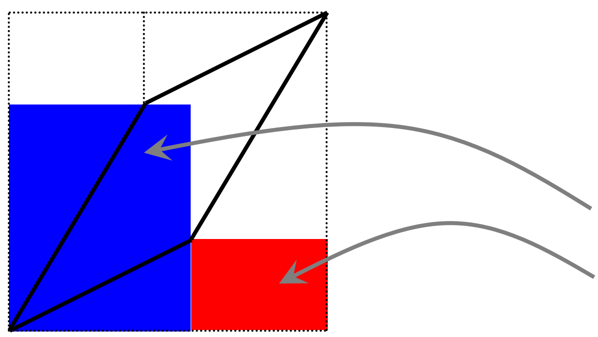
$\Rightarrow$ $|\mathbf{A}| = 14$

$\Rightarrow$ The area of the parallelogram is 14 cm^2.

This wedge $\wedge$ and the corresponding algebra are inventions as well. This algebra is called ***Grassmann Algebra***. Calculations which contain only wedge products (outer products) follow this Grassmann Algebra.

Until NOW all this is a repetition of the solution strategy of Grassmann, which we have discussed at the first lesson. Now the solution of worksheet problem 4 will follow.

Comparison of solution strategy you know from school … … with the solution strategy of Grassmann:



$$\mathbf{a} \wedge \mathbf{b} = 4\,\sigma_x \cdot 5\,\sigma_y - 3\,\sigma_x \cdot 2\,\sigma_y$$

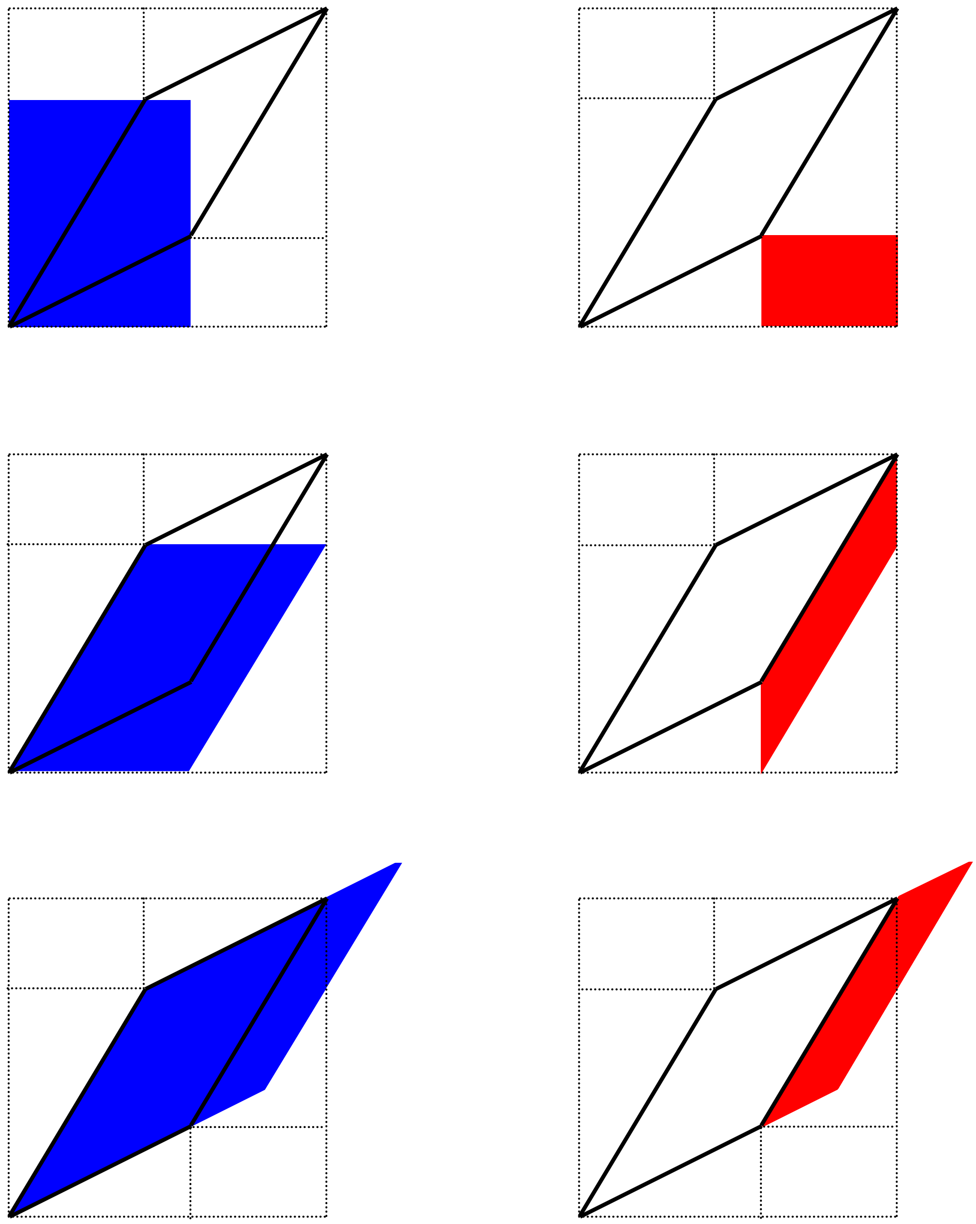
$$= 20\,\sigma_x\sigma_y - 6\,\sigma_x\sigma_y$$

$$= 14\,\sigma_x\sigma_y$$

$4\,\sigma_x \cdot 5\,\sigma_y = 20\,\sigma_x\sigma_y$

$3\,\sigma_x \cdot 2\,\sigma_y = 6\,\sigma_x\sigma_y$

It can be shown by parallel displacement that the area of the parallelogram is identical to the difference of the areas of the blue and red rectangles.



⇒ If the red smaller parallelogram of the right figure is subtracted from the blue parallelogram of the left figure, we will get the area of the original parallelogram $|\mathbf{a} \wedge \mathbf{b}|$. Thus the area of the original parallelogram is indeed identical to the difference of the areas of the blue and red rectangles which form the starting point of Grassmann’s calculation.

To find out how to split parallelograms with the help of an outer algebra in a mathematical correct and an algebraically consistent way mankind needed over 4½ thousand years. It all started in Mesopotamia and was a really long mathematical struggle until Hermann Grassmann was finally able to write in 1844, that by applying his theory of extensions ***algebra will gain a substantially different shape*** (in German: „Durch diese Anwendung [werde] auch die Algebra eine wesentlich veränderte Gestalt gewinnen.“ Quotation from Hermann Grassmann: Die Wissenschaft der extensiven Größe oder die Ausdehnungslehre. Erster Theil, die lineale Ausdehnungslehre enthaltend. Verlag von Otto Wigand, Leipzig 1844, p. 71).

Problem 5:

a) $\mathbf{a} = 9\ \sigma_x$
$\mathbf{b} = 3\ \sigma_x + 5\ \sigma_y$

$\mathbf{a}\ \mathbf{b} = (9\ \sigma_x)\ (3\ \sigma_x + 5\ \sigma_y)$

$= 27\ \sigma_x^2 + 45\ \sigma_x\sigma_y$

$= 27 + 45\ \sigma_x\sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 45\ \sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 45$

$\Rightarrow$ The area of the parallelogram is 45 cm^2.

b) $\mathbf{a} = 4.2\ \sigma_x + 1.6\ \sigma_y$
$\mathbf{b} = 3.2\ \sigma_x + 3.9\ \sigma_y$

$\mathbf{a}\ \mathbf{b} = (4.2\ \sigma_x + 1.6\ \sigma_y)\ (3.2\ \sigma_x + 3.9\ \sigma_y)$

$= 13.44\ \sigma_x^2 + 16.38\ \sigma_x\sigma_y + 5.12\ \sigma_y\sigma_x + 6.24\ \sigma_y^2$

$= 13.44 + 16.38\ \sigma_x\sigma_y - 5.12\ \sigma_x\sigma_y + 6.24$

$= 19.68 + 11.26\ \sigma_x\sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 11.26\ \sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 11.26$

$\Rightarrow$ The area of the parallelogram is 11.26 cm^2.

c) $\mathbf{a} = 6\ \sigma_x + 3\ \sigma_y$
$\mathbf{b} = -5\ \sigma_x + 10\ \sigma_y$

$\mathbf{a}\ \mathbf{b} = (6\ \sigma_x + 3\ \sigma_y)\ (-5\ \sigma_x + 10\ \sigma_y)$

$= -30\ \sigma_x^2 + 60\ \sigma_x\sigma_y - 15\ \sigma_y\sigma_x + 30\ \sigma_y^2$

$= -30 + 60\ \sigma_x\sigma_y + 15\ \sigma_x\sigma_y + 30$

$= 0 + 75\ \sigma_x\sigma_y$

$= 75\ \sigma_x\sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 75\ \sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 75$

$\Rightarrow$ The area of the parallelogram (which now is a square) is 75 cm^2.

d) $\mathbf{a} = 30\ \sigma_x + 15\ \sigma_y$
$\mathbf{b} = -40\ \sigma_x + 25\ \sigma_y$

$\mathbf{a}\ \mathbf{b} = (30\ \sigma_x + 15\ \sigma_y)\ (-40\ \sigma_x + 25\ \sigma_y)$

$= -1200\ \sigma_x^2 + 750\ \sigma_x\sigma_y - 600\ \sigma_y\sigma_x + 375\ \sigma_y^2$

$= -1200 + 750\ \sigma_x\sigma_y + 600\ \sigma_x\sigma_y + 375$

$= -825 + 1350\ \sigma_x\sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 1350\ \sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 1350$

$\Rightarrow$ The area of the parallelogram is 1350 cm^2.

Problem 9:

a) $$400\left[\frac{1.085^{12}-1}{0.085}\right] = 7\,819.699917 \approx 7\,819.6999$$

Please use decimal points when writing decimal numbers in English texts.
Decimal commas are only used in German texts.

b) $$400\left[\frac{1-\dfrac{1}{1.085^{12}}}{0.085}\right] = 2\,937.874428 \approx 2\,937.8744$$

Please write the equation sign (or equality sign) always on level with the main fraction line.

c) $$\frac{400\cdot 1.085}{1-\dfrac{1}{(1+0.085)^{12}}} = 695.180475 \approx 695.1805$$

If the digit to the right of the last digit you are keeping (the digit to the right of the last place-value digit) is 5 or greater than 5, then the last digit (the place-value digit) will be increased by one.

d) $$\sqrt[5]{\frac{1.56\cdot 10^{8}}{34\cdot(5+\ln 300)}} = 13.378939 \approx 13.3789$$

e) $$\frac{e^{\left(320\pi-10^{3}\right)}}{\ln\dfrac{1}{440}} = -\,33.232583 \approx -\,33.2326$$

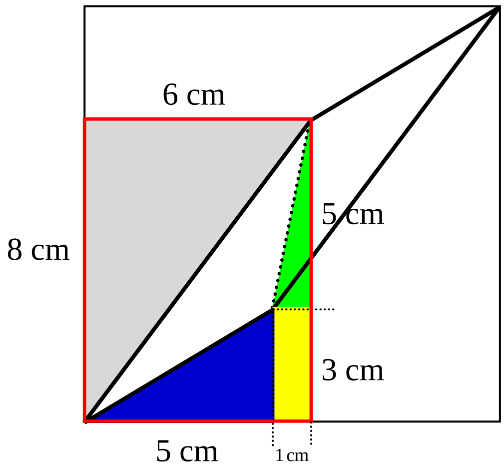
f) $$-\,2.95+\log_{10}\left[\frac{1-\dfrac{3}{5}}{\dfrac{27}{6}-3}\right] = -\,3.524032 \approx -\,3.5240$$

To show that the result is rounded to the nearer (i.e. to the nearest) ten-thousandth, please leave four decimal places and write a zero as last digit.
Thus the result will therefore be written as -3.5240.
The decimal number -3.524 is inaccurate because then the result is rounded to the nearer (or nearest) thousandth.

Supplement of Worksheet 1 – Answers

Problem 1:

a) Strategy of Hayate



If we do not choose the great black square but the smaller red rectangle as starting point, half of the area of the parallelogram can be calculated by subtracting the grey triangle, the blue triangle, the green triangle, and the yellow rectangle from the rectangle in red bold outline.

$$\frac{1}{2}A_{\text{parallelogram}} = A_{\text{rectangle in red bold outline}} - A_{\text{grey triangle}} - A_{\text{blue triangle}} - A_{\text{green triangle}} - A_{\text{yellow rectangle}}$$

$$= 6\,\text{cm}\cdot 8\,\text{cm} - \frac{1}{2}\cdot 6\,\text{cm}\cdot 8\,\text{cm} - \frac{1}{2}\cdot 5\,\text{cm}\cdot 3\,\text{cm} - \frac{1}{2}\cdot 1\,\text{cm}\cdot 5\,\text{cm} - 1\,\text{cm}\cdot 3\,\text{cm}$$

$$= 48\,\text{cm}^2 - 24\,\text{cm}^2 - 7.5\,\text{cm}^2 - 2.5\,\text{cm}^2 - 3\,\text{cm}^2$$

$$= 11\,\text{cm}^2$$

$$\Rightarrow\quad A_{\text{parallelogram}} = 2\cdot 11\,\text{cm}^2 = 22\,\text{cm}^2$$

b) Strategy of Grassmann:

$\mathbf{a} = 5\,\sigma_x + 3\,\sigma_y$

$\mathbf{b} = 6\,\sigma_x + 8\,\sigma_y$

$$\mathbf{a}\,\mathbf{b} = (5\,\sigma_x + 3\,\sigma_y)\,(6\,\sigma_x + 8\,\sigma_y)$$

$$= 5\cdot 6\,\sigma_x^2 + 5\cdot 8\,\sigma_x\sigma_y + 3\cdot 6\,\sigma_y\sigma_x + 3\cdot 8\,\sigma_y^2$$

$$= 30\,\sigma_x^2 + 40\,\sigma_x\sigma_y + 18\,\sigma_y\sigma_x + 24\,\sigma_y^2$$

$$= 30\cdot 1 + 40\,\sigma_x\sigma_y + 18\,(-\,\sigma_x\sigma_y) + 24\cdot 1$$

$= 30 + 40\,\sigma_x\sigma_y - 18\,\sigma_x\sigma_y + 24$

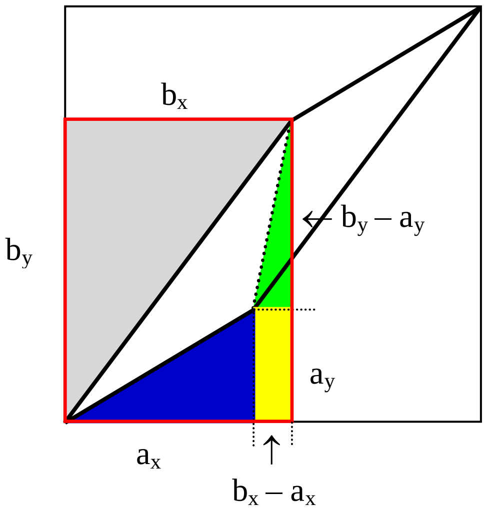
$= 54 + 22\,\sigma_x\sigma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 22\,\sigma_x\sigma_y$

$\Rightarrow \quad |\mathbf{A}| = |\mathbf{a} \wedge \mathbf{b}| = 22$

$\Rightarrow$ The area of the parallelogram is 22 cm^2.

Problem 2:

Generalized parallelogram:



$$\frac{1}{2}\,A_{\text{parallelogram}} = A_{\text{rectangle in red bold outline}} - A_{\text{grey triangle}} - A_{\text{blue triangle}} - A_{\text{greentriangle}} - A_{\text{yellow rectangle}}$$

$$= b_x b_y - \frac{1}{2}\,b_x b_y - \frac{1}{2}\,a_x a_y - \frac{1}{2}\cdot(b_x - a_x)\,(b_y - a_y) - (b_x - a_x)\,a_y$$

$$= \frac{1}{2}\,b_x b_y - \frac{1}{2}\,a_x a_y - \frac{1}{2}\cdot(b_x - a_x)\,(b_y - a_y) - (b_x - a_x)\,a_y$$

$$\Rightarrow \quad A_{\text{parallelogram}} = b_x b_y - a_x a_y - (b_x - a_x)\,(b_y - a_y) - 2\,(b_x - a_x)\,a_y$$

$$= b_x b_y - a_x a_y - (b_x b_y - b_x a_y - a_x b_y + a_x a_y) - 2\,b_x a_y + 2\,a_x a_y$$

$$= b_x b_y - a_x a_y - b_x b_y + b_x a_y + a_x b_y - a_x a_y - 2\,b_x a_y + 2\,a_x a_y$$

$$= b_x a_y + a_x b_y - 2\,b_x a_y$$

$= a_x b_y - a_y b_x$ QED (quod erat demonstrandum)

Mathematics for Business and Economics

Berlin School of Economics and Law

Worksheet 3 – Answers

Problem 1:

a) Equilibrium: $p_e = x_e + 15 = -2\,x_e + 60$

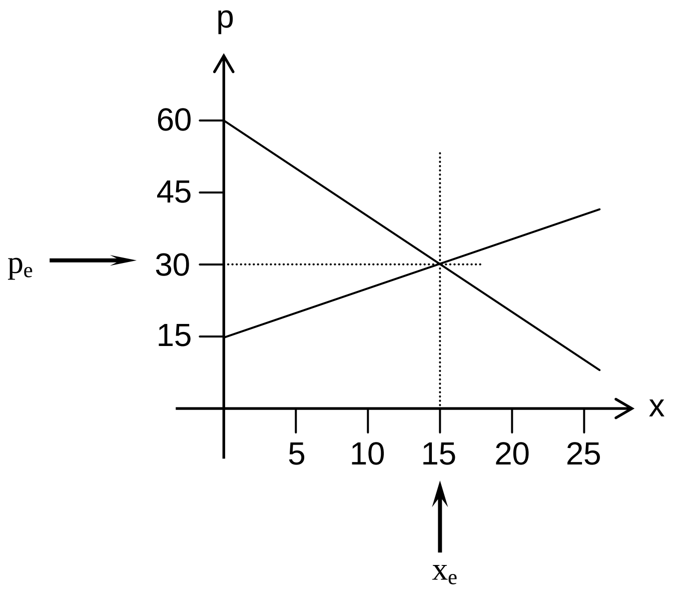
$3\,x_e = 45$

$x_e = 15$ (Equilibrium quantity)

$p_e = 15 + 15$ or $p_e = -2 \cdot 15 + 60$

$p_e = 30$ $\qquad$ $p_e = -30 + 60 = 30$ (Equilibrium price)

b) Graphical solution:



c) Supply: $p = x + 15 \Rightarrow x - p = -15 \Rightarrow 1\,x - 1\,p = -15$

Demand: $p = -2\,x + 60 \Rightarrow 2\,x + p = 60 \Rightarrow 2\,x + 1\,p = 60$

system of two linear equations

Coefficient vectors: $\mathbf{a} = 1\,\sigma_x + 2\,\sigma_y = \sigma_x + 2\,\sigma_y$

$\mathbf{b} = -1\,\sigma_x + 1\,\sigma_y = -\sigma_x + \sigma_y$

Resulting vector: $\mathbf{r} = -15\,\sigma_x + 60\,\sigma_y = -15\,\sigma_x + 60\,\sigma_y$

The following system of two linear equations has to be solved: $\mathbf{a}\,x + \mathbf{b}\,p = \mathbf{r}$

Outer products: $\mathbf{a} \wedge \mathbf{b} = \sigma_x\sigma_y - 2\,\sigma_y\sigma_x = \sigma_x\sigma_y + 2\,\sigma_x\sigma_y = 3\,\sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = -15\,\sigma_x\sigma_y - 60\,\sigma_y\sigma_x = -15\,\sigma_x\sigma_y + 60\,\sigma_x\sigma_y = 45\,\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = 60\,\sigma_x\sigma_y - 30\,\sigma_y\sigma_x = 60\,\sigma_x\sigma_y + 30\,\sigma_x\sigma_y = 90\,\sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \dfrac{45}{3} = 15$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \dfrac{90}{3} = 30$ (Equilibrium price)

Problem 2:

a) Equilibrium: $p_e = 3\,x_e + 40 = -x_e + 120$

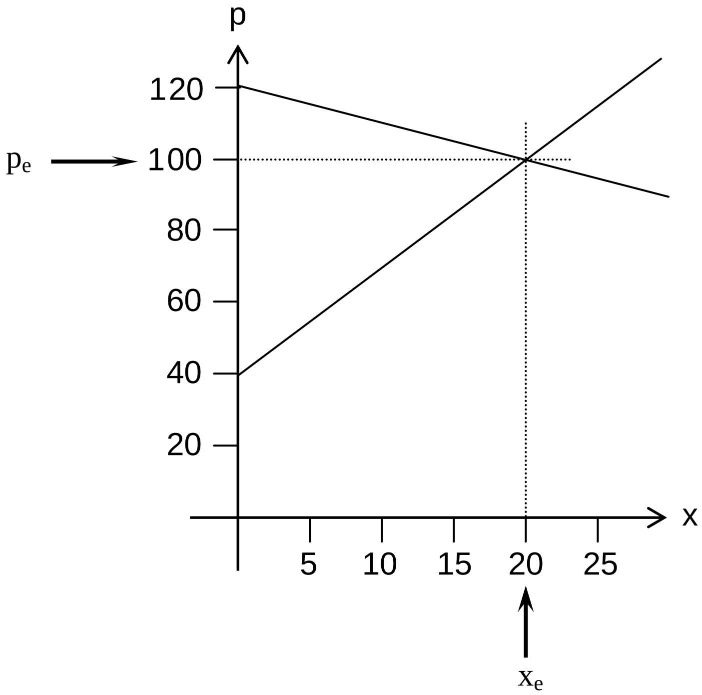
$4\,x_e = 80$

$x_e = 20$ (Equilibrium quantity)

$p_e = 3 \cdot 20 + 40$ or $p_e = -20 + 120$

$p_e = 60 + 40 = 100$ $\quad$ $p_e = 100$ (Equilibrium price)

Graphical solution:



b) Equilibrium: $p_e = 3\,x_e + 20 = -\dfrac{1}{2}\,x_e + 90$

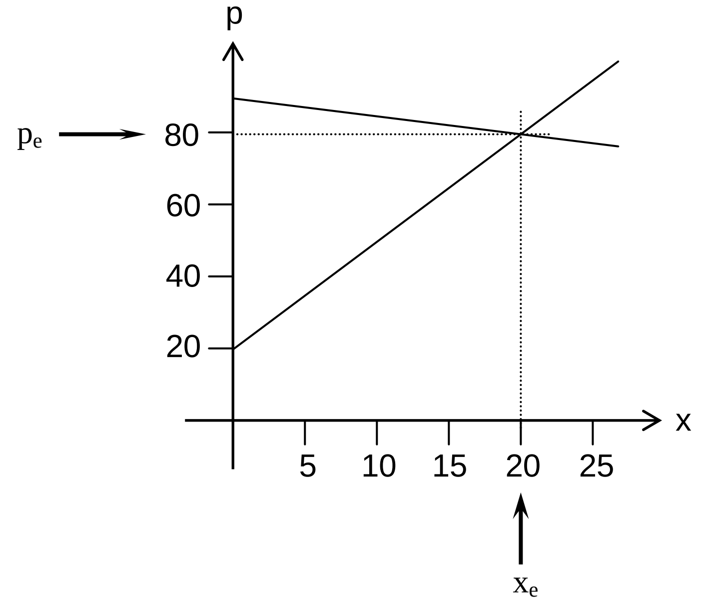
$3.5\,x_e = 70$

$x_e = 20$ (Equilibrium quantity)

$p_e = 3 \cdot 20 + 20$ or $p_e = -\dfrac{1}{2} \cdot 20 + 90$

$p_e = 60 + 20 = 80$ $\quad$ $p_e = -10 + 90 = 80$ (Equilibrium price)

Graphical solution:



c) Equilibrium: $p_e = \frac{1}{4} x_e + 400 = -\frac{1}{2} x_e + 1000$

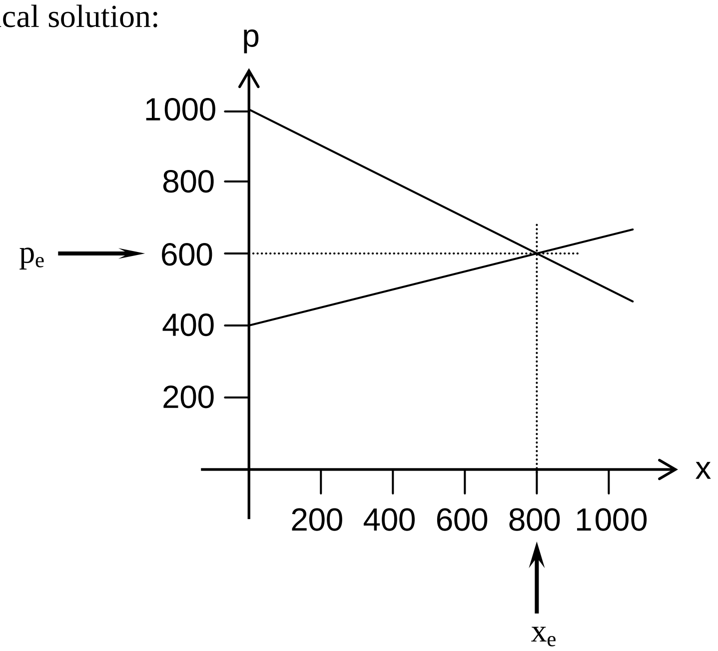
$$0.75\, x_e = 600$$

$$x_e = 800 \qquad \text{(Equilibrium quantity)}$$

$p_e = \frac{1}{4} \cdot 800 + 400$ or $p_e = -\frac{1}{2} \cdot 800 + 1000$

$p_e = 200 + 400 = 600$ $\quad$ $p_e = -400 + 1000 = 600$ (Equilibrium price)

Graphical solution:



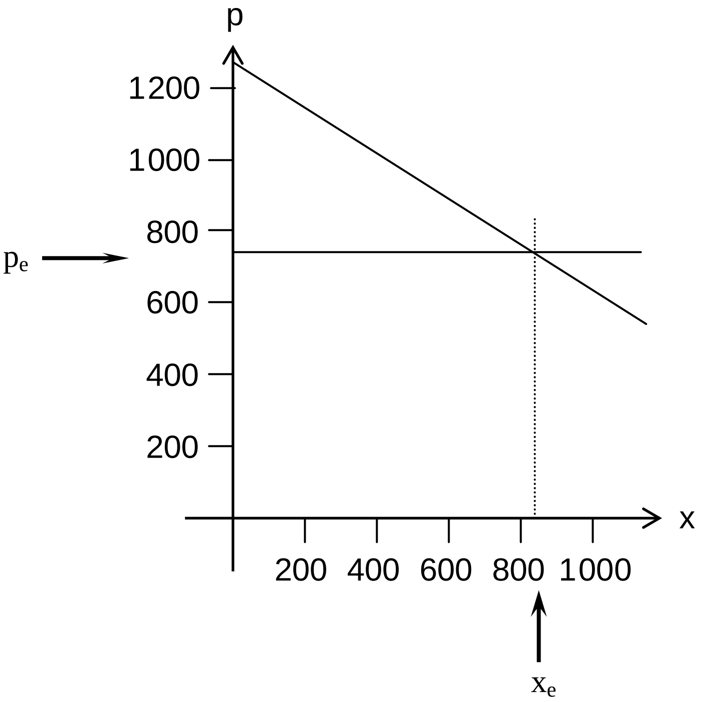
d) Equilibrium: $p_e = 750 = -\frac{5}{8} x_e + 1270$

$$0.625\, x_e = 520$$

$$x_e = 832 \quad \text{(Equilibrium quantity)}$$

$$p_e = 750 \quad \text{or} \quad p_e = -\frac{5}{8} \cdot 832 + 1270 = 750 \quad \text{(Equilibrium price)}$$

Graphical solution:



Problem 3:

Supply and demand functions form a system of two linear equations.

(2a) Supply: $p = 3x + 40 \Rightarrow 3x - p = -40 \Rightarrow \mathbf{3}\, x - \mathbf{1}\, p = \mathbf{-40}$

Demand: $p = -x + 120 \Rightarrow x + p = 120 \Rightarrow \mathbf{1}\, x + \mathbf{1}\, p = \mathbf{120}$

Pauli vectors: $\mathbf{a} = \mathbf{3}\, \sigma_x + \mathbf{1}\, \sigma_y = 3\, \sigma_x + \sigma_y$

$\mathbf{b} = \mathbf{-1}\, \sigma_x + \mathbf{1}\, \sigma_y = -\sigma_x + \sigma_y$

$\mathbf{r} = \mathbf{-40}\, \sigma_x + \mathbf{120}\, \sigma_y = -40\, \sigma_x + 120\, \sigma_y$

Outer products: $\mathbf{a} \wedge \mathbf{b} = 3\, \sigma_x\sigma_y - \sigma_y\sigma_x = 3\, \sigma_x\sigma_y + \sigma_x\sigma_y = 4\, \sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = -40\, \sigma_x\sigma_y - 120\, \sigma_y\sigma_x = -40\, \sigma_x\sigma_y + 120\, \sigma_x\sigma_y = 80\, \sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = 360\, \sigma_x\sigma_y - 40\, \sigma_y\sigma_x = 360\, \sigma_x\sigma_y + 40\, \sigma_x\sigma_y = 400\, \sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r} \wedge \mathbf{b}) = \frac{80}{4} = 20$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r}) = \frac{400}{4} = 100$ (Equilibrium price)

(2b) Supply: $p = 3x + 20 \quad \Rightarrow \quad 3x - p = -20 \quad \Rightarrow \quad 3x - 1p = -20$

Demand: $p = -\frac{1}{2}x + 90 \quad \Rightarrow \quad \frac{1}{2}x + p = 90 \quad \Rightarrow \quad \frac{1}{2}x + 1p = 90$

Pauli vectors: $\mathbf{a} = 3\,\sigma_x + 0.5\,\sigma_y$

$\mathbf{b} = -\sigma_x + \sigma_y$

$\mathbf{r} = -20\,\sigma_x + 90\,\sigma_y$

Outer products: $\mathbf{a} \wedge \mathbf{b} = 3\,\sigma_x\sigma_y - 0.5\,\sigma_y\sigma_x = 3\,\sigma_x\sigma_y + 0.5\sigma_x\sigma_y = 3.5\,\sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = -20\,\sigma_x\sigma_y - 90\,\sigma_y\sigma_x = -20\,\sigma_x\sigma_y + 90\,\sigma_x\sigma_y = 70\,\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = 270\,\sigma_x\sigma_y - 10\,\sigma_y\sigma_x = 270\,\sigma_x\sigma_y + 10\,\sigma_x\sigma_y = 280\,\sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{r} \wedge \mathbf{b}) = \frac{70}{3.5} = 20$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{a} \wedge \mathbf{r}) = \frac{280}{3.5} = 80$ (Equilibrium price)

(2c) Supply: $p = \frac{1}{4}x + 400 \quad \Rightarrow \quad \frac{1}{4}x - p = -400 \quad \Rightarrow \quad \frac{1}{4}x - 1p = -400$

Demand: $p = -\frac{1}{2}x + 1000 \quad \Rightarrow \quad \frac{1}{2}x + p = 1000 \quad \Rightarrow \quad \frac{1}{2}x + 1p = 1000$

Pauli vectors: $\mathbf{a} = 0.25\,\sigma_x + 0.5\,\sigma_y$

$\mathbf{b} = -\sigma_x + \sigma_y$

$\mathbf{r} = -400\,\sigma_x + 1000\,\sigma_y$

Outer products: $\mathbf{a} \wedge \mathbf{b} = 0.25\,\sigma_x\sigma_y - 0.5\,\sigma_y\sigma_x = 0.25\,\sigma_x\sigma_y + 0.5\sigma_x\sigma_y = 0.75\,\sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = -400\,\sigma_x\sigma_y - 1000\,\sigma_y\sigma_x = -400\,\sigma_x\sigma_y + 1000\,\sigma_x\sigma_y = 600\,\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = 250\,\sigma_x\sigma_y - 200\,\sigma_y\sigma_x = 250\,\sigma_x\sigma_y + 200\,\sigma_x\sigma_y = 450\,\sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{r} \wedge \mathbf{b}) = \frac{600}{0.75} = 800$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{a} \wedge \mathbf{r}) = \frac{450}{0.75} = 600$ (Equilibrium price)

(2d) Supply: $p = 750 \quad \Rightarrow \quad p = 750 \quad \Rightarrow \quad 0x + 1p = 750$

Demand: $p = -\frac{5}{8}x + 1270 \quad \Rightarrow \quad \frac{5}{8}x + p = 1270 \quad \Rightarrow \quad \frac{5}{8}x + 1p = 1270$

Pauli vectors: $\mathbf{a} = 0\,\sigma_x + 0.625\,\sigma_y = 0.625\,\sigma_y$

$\mathbf{b} = \sigma_x + \sigma_y$

$\mathbf{r} = 750\,\sigma_x + 1270\,\sigma_y$

Outer products: $\mathbf{a} \wedge \mathbf{b} = 0.625\ \sigma_y\sigma_x = -\ 0.625\ \sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = 750\ \sigma_x\sigma_y + 1270\ \sigma_y\sigma_x = 750\ \sigma_x\sigma_y - 1270\ \sigma_x\sigma_y = -\ 520\ \sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = 468.75\ \sigma_y\sigma_x = -\ 468.75\ \sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r} \wedge \mathbf{b}) = \frac{-520}{-0.625} = 832$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{a} \wedge \mathbf{r}) = \frac{-468.75}{-0.625} = 750$ (Equilibrium price)

Problem 4:

a) Equilibrium:

$$x_e = \frac{1}{3} p_e - 4 = -2\ p_e + 206$$

$$\frac{1}{3} p_e = -2\ p_e + 210$$

$$\frac{7}{3} p_e = 210$$

$$p_e = 90 \quad \text{(Equilibrium price)}$$

$$x_e = \frac{1}{3} \cdot 90 - 4 \qquad \text{or} \qquad x_e = -2 \cdot 90 + 206$$

$$x_e = 30 - 4 = 26 \qquad\qquad x_e = -180 + 206 = 26 \quad \text{(Equilibrium quantity)}$$

b) Equilibrium:

$$x_e = \frac{2}{5} (p_e - 80) = 319 - \frac{5}{7} p_e$$

$$\frac{2}{5} p_e - 32 = 319 - \frac{5}{7} p_e$$

$$\frac{14+25}{35} p_e = 351$$

$$p_e = 351 \cdot \frac{35}{39} = 315 \quad \text{(Equilibrium price)}$$

$$x_e = \frac{2}{5} (315 - 80) \qquad \text{or} \qquad x_e = 319 - \frac{5}{7} \cdot 315$$

$$x_e = \frac{2}{5} \cdot 235 = 94 \qquad\qquad x_e = 319 - 225 = 94 \quad \text{(Equilibrium quantity)}$$

Problem 5:

Supply and demand functions form a system of two linear equations.

a) Supply: $x = \frac{1}{3} p - 4 \quad \Rightarrow \quad x - \frac{1}{3} p = -\ 4$

Demand: $x = -2\ p + 206 \quad \Rightarrow \quad x + 2\ p = 206$

Pauli vectors: $\mathbf{a} = \sigma_x + \sigma_y$

$\mathbf{b} = -\frac{1}{3}\sigma_x + 2\,\sigma_y$

$\mathbf{r} = -4\,\sigma_x + 206\,\sigma_y$

Outer products: $\mathbf{a} \wedge \mathbf{b} = \left(2 + \frac{1}{3}\right)\sigma_x\sigma_y = \frac{7}{3}\sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = \left(-8 + \frac{206}{3}\right)\sigma_x\sigma_y = \frac{182}{3}\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = (206 + 4)\,\sigma_x\sigma_y = 210\,\sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \dfrac{\frac{182}{3}}{\frac{7}{3}} = \frac{182}{3}\cdot\frac{3}{7} = \frac{182}{7} = 26$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \dfrac{210}{\frac{7}{3}} = 210\cdot\frac{3}{7} = 90$ (Equilibrium price)

b) Supply: $x = \frac{2}{5}(p - 80) = \frac{2}{5}p - 32 \quad \Rightarrow \quad x - \frac{2}{5}p = -32$

Demand: $x = 319 - \frac{5}{7}p \quad \Rightarrow \quad x + \frac{5}{7}p = 319$

Pauli vectors: $\mathbf{a} = \sigma_x + \sigma_y$

$\mathbf{b} = -\frac{2}{5}\sigma_x + \frac{5}{7}\sigma_y$

$\mathbf{r} = -32\,\sigma_x + 319\,\sigma_y$

Outer products: $\mathbf{a} \wedge \mathbf{b} = \left(\frac{5}{7} + \frac{2}{5}\right)\sigma_x\sigma_y = \frac{39}{35}\sigma_x\sigma_y$

$\mathbf{r} \wedge \mathbf{b} = \left(-\frac{160}{7} + \frac{638}{5}\right)\sigma_x\sigma_y = \frac{3666}{35}\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r} = (319 + 32)\,\sigma_x\sigma_y = 351\,\sigma_x\sigma_y$

Solutions: $x_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \dfrac{\frac{3666}{35}}{\frac{39}{35}} = \frac{3666}{35}\cdot\frac{35}{39} = \frac{3666}{39} = 94$ (Equilibrium quantity)

$p_e = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \dfrac{351}{\frac{39}{35}} = 351\cdot\frac{35}{39} = 315$ (Equilibrium price)

HWR Berlin, Wintersemester 2017/2018

Mathematics for Business and Economics

Berlin School of Economics and Law

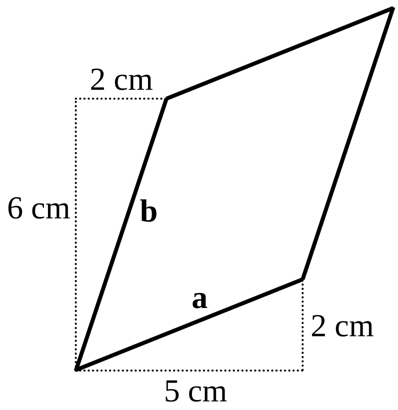
Worksheet 8 – Answers

Problem 1:

a) $\mathbf{a} = 5\,\sigma_x + 2\,\sigma_y$

$\mathbf{b} = 2\,\sigma_x + 6\,\sigma_y$

Sketch:



$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (5\,\sigma_x + 2\,\sigma_y)\,(2\,\sigma_x + 6\,\sigma_y)\\ &= 5\cdot 2\,\sigma_x^2 + 5\cdot 6\,\sigma_x\sigma_y + 2\cdot 2\,\sigma_y\sigma_x + 2\cdot 6\,\sigma_y^2\\ &= 10\,\sigma_x^2 + 30\,\sigma_x\sigma_y + 4\,\sigma_y\sigma_x + 12\,\sigma_y^2\\ &= 10\cdot 1 + 30\,\sigma_x\sigma_y + 4\,(-\sigma_x\sigma_y) + 12\cdot 1\\ &= 10 + 30\,\sigma_x\sigma_y - 4\,\sigma_x\sigma_y + 12\\ &= 22 + 26\,\sigma_x\sigma_y\end{aligned}$$

$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 26\,\sigma_x\sigma_y$

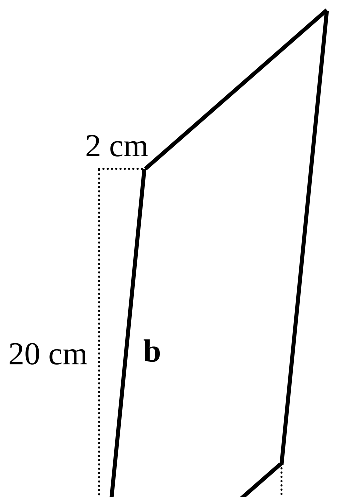
$\Rightarrow\ |\mathbf{A}| = 26$

$\Rightarrow$ The area of the parallelogram is 26 cm^2.

b) $\mathbf{a} = 8\,\sigma_x + 7\,\sigma_y$

$\mathbf{b} = 2\,\sigma_x + 20\,\sigma_y$

Sketch:



$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (8\,\sigma_x + 7\,\sigma_y)\,(2\,\sigma_x + 20\,\sigma_y)\\ &= 8\cdot 2\,\sigma_x^2 + 8\cdot 20\,\sigma_x\sigma_y + 7\cdot 2\,\sigma_y\sigma_x + 7\cdot 20\,\sigma_y^2\\ &= 16\,\sigma_x^2 + 160\,\sigma_x\sigma_y + 14\,\sigma_y\sigma_x + 140\,\sigma_y^2\\ &= 16\cdot 1 + 160\,\sigma_x\sigma_y + 14\,(-\sigma_x\sigma_y) + 140\cdot 1\\ &= 16 + 160\,\sigma_x\sigma_y - 14\,\sigma_x\sigma_y + 140\\ &= 156 + 146\,\sigma_x\sigma_y\end{aligned}$$

$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 146\,\sigma_x\sigma_y$

$\Rightarrow\ |\mathbf{A}| = 146$

$\Rightarrow$ The area of the parallelogram is 146 cm^2.

c) $\mathbf{a} = 5\,\sigma_x - 5\,\sigma_y$

$\mathbf{b} = 3\,\sigma_x + 7\,\sigma_y$

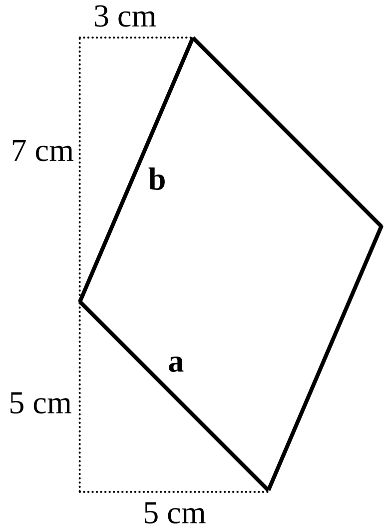
$$\mathbf{a\,b} = (5\,\sigma_x - 5\,\sigma_y)\,(3\,\sigma_x + 7\,\sigma_y)$$
$$= 5 \cdot 3\,\sigma_x^2 + 5 \cdot 7\,\sigma_x\sigma_y - 5 \cdot 3\,\sigma_y\sigma_x - 5 \cdot 7\,\sigma_y^2$$
$$= 15\,\sigma_x^2 + 35\,\sigma_x\sigma_y - 15\,\sigma_y\sigma_x - 35\,\sigma_y^2$$
$$= 15 \cdot 1 + 35\,\sigma_x\sigma_y - 15\,(-\,\sigma_x\sigma_y) - 35 \cdot 1$$
$$= 15 + 35\,\sigma_x\sigma_y + 15\,\sigma_x\sigma_y - 35$$
$$= -\,20 + 50\,\sigma_x\sigma_y$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 50\,\sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 50$

$\Rightarrow$ The area of the parallelogram is 50 cm^2.

Sketch:



d) $\mathbf{a} = 4\,\sigma_x + 16\,\sigma_y$

$\mathbf{b} = 9\,\sigma_x + 2\,\sigma_y$

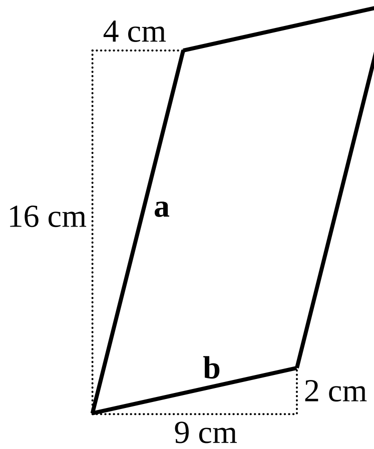
$$\mathbf{a\,b} = (4\,\sigma_x + 16\,\sigma_y)\,(9\,\sigma_x + 2\,\sigma_y)$$
$$= 4 \cdot 9\,\sigma_x^2 + 4 \cdot 2\,\sigma_x\sigma_y + 16 \cdot 9\,\sigma_y\sigma_x + 16 \cdot 2\,\sigma_y^2$$
$$= 36\,\sigma_x^2 + 8\,\sigma_x\sigma_y + 144\,\sigma_y\sigma_x + 32\,\sigma_y^2$$
$$= 36 \cdot 1 + 8\,\sigma_x\sigma_y + 144\,(-\,\sigma_x\sigma_y) + 32 \cdot 1$$
$$= 36 + 8\,\sigma_x\sigma_y - 144\,\sigma_x\sigma_y + 32$$
$$= 68 - 136\,\sigma_x\sigma_y$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = -\,136\,\sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 136$

$\Rightarrow$ The area of the parallelogram is 136 cm^2.

Sketch:



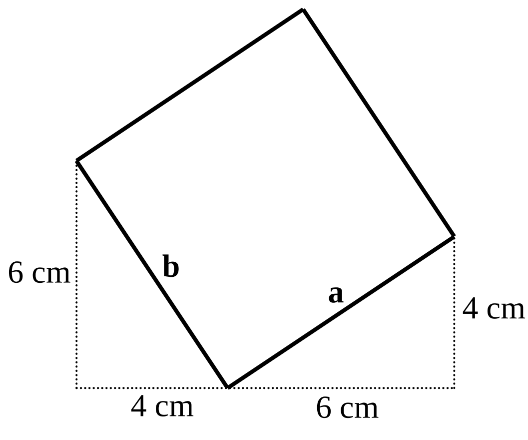
Problem 2:

a) $\mathbf{a} = 6\,\sigma_x + 4\,\sigma_y$

$\mathbf{b} = -\,4\,\sigma_x + 6\,\sigma_y$

$$\mathbf{a\,b} = (6\,\sigma_x + 4\,\sigma_y)\,(-\,4\,\sigma_x + 6\,\sigma_y)$$
$$= 6 \cdot (-\,4)\,\sigma_x^2 + 6 \cdot 6\,\sigma_x\sigma_y + 4 \cdot (-\,4)\,\sigma_y\sigma_x + 4 \cdot 6\,\sigma_y^2$$
$$= -\,24\,\sigma_x^2 + 36\,\sigma_x\sigma_y - 16\,\sigma_y\sigma_x + 24\,\sigma_y^2$$
$$= -\,24 \cdot 1 + 36\,\sigma_x\sigma_y - 16\,(-\,\sigma_x\sigma_y) + 24 \cdot 1$$
$$= -\,24 + 36\,\sigma_x\sigma_y + 16\,\sigma_x\sigma_y + 24$$
$$= 0 + 52\,\sigma_x\sigma_y$$
$$= 52\,\sigma_x\sigma_y$$

Sketch:



$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 52\, \sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 52$

$\Rightarrow$ The area of the parallelogram is 52 cm^2.

As the sides of the parallelogram are perpendicular to each other and have the same length, it is a square.

b) $\mathbf{a} = -\,4.8\, \sigma_x - 3.4\, \sigma_y$

$\mathbf{b} = -\,5.1\, \sigma_x + 7.2\, \sigma_y$

$\mathbf{a}\,\mathbf{b} = (-\,4.8\, \sigma_x - 3.4\, \sigma_y)\,(-\,5.1\, \sigma_x + 7.2\, \sigma_y)$

$= -\,4.8 \cdot (-\,5.1)\, \sigma_x^2 - 4.8 \cdot 7.2\, \sigma_x\sigma_y - 3.4 \cdot (-\,5.1)\, \sigma_y\sigma_x - 3.4 \cdot 7.2\, \sigma_y^2$

$= 24.48\, \sigma_x^2 - 34.56\, \sigma_x\sigma_y + 17.34\, \sigma_y\sigma_x - 24.48\, \sigma_y^2$

$= 24.48 \cdot 1 - 34.56\, \sigma_x\sigma_y + 17.34\, (-\, \sigma_x\sigma_y) - 24.48 \cdot 1$

$= 24.48 - 34.36\, \sigma_x\sigma_y - 17.34\, \sigma_x\sigma_y - 24.48$

$= 0 - 51.90\, \sigma_x\sigma_y$

$= -\,51.90\, \sigma_x\sigma_y$

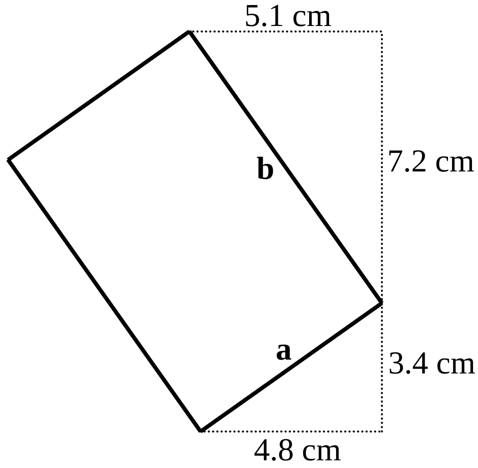
$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = -\,51.90\, \sigma_x\sigma_y$

$\Rightarrow$ $|\mathbf{A}| = 51.90$

$\Rightarrow$ The area of the parallelogram is 51.90 cm^2.

As the sides of the parallelogram are perpendicular to each other, it is a rectangle.

Sketch:



c) $\mathbf{a} = 4\, \sigma_x + 3\, \sigma_y$

$\mathbf{b} = 12\, \sigma_x + 9\, \sigma_y$

$\mathbf{a}\,\mathbf{b} = (4\, \sigma_x + 3\, \sigma_y)\,(12\, \sigma_x + 9\, \sigma_y)$

$= 4 \cdot 12\, \sigma_x^2 + 4 \cdot 9\, \sigma_x\sigma_y + 3 \cdot 12\, \sigma_y\sigma_x + 3 \cdot 9\, \sigma_y^2$

$= 48\, \sigma_x^2 + 36\, \sigma_x\sigma_y + 36\, \sigma_y\sigma_x + 27\, \sigma_y^2$

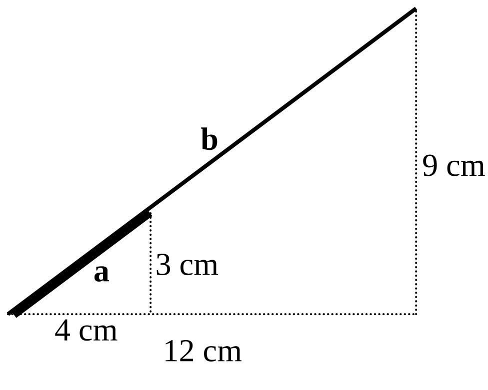
$= 48 \cdot 1 + 36\, \sigma_x\sigma_y + 36\, (-\, \sigma_x\sigma_y) + 27 \cdot 1$

$= 48 + 36\, \sigma_x\sigma_y - 36\, \sigma_x\sigma_y + 27$

$= 75 + 0\, \sigma_x\sigma_y$

$= 75$

Sketch:



$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 0\, \sigma_x\sigma_y = 0$

$\Rightarrow$ $|\mathbf{A}| = 0$

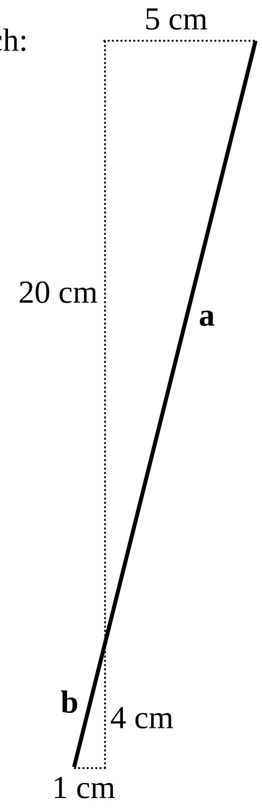
$\Rightarrow$ The area of the parallelogram equals 0 cm^2. Thus there is no area.

It is not possible to form a parallelogram, because all sides are parallel.

d) $\mathbf{a} = 5\,\sigma_x + 20\,\sigma_y$

$\mathbf{b} = -\sigma_x - 4\,\sigma_y$

Sketch:



$$
\begin{aligned}
\mathbf{a}\,\mathbf{b} &= (5\,\sigma_x + 20\,\sigma_y)\,(-\sigma_x - 4\,\sigma_y) \\
&= 5 \cdot (-1)\,\sigma_x^2 + 5 \cdot (-4)\,\sigma_x\sigma_y + 20 \cdot (-1)\,\sigma_y\sigma_x + 20 \cdot (-4)\,\sigma_y^2 \\
&= -5\,\sigma_x^2 + -20\,\sigma_x\sigma_y - 20\,\sigma_y\sigma_x - 80\,\sigma_y^2 \\
&= -5 \cdot 1 - 20\,\sigma_x\sigma_y - 20\,(-\sigma_x\sigma_y) - 80 \cdot 1 \\
&= -5 - 20\,\sigma_x\sigma_y + 20\,\sigma_x\sigma_y - 80 \\
&= -85 + 0\,\sigma_x\sigma_y \\
&= -85
\end{aligned}
$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 0\,\sigma_x\sigma_y = 0$

$\Rightarrow$ $|\mathbf{A}| = 0$

$\Rightarrow$ The area of the parallelogram equals 0 cm^2.
Thus there is no area.

It is not possible to form a parallelogram, because all sides are parallel.

Problem 3:

a) $3\,x + 8\,y = 28$
$6\,x + 2\,y = 28$ $\Rightarrow$ $\mathbf{a} = 3\,\sigma_x + 6\,\sigma_y$

$\mathbf{b} = 8\,\sigma_x + 2\,\sigma_y$

$\mathbf{r} = 28\,\sigma_x + 28\,\sigma_y$

$\Rightarrow$
$$
\begin{aligned}
\mathbf{a}\,\mathbf{b} &= (3\,\sigma_x + 6\,\sigma_y)\,(8\,\sigma_x + 2\,\sigma_y) \\
&= 24\,\sigma_x^2 + 6\,\sigma_x\sigma_y + 48\,\sigma_y\sigma_x + 12\,\sigma_y^2 \\
&= 36 - 42\,\sigma_x\sigma_y
\end{aligned}
$$

$\mathbf{a} \wedge \mathbf{b} = -42\,\sigma_x\sigma_y$

$\Rightarrow$
$$
\begin{aligned}
\mathbf{r}\,\mathbf{b} &= (28\,\sigma_x + 28\,\sigma_y)\,(8\,\sigma_x + 2\,\sigma_y) \\
&= 224\,\sigma_x^2 + 56\,\sigma_x\sigma_y + 224\,\sigma_y\sigma_x + 56\,\sigma_y^2 \\
&= 280 - 168\,\sigma_x\sigma_y
\end{aligned}
$$

$\mathbf{r} \wedge \mathbf{b} = -168\,\sigma_x\sigma_y$

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b})\,x &= \mathbf{r} \wedge \mathbf{b} \\
-42\,\sigma_x\sigma_y\,x &= -168\,\sigma_x\sigma_y \\
\Rightarrow \quad x &= 4
\end{aligned}
$$

$\Rightarrow$
$$
\begin{aligned}
\mathbf{a}\,\mathbf{r} &= (3\,\sigma_x + 6\,\sigma_y)\,(28\,\sigma_x + 28\,\sigma_y) \\
&= 84\,\sigma_x^2 + 84\,\sigma_x\sigma_y + 168\,\sigma_y\sigma_x + 168\,\sigma_y^2 \\
&= 252 - 84\,\sigma_x\sigma_y
\end{aligned}
$$

$\mathbf{a} \wedge \mathbf{r} = -84\,\sigma_x\sigma_y$

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b})\,y &= \mathbf{a} \wedge \mathbf{r} \\
-42\,\sigma_x\sigma_y\,y &= -84\,\sigma_x\sigma_y \\
\Rightarrow \quad y &= 2
\end{aligned}
$$

Check: $3 \cdot 4 + 8 \cdot 2 = 12 + 16 = 28$
$6 \cdot 4 + 2 \cdot 2 = 24 + 4 = 28$

b) $4x + 9y = 29$ $\Rightarrow$ $\mathbf{a} = 4\sigma_x + 5\sigma_y$
$5x + 6y = 31$ $\quad\mathbf{b} = 9\sigma_x + 6\sigma_y$
$\quad\mathbf{r} = 29\sigma_x + 31\sigma_y$

$$\Rightarrow \quad \mathbf{a\,b} = (4\sigma_x + 5\sigma_y)(9\sigma_x + 6\sigma_y)$$
$$= 36\sigma_x^2 + 24\sigma_x\sigma_y + 45\sigma_y\sigma_x + 30\sigma_y^2$$
$$= 66 - 21\sigma_x\sigma_y$$

$$\mathbf{a} \wedge \mathbf{b} = -21\sigma_x\sigma_y$$

$$\Rightarrow \quad \mathbf{r\,b} = (29\sigma_x + 31\sigma_y)(9\sigma_x + 6\sigma_y)$$
$$= 261\sigma_x^2 + 174\sigma_x\sigma_y + 279\sigma_y\sigma_x + 186\sigma_y^2$$
$$= 447 - 105\sigma_x\sigma_y$$

$$\mathbf{r} \wedge \mathbf{b} = -105\sigma_x\sigma_y$$

$$(\mathbf{a} \wedge \mathbf{b})\,x = \mathbf{r} \wedge \mathbf{b}$$
$$-21\sigma_x\sigma_y\,x = -105\sigma_x\sigma_y$$
$$\Rightarrow \quad x = 5$$

$$\Rightarrow \quad \mathbf{a\,r} = (4\sigma_x + 5\sigma_y)(29\sigma_x + 31\sigma_y)$$
$$= 116\sigma_x^2 + 124\sigma_x\sigma_y + 145\sigma_y\sigma_x + 155\sigma_y^2$$
$$= 271 - 21\sigma_x\sigma_y$$

$$\mathbf{a} \wedge \mathbf{r} = -21\sigma_x\sigma_y$$

$$(\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r}$$
$$-21\sigma_x\sigma_y\,y = -21\sigma_x\sigma_y$$
$$\Rightarrow \quad y = 1$$

Check: $4 \cdot 5 + 9 \cdot 1 = 20 + 9 = 29$
$5 \cdot 5 + 6 \cdot 1 = 25 + 6 = 31$

c) $6x + 4y = 6$ $\Rightarrow$ $\mathbf{a} = 6\sigma_x + 2\sigma_y$
$2x + y = 3$ $\quad\mathbf{b} = 4\sigma_x + \sigma_y$
$\quad\mathbf{r} = 6\sigma_x + 3\sigma_y$

$$\Rightarrow \quad \mathbf{a\,b} = (6\sigma_x + 2\sigma_y)(4\sigma_x + \sigma_y)$$
$$= 24\sigma_x^2 + 6\sigma_x\sigma_y + 8\sigma_y\sigma_x + 2\sigma_y^2$$
$$= 26 - 2\sigma_x\sigma_y$$

$$\mathbf{a} \wedge \mathbf{b} = -2\sigma_x\sigma_y$$

$$\Rightarrow \quad \mathbf{r\,b} = (6\sigma_x + 3\sigma_y)(4\sigma_x + \sigma_y)$$
$$= 24\sigma_x^2 + 6\sigma_x\sigma_y + 12\sigma_y\sigma_x + 3\sigma_y^2$$
$$= 27 - 6\sigma_x\sigma_y$$

$$\mathbf{r} \wedge \mathbf{b} = -6\sigma_x\sigma_y$$

$$(\mathbf{a} \wedge \mathbf{b})\,x = \mathbf{r} \wedge \mathbf{b}$$
$$-2\sigma_x\sigma_y\,x = -6\sigma_x\sigma_y$$
$$\Rightarrow \quad x = 3$$

$$\Rightarrow \quad \mathbf{a\,r} = (6\sigma_x + 2\sigma_y)(6\sigma_x + 3\sigma_y)$$
$$= 36\sigma_x^2 + 18\sigma_x\sigma_y + 12\sigma_y\sigma_x + 6\sigma_y^2$$
$$= 42 + 6\sigma_x\sigma_y$$

$$\mathbf{a} \wedge \mathbf{r} = 6\sigma_x\sigma_y$$

$$(\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r}$$
$$-2\sigma_x\sigma_y\,y = 6\sigma_x\sigma_y$$
$$\Rightarrow \quad y = -3$$

Check: $6 \cdot 3 + 4 \cdot (-3) = 18 - 12 = 6$

$2 \cdot 3 + (-3) = 6 - 3 = 3$

d) $5x - 2y = 6$, $-2x - 3y = 28$ $\Rightarrow$ $\mathbf{a} = 5\,\sigma_x - 2\,\sigma_y$, $\mathbf{b} = -2\,\sigma_x - 3\,\sigma_y$, $\mathbf{r} = 6\,\sigma_x + 28\,\sigma_y$

$$\Rightarrow \mathbf{a}\,\mathbf{b} = (5\,\sigma_x - 2\,\sigma_y)(-2\,\sigma_x - 3\,\sigma_y) = -10\,\sigma_x^2 - 15\,\sigma_x\sigma_y + 4\,\sigma_y\sigma_x + 6\,\sigma_y^2 = -4 - 19\,\sigma_x\sigma_y$$

$$\mathbf{a} \wedge \mathbf{b} = -19\,\sigma_x\sigma_y$$

$$\Rightarrow \mathbf{r}\,\mathbf{b} = (6\,\sigma_x + 28\,\sigma_y)(-2\,\sigma_x - 3\,\sigma_y) = -12\,\sigma_x^2 - 18\,\sigma_x\sigma_y - 56\,\sigma_y\sigma_x - 84\,\sigma_y^2 = -96 + 38\,\sigma_x\sigma_y$$

$$\mathbf{r} \wedge \mathbf{b} = 38\,\sigma_x\sigma_y$$

$$(\mathbf{a} \wedge \mathbf{b})\,x = \mathbf{r} \wedge \mathbf{b}$$
$$-19\,\sigma_x\sigma_y\,x = 38\,\sigma_x\sigma_y$$
$$\Rightarrow x = -2$$

$$\Rightarrow \mathbf{a}\,\mathbf{r} = (5\,\sigma_x - 2\,\sigma_y)(6\,\sigma_x + 28\,\sigma_y) = 30\,\sigma_x^2 + 140\,\sigma_x\sigma_y - 12\,\sigma_y\sigma_x - 56\,\sigma_y^2 = -26 + 152\,\sigma_x\sigma_y$$

$$\mathbf{a} \wedge \mathbf{r} = 152\,\sigma_x\sigma_y$$

$$(\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r}$$
$$-19\,\sigma_x\sigma_y\,y = 152\,\sigma_x\sigma_y$$
$$\Rightarrow y = -8$$

Check: $5 \cdot (-2) - 2 \cdot (-8) = -10 + 16 = 6$

$-2 \cdot (-2) - 3 \cdot (-8) = 4 + 24 = 28$

Problem 4:

The system of two linear equations of this text problem is identical to the system of linear equations of problem 3a. Therefore the results of that problem can be used.

$3x + 8y = 28$, $6x + 2y = 28$ $\Rightarrow$ $\mathbf{a} = 3\,\sigma_x + 6\,\sigma_y$, $\mathbf{b} = 8\,\sigma_x + 2\,\sigma_y$, $\mathbf{r} = 28\,\sigma_x + 28\,\sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = -42\,\sigma_x\sigma_y$, $\mathbf{r} \wedge \mathbf{b} = -168\,\sigma_x\sigma_y$, $\mathbf{a} \wedge \mathbf{r} = -84\,\sigma_x\sigma_y$

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{r} \wedge \mathbf{b}) = \frac{-168}{-42} = 4 \qquad y = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{a} \wedge \mathbf{r}) = \frac{-84}{-42} = 2$$

Check:

		4
		2
3	8	28
6	2	28

If 28 units of the first raw material R_1 and 28 units of the second raw material R_2 are consumed in the production process, 4 units of the first final product P_1 and 2 units of the second final product P_2 will be produced.

Problem 5:

$$\begin{aligned} 2x + 7y &= 2050 \\ 5x + y &= 1000 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 2\,\sigma_x + 5\,\sigma_y \\ \mathbf{b} &= 7\,\sigma_x + \sigma_y \\ \mathbf{r} &= 2050\,\sigma_x + 1000\,\sigma_y \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} \wedge \mathbf{b} &= -33\,\sigma_x\sigma_y \\ \mathbf{r} \wedge \mathbf{b} &= -4950\,\sigma_x\sigma_y \\ \mathbf{a} \wedge \mathbf{r} &= -8250\,\sigma_x\sigma_y \end{aligned}$$

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \frac{-4950}{-33} = 150 \qquad y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \frac{-8250}{-33} = 250$$

Check:		
		150
		250
2	7	2050
5	1	1000

If 2050 units of the first raw material R_1 and 1000 units of the second raw material R_2 are consumed in the production process, 150 units of the first final product P_1 and 250 units of the second final product P_2 will be produced.

Problem 6:

first quarter ↓, second quarter ↓

$$\begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}}_{\mathbf{P}} = \underbrace{\begin{bmatrix} 33\,000 & 32\,000 \\ 38\,000 & 25\,000 \end{bmatrix}}_{\mathbf{R}}$$

R .... matrix of quarterly consumption of raw materials (consumption matrix)

P .... matrix of quarterly production (production matrix)

$$\begin{aligned} 4x_1 + 3y_1 &= 33000 \\ x_1 + 5y_1 &= 38000 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 4\,\sigma_x + \sigma_y \\ \mathbf{b} &= 3\,\sigma_x + 5\,\sigma_y \\ \mathbf{r_1} &= 33000\,\sigma_x + 38000\,\sigma_y \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} \wedge \mathbf{b} &= 17\,\sigma_x\sigma_y \\ \mathbf{r_1} \wedge \mathbf{b} &= 51000\,\sigma_x\sigma_y \\ \mathbf{a} \wedge \mathbf{r_1} &= 119000\,\sigma_x\sigma_y \end{aligned}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{51000}{17} = 3000 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{119000}{17} = 7000$$

$$\begin{aligned} 4x_2 + 3y_2 &= 32000 \\ x_2 + 5y_2 &= 25000 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 4\,\sigma_x + \sigma_y \\ \mathbf{b} &= 3\,\sigma_x + 5\,\sigma_y \\ \mathbf{r_2} &= 32000\,\sigma_x + 25000\,\sigma_y \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} \wedge \mathbf{b} &= 17\,\sigma_x\sigma_y \\ \mathbf{r_2} \wedge \mathbf{b} &= 85000\,\sigma_x\sigma_y \\ \mathbf{a} \wedge \mathbf{r_2} &= 68000\,\sigma_x\sigma_y \end{aligned}$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{85000}{17} = 5000 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{68000}{17} = 4000$$

$\Rightarrow$ matrix of quarterly production: $\mathbf{P} = \begin{bmatrix} 3\,000 & 5\,000 \\ 7\,000 & 4\,000 \end{bmatrix}$

Check:

		3000	5000
		7000	4000
4	3	33000	32000
1	5	38000	25000

3000 units of the first final product P_1 and 7000 units of the second final product P_2 will be produced in the first quarter.

5000 units of the first final product P_1 and 4000 units of the second final product P_2 will be produced in the second quarter.

Problem 7:

$$\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 42 & 28 \\ 23 & 26 \end{bmatrix} \qquad \mathbf{A}\,\mathbf{B} = \mathbf{D}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$8\,x_1 + 2\,y_1 = 42 \quad \Rightarrow \quad \mathbf{a} = 8\,\sigma_x + 4\,\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 16\,\sigma_x\sigma_y$

$4\,x_1 + 3\,y_1 = 23 \quad \mathbf{b} = 2\,\sigma_x + 3\,\sigma_y \quad \mathbf{r_1} \wedge \mathbf{b} = 80\,\sigma_x\sigma_y$

$\mathbf{r_1} = 42\,\sigma_x + 23\,\sigma_y \quad \mathbf{a} \wedge \mathbf{r_1} = 16\,\sigma_x\sigma_y$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{80}{16} = 5 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{16}{16} = 1$$

$8\,x_2 + 2\,y_2 = 28 \quad \Rightarrow \quad \mathbf{a} = 8\,\sigma_x + 4\,\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 16\,\sigma_x\sigma_y$

$4\,x_2 + 3\,y_2 = 26 \quad \mathbf{b} = 2\,\sigma_x + 3\,\sigma_y \quad \mathbf{r_2} \wedge \mathbf{b} = 32\,\sigma_x\sigma_y$

$\mathbf{r_2} = 28\,\sigma_x + 26\,\sigma_y \quad \mathbf{a} \wedge \mathbf{r_2} = 96\,\sigma_x\sigma_y$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{32}{16} = 2 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{96}{16} = 6$$

Check:

		5	2
		1	6
8	2	42	28
4	3	23	26

$\Rightarrow$ demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix}$

Problem 8:

$$\underbrace{\begin{bmatrix} 9 & 3 \\ 2 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 48 & 21 & 84 \\ 12 & 14 & 32 \end{bmatrix}}_{\mathbf{D}}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$9\,x_1 + 3\,y_1 = 48$ $\quad\Rightarrow\quad$ $\mathbf{a} = 9\,\sigma_x + 2\,\sigma_y$ $\quad\Rightarrow\quad$ $\mathbf{a} \wedge \mathbf{b} = 12\,\sigma_x\sigma_y$

$2\,x_1 + 2\,y_1 = 12$ $\quad$ $\mathbf{b} = 3\,\sigma_x + 2\,\sigma_y$ $\quad$ $\mathbf{r_1} \wedge \mathbf{b} = 60\,\sigma_x\sigma_y$

$\mathbf{r_1} = 48\,\sigma_x + 12\,\sigma_y$ $\quad$ $\mathbf{a} \wedge \mathbf{r_1} = 12\,\sigma_x\sigma_y$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{60}{12} = 5 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{12}{12} = 1$$

$9\,x_2 + 3\,y_2 = 21$ $\quad\Rightarrow\quad$ $\mathbf{a} = 9\,\sigma_x + 2\,\sigma_y$ $\quad\Rightarrow\quad$ $\mathbf{a} \wedge \mathbf{b} = 12\,\sigma_x\sigma_y$

$2\,x_2 + 2\,y_2 = 14$ $\quad$ $\mathbf{b} = 3\,\sigma_x + 2\,\sigma_y$ $\quad$ $\mathbf{r_2} \wedge \mathbf{b} = 0\,\sigma_x\sigma_y$

$\mathbf{r_2} = 21\,\sigma_x + 14\,\sigma_y$ $\quad$ $\mathbf{a} \wedge \mathbf{r_2} = 84\,\sigma_x\sigma_y$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{0}{12} = 0 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{84}{12} = 7$$

$9\,x_3 + 3\,y_3 = 84$ $\quad\Rightarrow\quad$ $\mathbf{a} = 9\,\sigma_x + 2\,\sigma_y$ $\quad\Rightarrow\quad$ $\mathbf{a} \wedge \mathbf{b} = 12\,\sigma_x\sigma_y$

$2\,x_3 + 2\,y_3 = 32$ $\quad$ $\mathbf{b} = 3\,\sigma_x + 2\,\sigma_y$ $\quad$ $\mathbf{r_3} \wedge \mathbf{b} = 72\,\sigma_x\sigma_y$

$\mathbf{r_3} = 84\,\sigma_x + 32\,\sigma_y$ $\quad$ $\mathbf{a} \wedge \mathbf{r_3} = 120\,\sigma_x\sigma_y$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b}) = \frac{72}{12} = 6 \qquad y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3}) = \frac{120}{12} = 10$$

Check:		5	0	6
		1	7	10
9	3	48	21	84
2	2	12	14	32

$\Rightarrow$ demand matrix of the second production step: $\quad \mathbf{B} = \begin{bmatrix} 5 & 0 & 6 \\ 1 & 7 & 10 \end{bmatrix}$

Problem 9:

First part of problem 9: Consumption of exactly one unit of the first raw material R_1

$7\,x + 5\,y = 1$ $\quad\Rightarrow\quad$ $\mathbf{a} = 7\,\sigma_x + 4\,\sigma_y$ $\quad\Rightarrow\quad$ $\mathbf{a} \wedge \mathbf{b} = 1\,\sigma_x\sigma_y = \sigma_x\sigma_y$

$4\,x + 3\,y = 0$ $\quad$ $\mathbf{b} = 5\,\sigma_x + 3\,\sigma_y$ $\quad$ $\mathbf{r_1} \wedge \mathbf{b} = 3\,\sigma_x\sigma_y$

$\mathbf{r_1} = 1\,\sigma_x = \sigma_x$ $\quad$ $\mathbf{a} \wedge \mathbf{r_1} = -\,4\,\sigma_x\sigma_y$

$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_1} \wedge \mathbf{b}) = \dfrac{3}{1} = 3 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_1}) = \dfrac{-4}{1} = -4$

Economic interpretation:

If exactly one unit of the first raw material R_1 had been consumed in the production process, 3 units of the first final product P_1 and (-4) units of the second final product P_2 would have been produced. However, the production of a negative number of final products is problematic.

Producing (-4) units means that in addition to an already produced quantity (-4) units are added. Mathematically, the negative number “minus four” is added or alternatively, the positive number “four” is subtracted. Thus after the production process the quantity is reduced by four units.

Therefore these four units will not be produced, but consumed and (in theory completely) split again into the initial raw materials R_1 and R_2.

The correct economic interpretation will then be:

If exactly one unit of the first raw material R_1 had been consumed in the production process, 3 units of the first final product P_1 would have been produced and additionally 4 units of the second final product P_2 would have been consumed.

Second part of problem 9: Consumption of exactly one unit of the second raw material R_2

$$\begin{array}{lclcl}
7x + 5y = 0 & \Rightarrow & \mathbf{a} = 7\,\sigma_x + 4\,\sigma_y & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 1\,\sigma_x\sigma_y = \sigma_x\sigma_y \\
4x + 3y = 1 & & \mathbf{b} = 5\,\sigma_x + 3\,\sigma_y & & \mathbf{r_2} \wedge \mathbf{b} = -5\,\sigma_x\sigma_y \\
 & & \mathbf{r_2} = 1\,\sigma_y = \sigma_y & & \mathbf{a} \wedge \mathbf{r_2} = 7\,\sigma_x\sigma_y
\end{array}$$

$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_2} \wedge \mathbf{b}) = \dfrac{-5}{1} = -5 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_2}) = \dfrac{-4}{1} = 7$

Economic interpretation:

If exactly one unit of the second raw material R_2 had been consumed in the production process, in addition 5 units of the first final product P_1 would have been consumed and 7 units of the second final product P_2 would have been produced.

As a complete splitting of products into the initial raw materials is hardly possible (and then usually connected with higher costs), negative production quantities or a negative output will only very rarely be part of realistic economical situations.

But **mathematically** the results just found are of enormous importance, which can be seen at the following check of the results.

Check:

			3	– 5	inverse $\mathbf{A}^{-1}$ of matrix **A**
			– 4	7	
initial matrix **A**	7	5	1	0	identity matrix **I**
	4	3	0	1	

Mathematical interpretation:

The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$.

Problem 10:

First part of problem 10: Consumption of exactly one unit of the first raw material R_1

$$\begin{array}{lll} 10\,x + 12\,y = 1 & \Rightarrow\ \mathbf{a} = 10\,\sigma_x + 4\,\sigma_y & \Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 2\,\sigma_x\sigma_y \\ \ \ 4\,x + \ \ 5\,y = 0 & \quad\ \mathbf{b} = 12\,\sigma_x + 5\,\sigma_y & \quad\ \mathbf{r_1} \wedge \mathbf{b} = 5\,\sigma_x\sigma_y \\ & \quad\ \mathbf{r_1} = 1\,\sigma_x = \sigma_x & \quad\ \mathbf{a} \wedge \mathbf{r_1} = -4\,\sigma_x\sigma_y \end{array}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{5}{2} = 2.5 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-4}{2} = -2$$

Economic interpretation:

If exactly one unit of the first raw material R_1 had been consumed in the production process, 2.5 units of the first final product P_1 would have been produced and additionally 2 units of the second final product P_2 would have been consumed.

Second part of problem 10: Consumption of exactly one unit of the second raw material R_2

$$\begin{array}{lll} 10\,x + 12\,y = 0 & \Rightarrow\ \mathbf{a} = 10\,\sigma_x + 4\,\sigma_y & \Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 2\,\sigma_x\sigma_y \\ \ \ 4\,x + \ \ 5\,y = 1 & \quad\ \mathbf{b} = 12\,\sigma_x + 5\,\sigma_y & \quad\ \mathbf{r_2} \wedge \mathbf{b} = -12\,\sigma_x\sigma_y \\ & \quad\ \mathbf{r_2} = 1\,\sigma_y = \sigma_y & \quad\ \mathbf{a} \wedge \mathbf{r_2} = 10\,\sigma_x\sigma_y \end{array}$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-12}{2} = -6 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{10}{2} = 5$$

Economic interpretation:

If exactly one unit of the second raw material R_2 had been consumed in the production process, in addition 6 units of the first final product P_1 would have been consumed and 5 units of the second final product P_2 would have been produced.

Check:

			2.5	– 6	inverse $\mathbf{A}^{-1}$ of matrix **A**
			– 2	5	
initial matrix **A**	10	12	1	0	identity matrix **I**
	4	5	0	1	

Result:

The inverse of the initial demand matrix $\mathbf{A} = \begin{bmatrix} 10 & 12 \\ 4 & 5 \end{bmatrix}$ is $\mathbf{A}^{-1} = \begin{bmatrix} 2.5 & -6 \\ -2 & 5 \end{bmatrix}$.

Problem 11:

a)

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix} \quad \Rightarrow \quad \mathbf{a} = 5\,\sigma_x + 9\,\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = -\,\sigma_x\sigma_y$$

$\mathbf{b} = 4\,\sigma_x + 7\,\sigma_y$ $\qquad$ $\mathbf{r_1} \wedge \mathbf{b} = 7\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{r_2} \wedge \mathbf{b} = -\,4\,\sigma_x\sigma_y$

$\mathbf{r_1} = \sigma_x$ $\qquad$ $\mathbf{a} \wedge \mathbf{r_1} = -\,9\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{a} \wedge \mathbf{r_2} = 5\,\sigma_x\sigma_y$

$\mathbf{r_2} = \sigma_y$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{7}{-1} = -\,7 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-9}{-1} = 9$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-4}{-1} = 4 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{5}{-1} = -\,5$$

Check:

		– 7	4
		9	– 5
5	4	1	0
9	7	0	1

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} -7 & 4 \\ 9 & -5 \end{bmatrix}$$

b)

$$\mathbf{B} = \begin{bmatrix} 10 & 4 \\ 19 & 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{a} = 10\,\sigma_x + 19\,\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 4\,\sigma_x\sigma_y$$

$\mathbf{b} = 4\,\sigma_x + 8\,\sigma_y$ $\qquad$ $\mathbf{r_1} \wedge \mathbf{b} = 8\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{r_2} \wedge \mathbf{b} = -\,4\,\sigma_x\sigma_y$

$\mathbf{r_1} = \sigma_x$ $\qquad$ $\mathbf{a} \wedge \mathbf{r_1} = -\,19\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{a} \wedge \mathbf{r_2} = 10\,\sigma_x\sigma_y$

$\mathbf{r_2} = \sigma_y$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{8}{4} = 2 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-19}{4} = -\,4.75$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-4}{4} = -\,1 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{10}{4} = 2.5$$

Check:

		2	– 1
		– 4.75	2.5
10	4	1	0
19	8	0	1

$$\Rightarrow \quad \mathbf{B}^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -19 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4.75 & 2.5 \end{bmatrix}$$

c) $\mathbf{C} = \begin{bmatrix} 10 & 6 \\ 20 & 13 \end{bmatrix}$ $\Rightarrow$ $\mathbf{a} = 10\,\sigma_x + 20\,\sigma_y$, $\mathbf{b} = 6\,\sigma_x + 13\,\sigma_y$, $\mathbf{r_1} = \sigma_x$, $\mathbf{r_2} = \sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 10\,\sigma_x\sigma_y$

$\mathbf{r_1} \wedge \mathbf{b} = 13\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{r_2} \wedge \mathbf{b} = -13\,\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r_1} = -20\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{a} \wedge \mathbf{r_2} = 10\,\sigma_x\sigma_y$

$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{13}{10} = 1.3$ $\qquad$ $y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-20}{10} = -2$

$x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-6}{10} = -0.6$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{10}{10} = 1$

Check:

		1.3	−0.6
		−2	1
10	6	1	0
20	13	0	1

$$\Rightarrow \quad \mathbf{C}^{-1} = \frac{1}{10}\begin{bmatrix} 13 & -6 \\ -20 & 10 \end{bmatrix} = \begin{bmatrix} 1.3 & -0.6 \\ -2 & 1 \end{bmatrix}$$

d) $\mathbf{D} = \begin{bmatrix} 0 & -2.5 \\ 0.2 & 3.4 \end{bmatrix}$ $\Rightarrow$ $\mathbf{a} = 0.2\,\sigma_y$, $\mathbf{b} = -2.5\,\sigma_x + 3.4\,\sigma_y$, $\mathbf{r_1} = \sigma_x$, $\mathbf{r_2} = \sigma_y$ $\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 0.5\,\sigma_x\sigma_y$

$\mathbf{r_1} \wedge \mathbf{b} = 3.4\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{r_2} \wedge \mathbf{b} = 2.5\,\sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{r_1} = -0.2\,\sigma_x\sigma_y$ $\qquad$ $\mathbf{a} \wedge \mathbf{r_2} = 0$

$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_1} \wedge \mathbf{b}) = \frac{3.4}{0.5} = 2 \cdot 3.4 = 6.8$ $\qquad$ $y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_1}) = \frac{-0.2}{0.5} = 2 \cdot (-0.2) = -0.4$

$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_2} \wedge \mathbf{b}) = \frac{2.5}{0.5} = 2 \cdot 2.5 = 5$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_2}) = \frac{0}{0.5} = 2 \cdot 0 = 0$

Check:

		6.8	5
		−0.4	0
0	−2.5	1	0
0.2	3.4	0	1

$$\Rightarrow \quad \mathbf{D}^{-1} = 2 \cdot \begin{bmatrix} 3.4 & 2.5 \\ -0.2 & 0 \end{bmatrix} = \begin{bmatrix} 6.8 & 5 \\ -0.4 & 0 \end{bmatrix}$$

HWR Berlin, Wintersemester 2017/2018

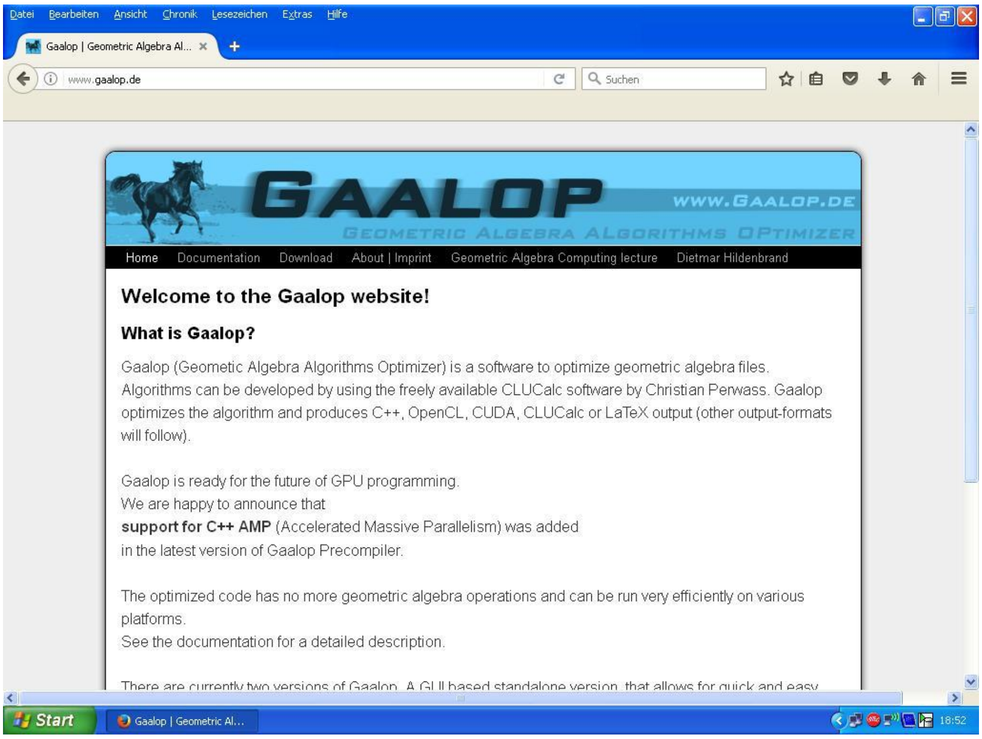
Mathematics for Business and Economics

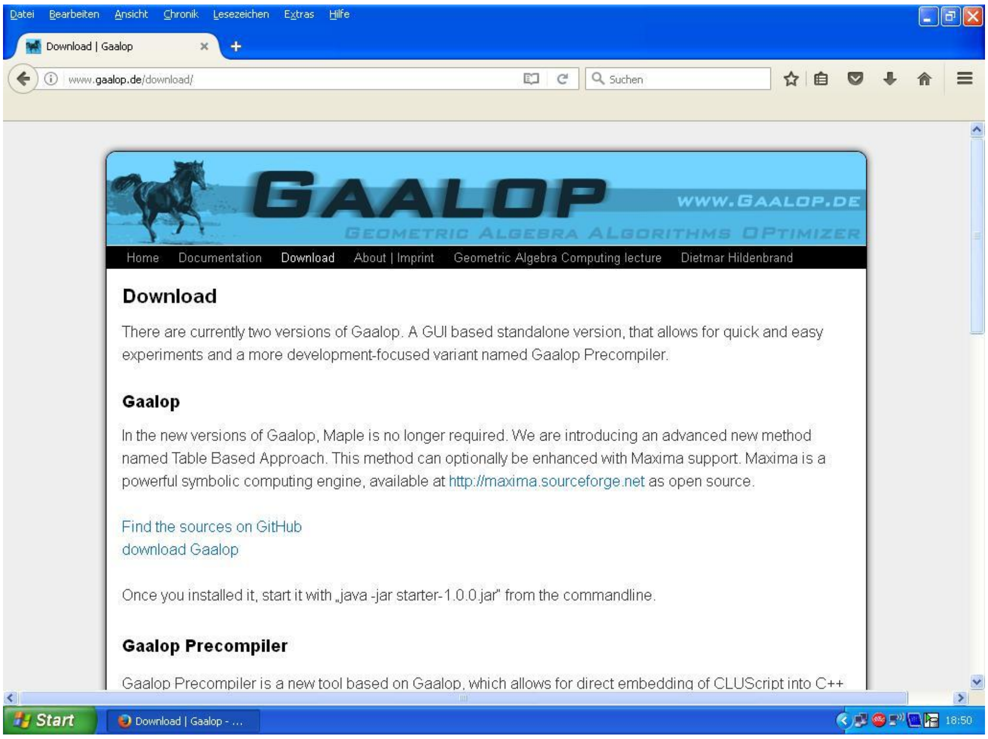
Berlin School of Economics and Law

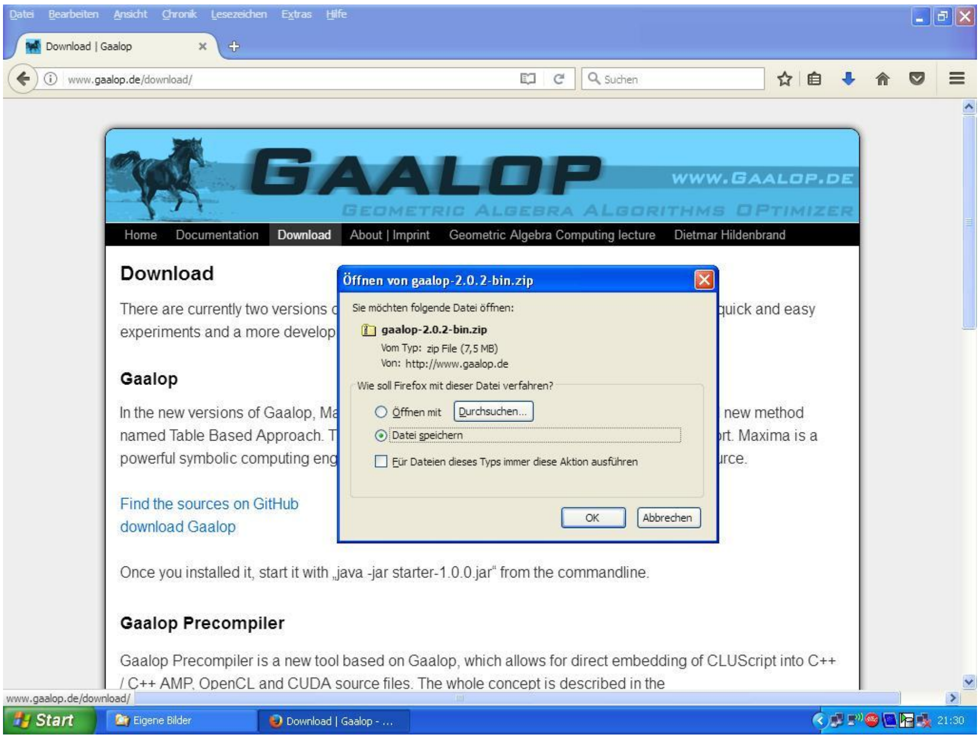
Worksheet 9 – Answers

Problem 1:

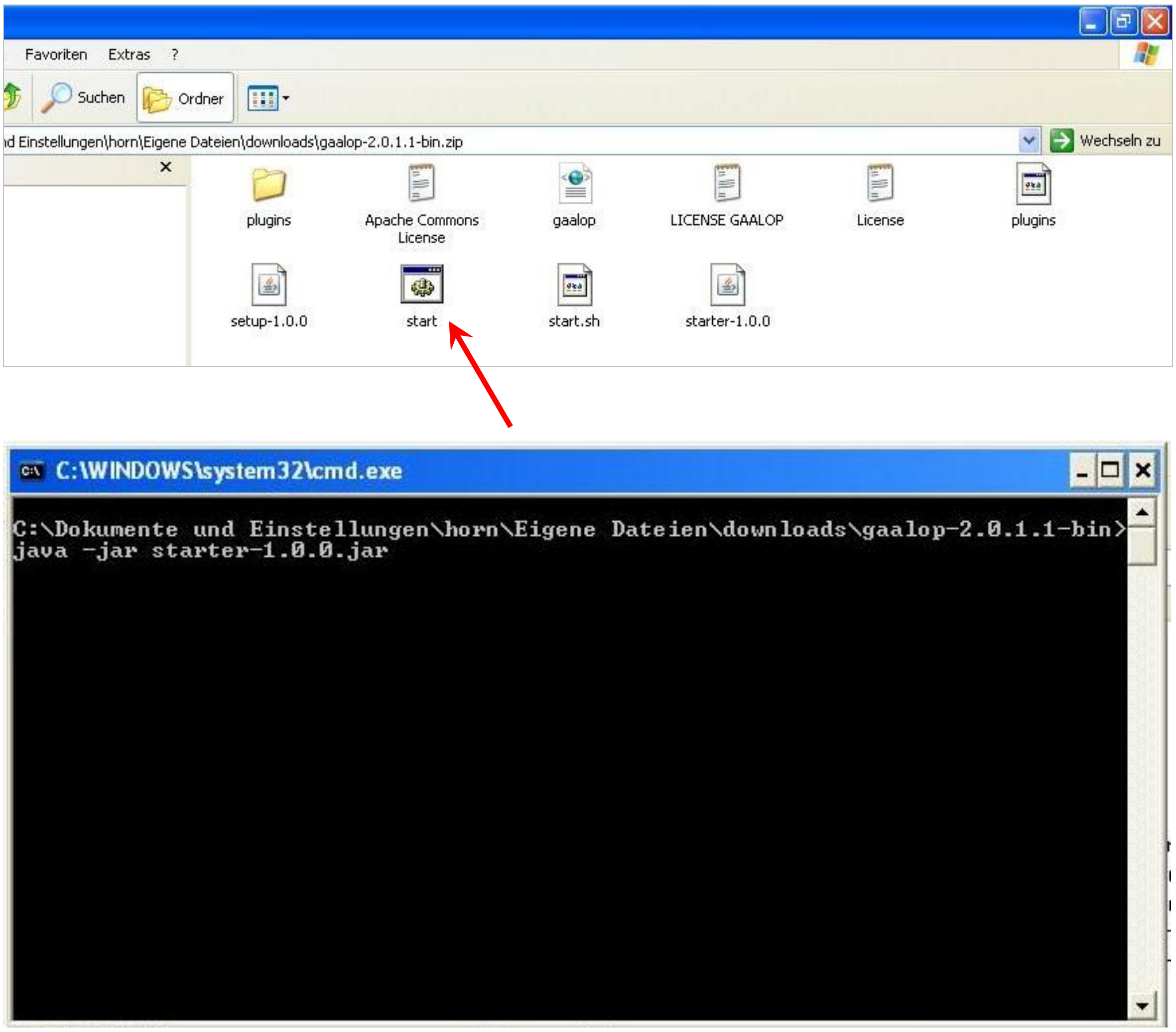
a) Downloading GAALOP:





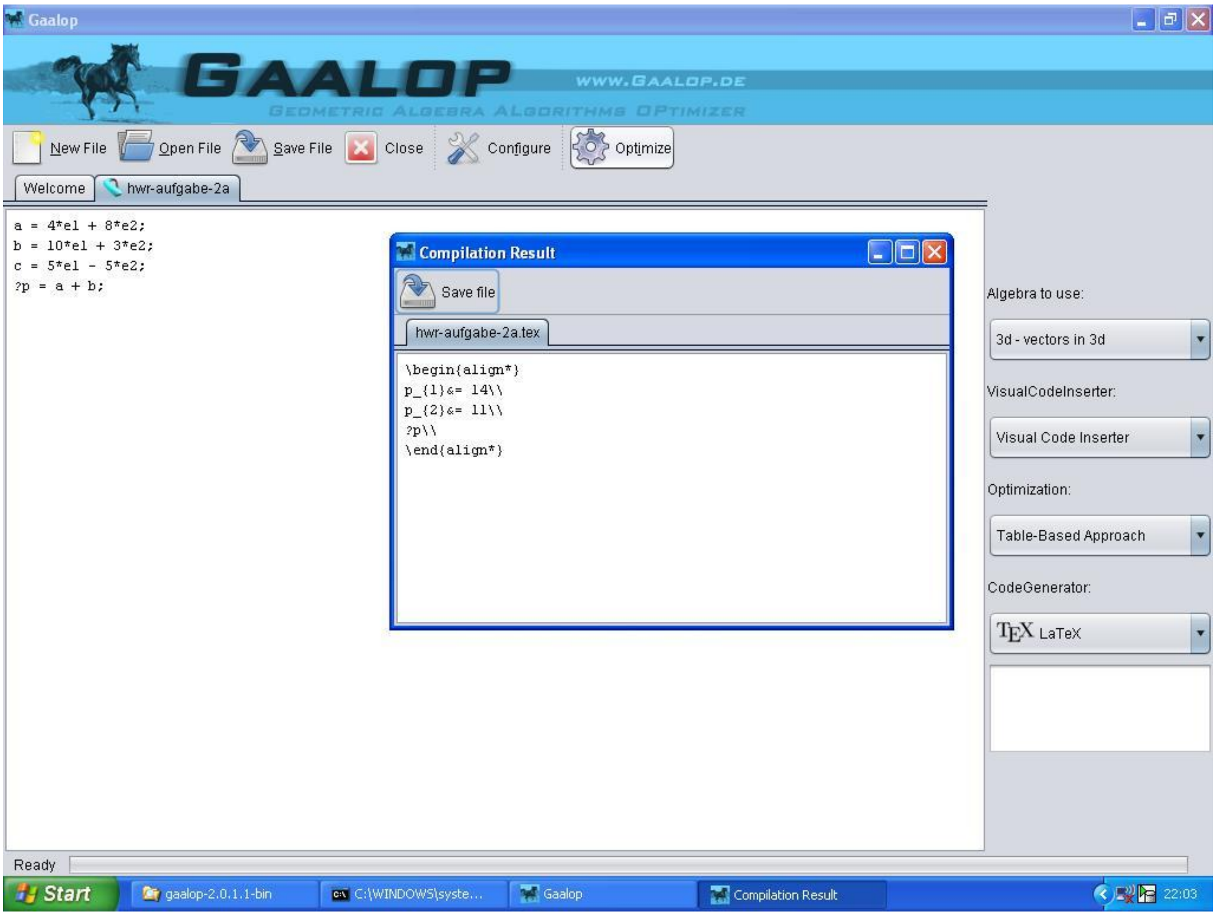


b) Starting GAALOP



Problem 2:

a)



Elements of Pauli Algebra (Geometric Algebra of three-dimensional space) have eight components. The compiler field shows only nonzero components as compilation result. Thus components which are not shown in the compiler field must be 0.

Therefore six of the eight components of problem 2a) are zero and the following interpretation of the results can be given:

numbers:

scalar component without directions	p_{0} &= 0

oriented line elements (or directed line segments):

vector component into σ_x-direction	p_{1} &= 14 $\rightarrow$ 14 σ_x
vector component into σ_y-direction	p_{2} &= 11 $\rightarrow$ 11 σ_y
vector component into σ_z-direction:	p_{3} &= 0

oriented area elements:

bivector component into $\sigma_x\sigma_y$-direction p_{4} &= 0

bivector component into $\sigma_x\sigma_z$-direction p_{5} &= 0

bivector component into $\sigma_y\sigma_z$-direction p_{6} &= 0

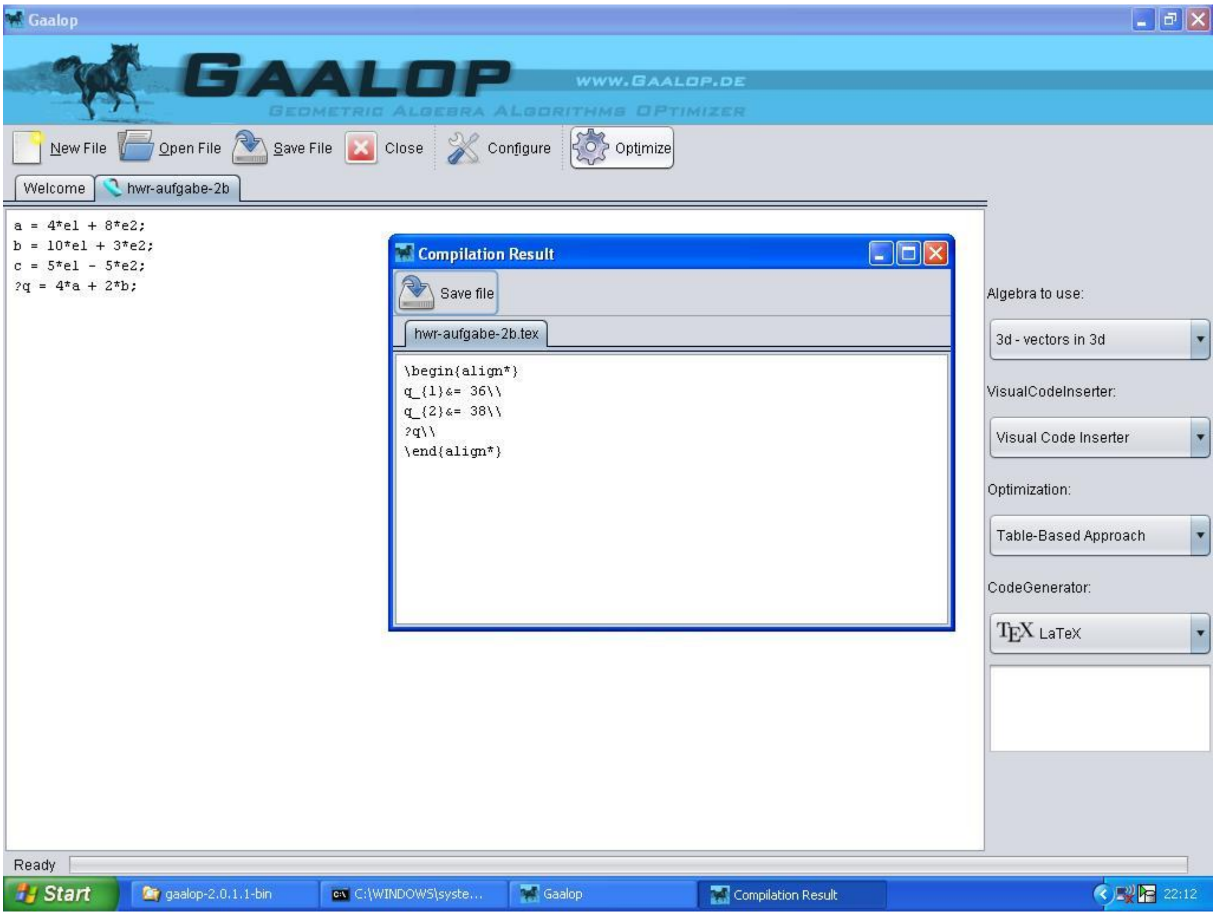
oriented volume element:

trivector component into $\sigma_x\sigma_y\sigma_z$-direction p_{7} &= 0

result in Pauli notation:

$$\mathbf{p} = (4\,\sigma_x + 8\,\sigma_y) + (10\,\sigma_x + 3\,\sigma_y) = 14\,\sigma_x + 11\,\sigma_y$$

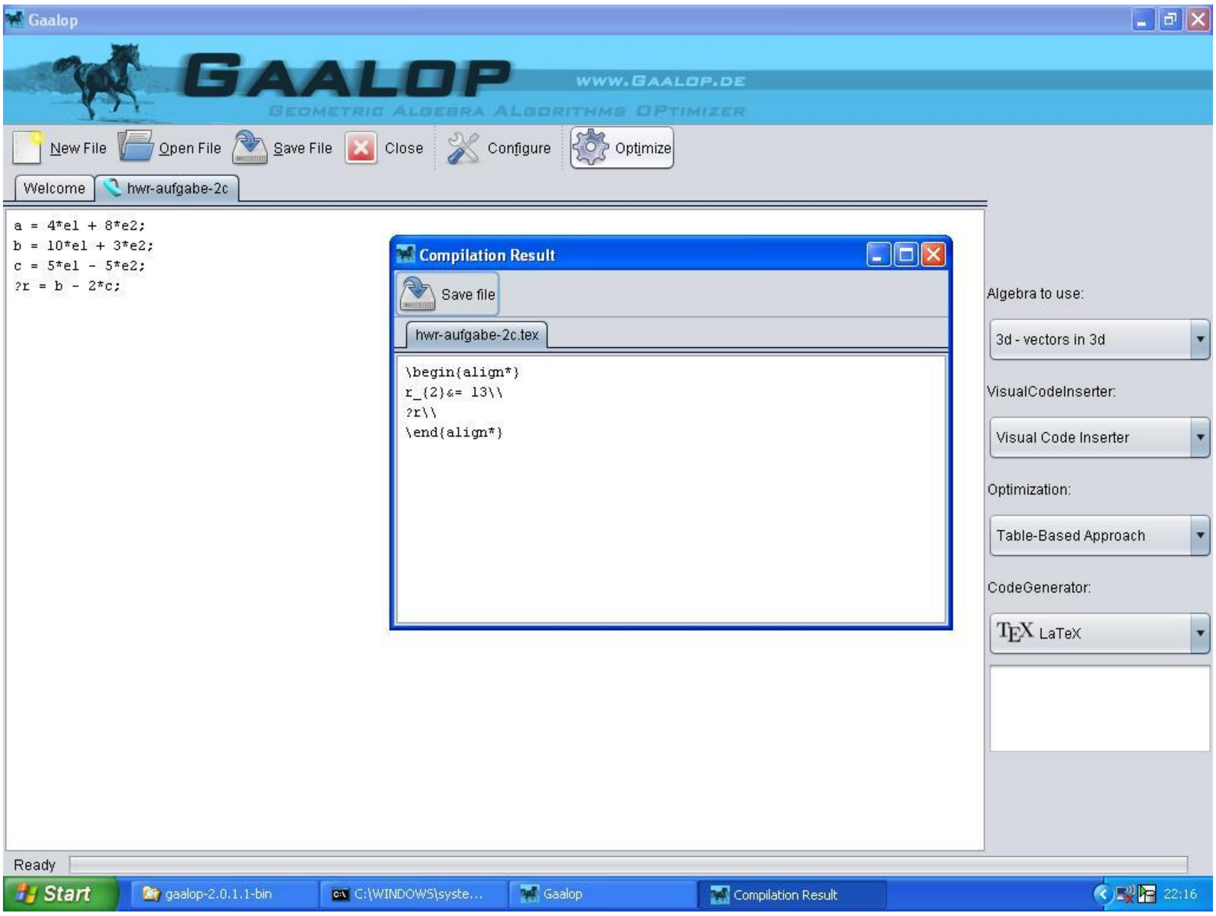
b)



result in Pauli notation:

$$\mathbf{q} = 4 \cdot (4\,\sigma_x + 8\,\sigma_y) + 2 \cdot (10\,\sigma_x + 3\,\sigma_y) = 36\,\sigma_x + 38\,\sigma_y$$

c)

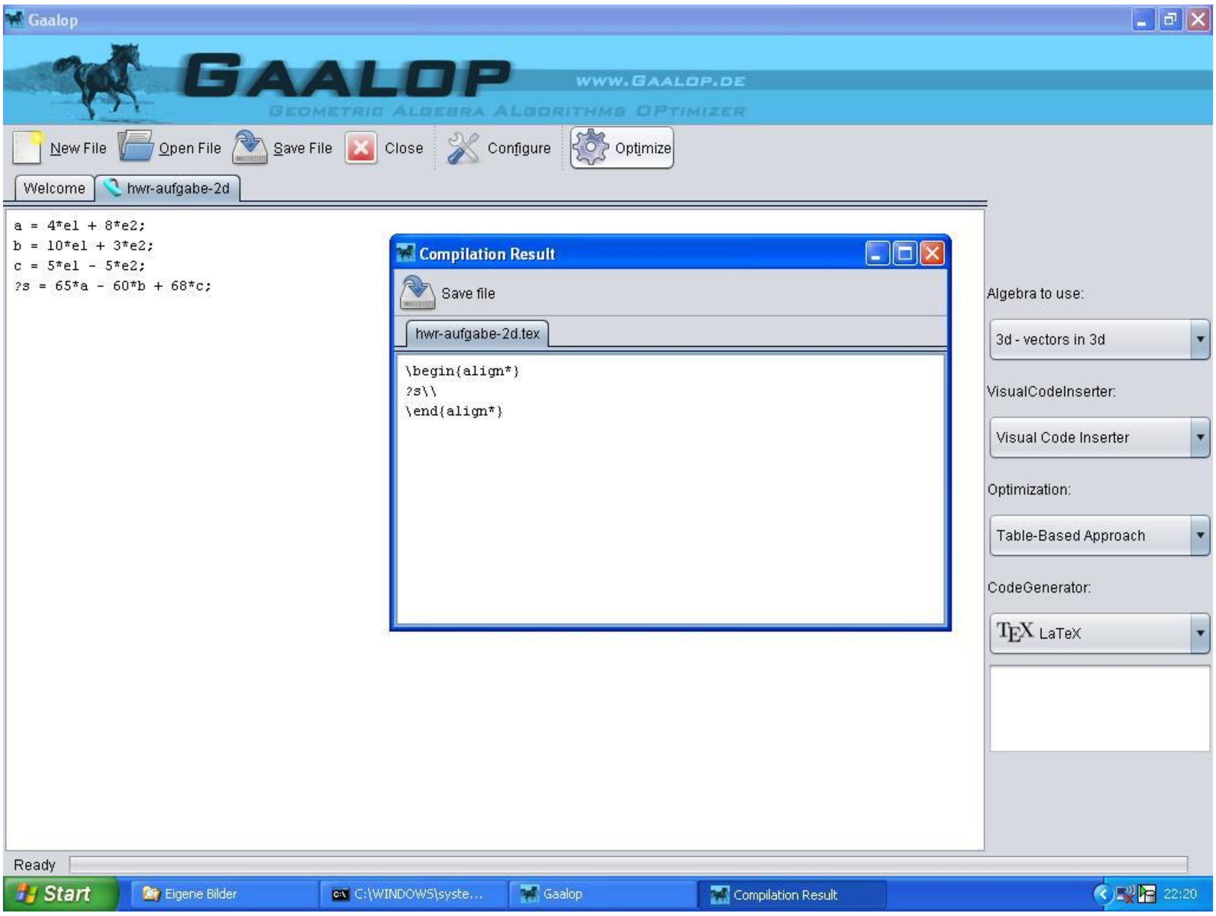


result in Pauli notation:

$\mathbf{r} = (10\ \sigma_x + 3\ \sigma_y) - 2 \cdot (5\ \sigma_x - 5\ \sigma_y) = 0\ \sigma_x + 13\ \sigma_y = 13\ \sigma_y$

As no σ_x-component r_{1} is shown in the compiler field, this component must be zero: r_{1}&= 0 .

d)



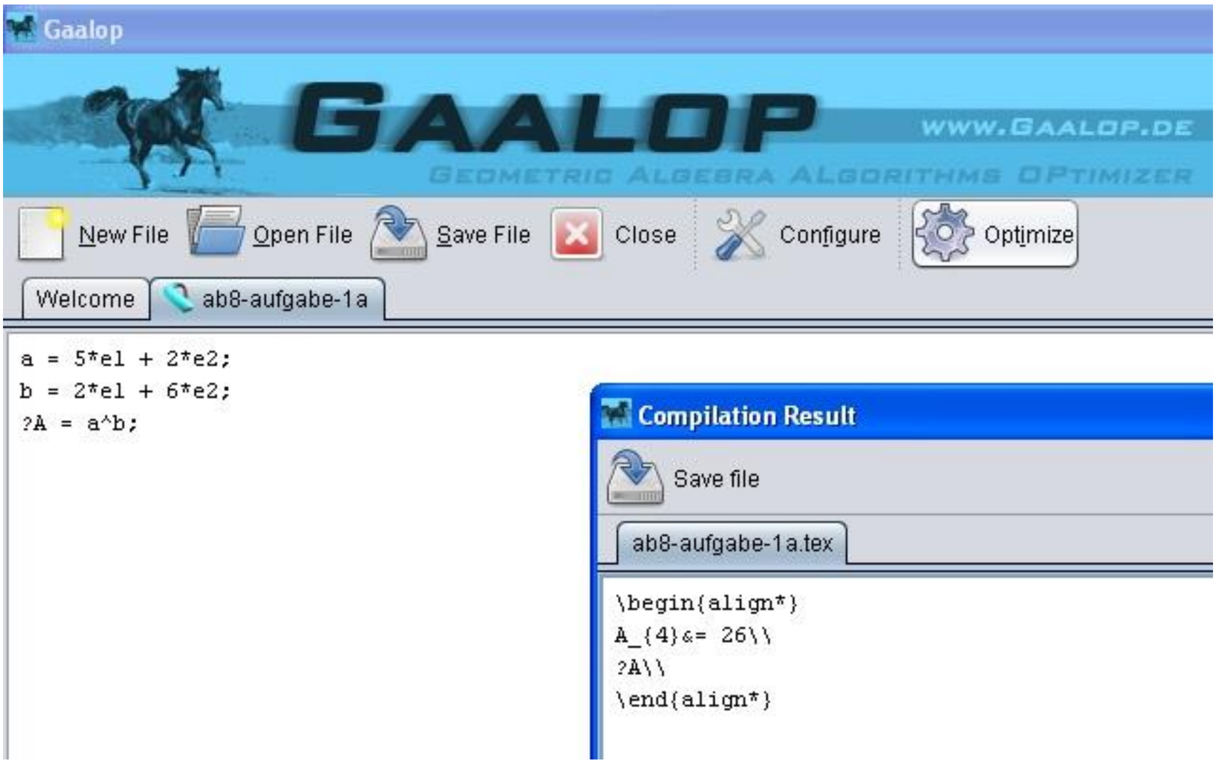
As neither a σ_x-component s_{1} nor a σ_y-component s_{2} is shown in the compiler field, both components must be zero: s_{1} &= 0
s_{2} &= 0 .

result in Pauli notation:

$\mathbf{s} = 65 \cdot (4\,\sigma_x + 8\,\sigma_y) - 60 \cdot (10\,\sigma_x + 3\,\sigma_y) + 68 \cdot (5\,\sigma_x - 5\,\sigma_y) = 0\,\sigma_x + 0\,\sigma_y = 0$

Problem 3 → Problem 1 of worksheet 8:

1) a)

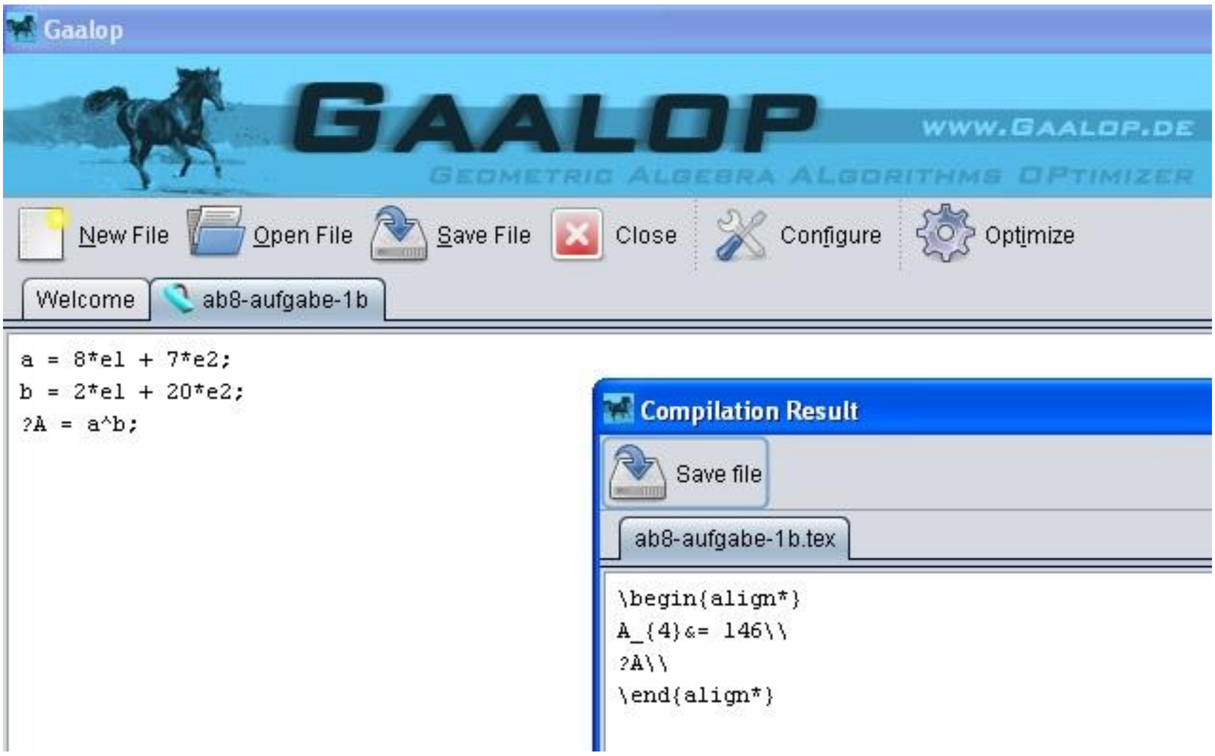


result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = 26\ \sigma_x\sigma_y$ $\Rightarrow$ $|\mathbf{A}| = 26$

$\Rightarrow$ The area of the parallelogram is 26 cm^2.

1) b)

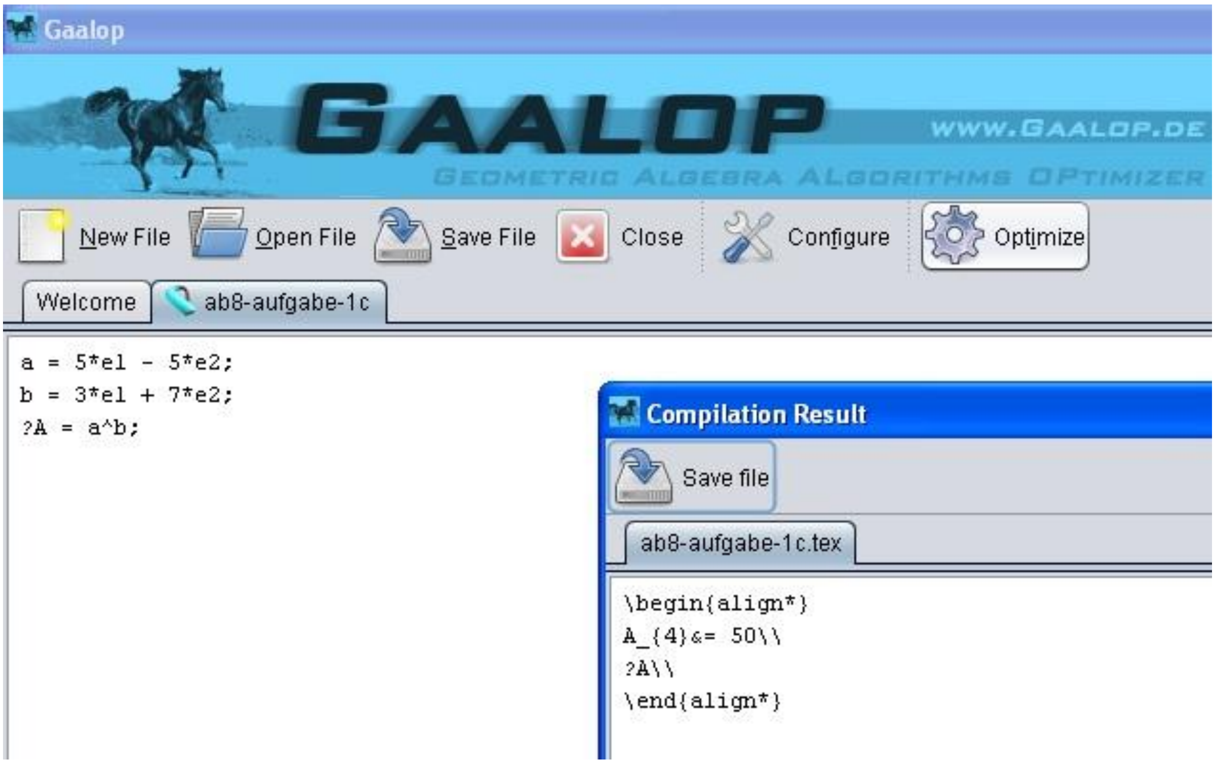


result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = 146\ \sigma_x\sigma_y$ $\Rightarrow$ $|\mathbf{A}| = 146$

$\Rightarrow$ The area of the parallelogram is 146 cm^2.

1) c)

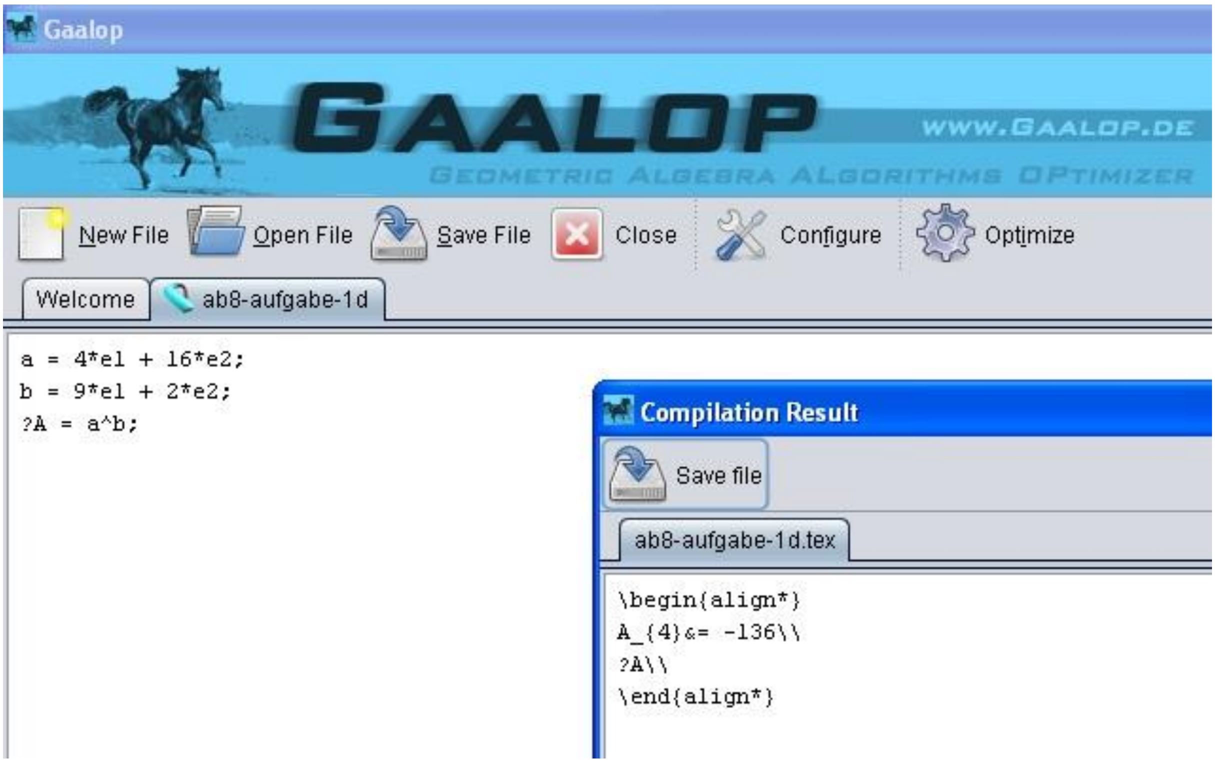


result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = 50\ \sigma_x\sigma_y$ $\Rightarrow$ $|\mathbf{A}| = 50$

$\Rightarrow$ The area of the parallelogram is 50 cm^2.

1) d)



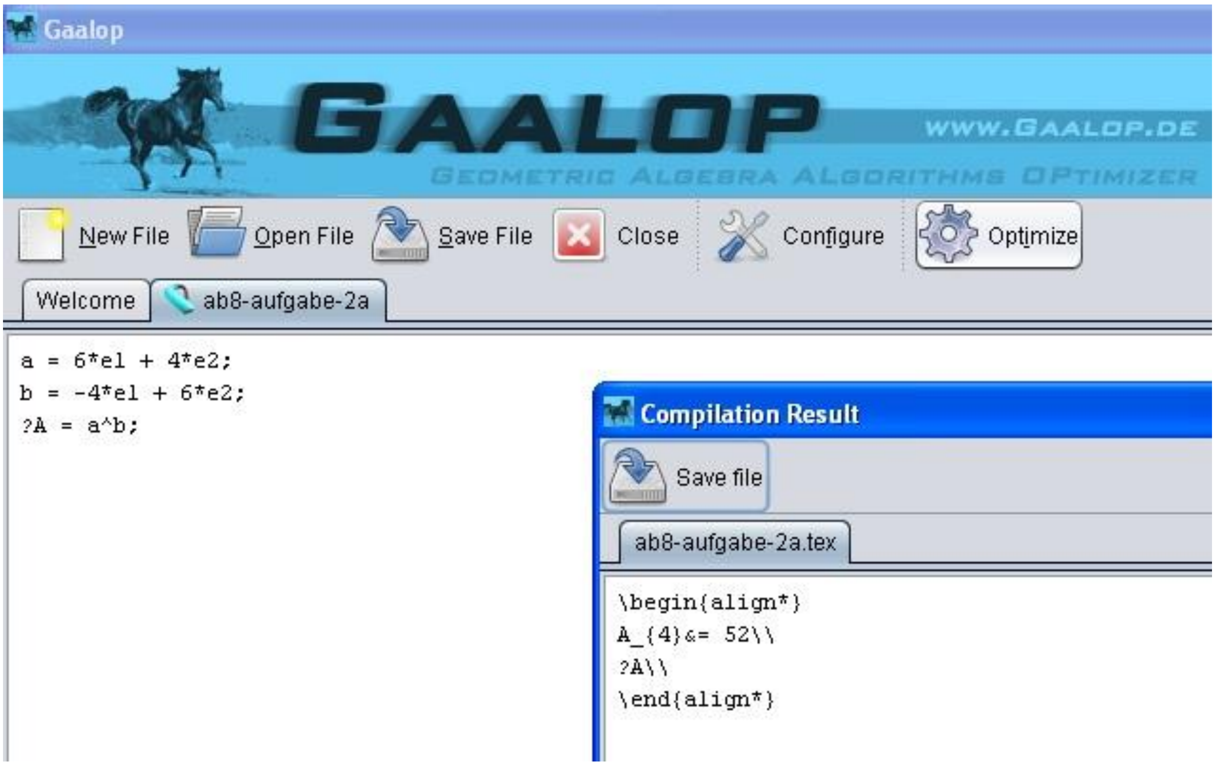
result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = -136\ \sigma_x\sigma_y$ $\Rightarrow$ $|\mathbf{A}| = 136$

$\Rightarrow$ The area of the parallelogram is 136 cm^2.

Problem 3 → Problem 2 of worksheet 8:

2) a)

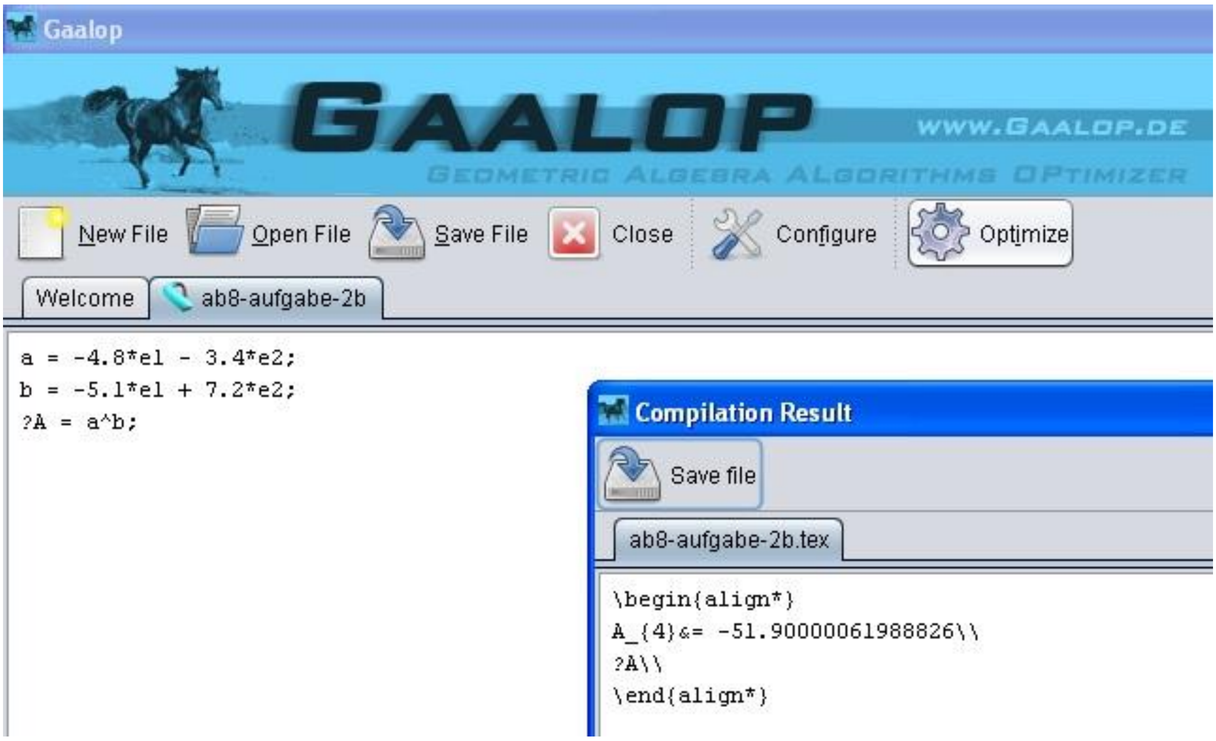


result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = 52\ \sigma_x\sigma_y \qquad \Rightarrow \quad |\mathbf{A}| = 52$

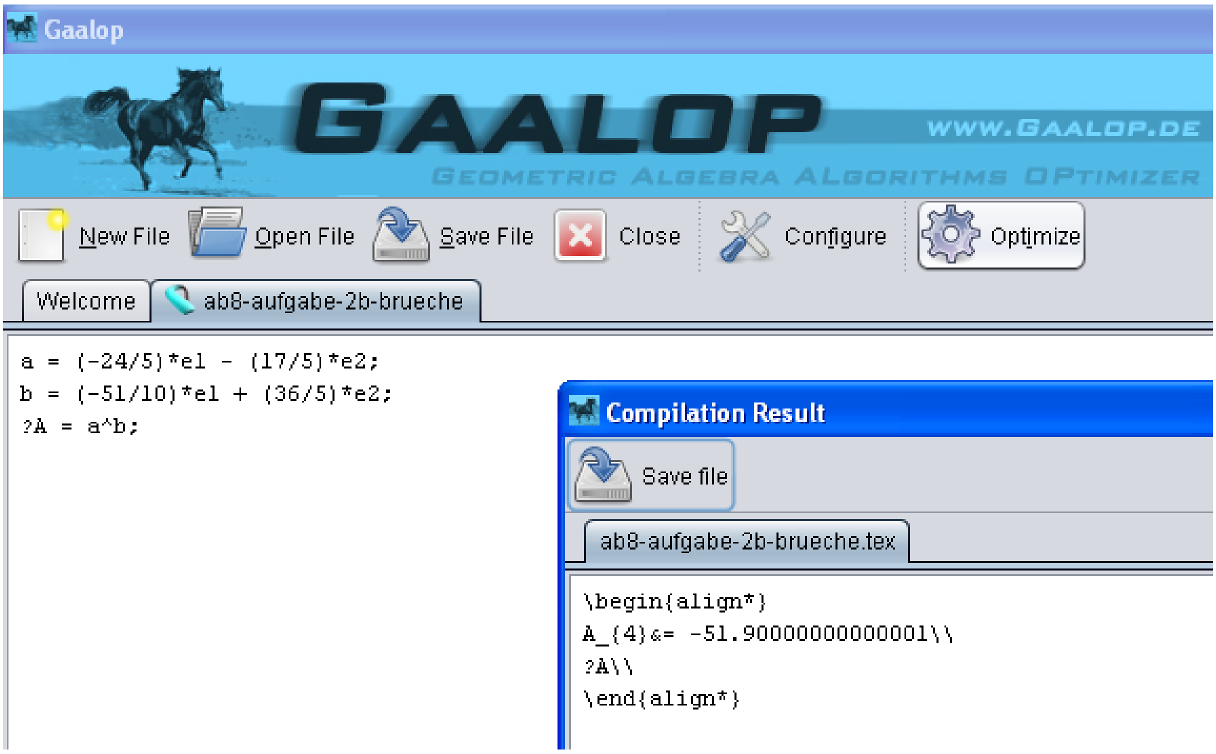
$\Rightarrow$ The area of the square is 52 cm^2.

2) b) Solution with slightly inexact result:



Short remark: As GAALOP is still in the development stage, the program shows unwelcome rounding inaccuracies (which will be removed in newer versions of GAALOP). Therefore the last digits of the result should be ignored. Nevertheless, a correct result can be found if the coefficients are given as fractions of integers (see following solution).

Solution with a more accurate result:

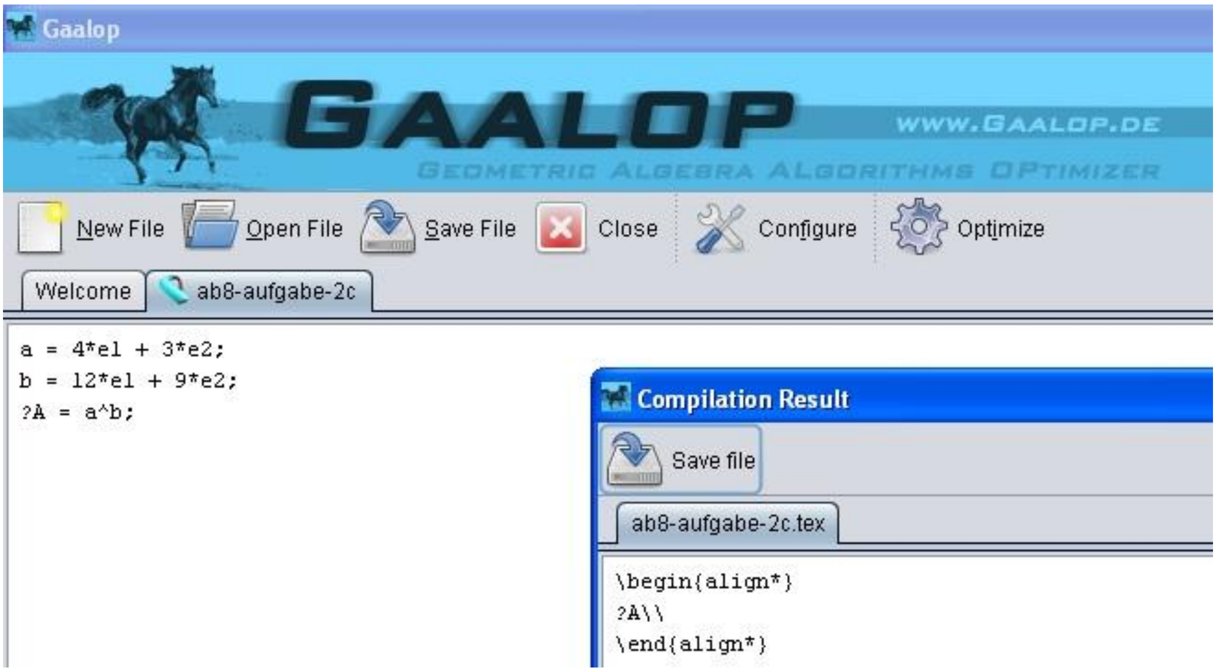


result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = -51.90\ \sigma_x\sigma_y \qquad \Rightarrow \quad |\mathbf{A}| = 51.90$

$\Rightarrow$ The area of the rectangle is 51.90 cm^2.

2) c)



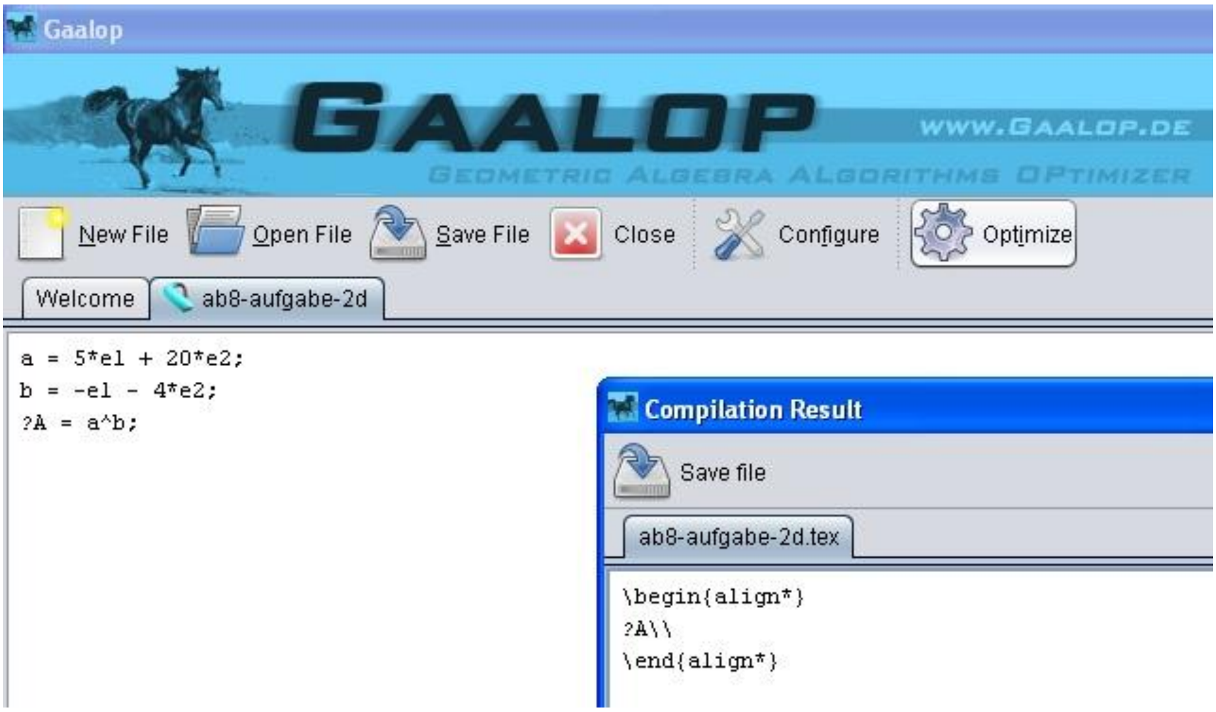
As no $\sigma_x\sigma_y$-component A_{4} is shown in the compiler field, this component must be zero: A_{4} &= 0

result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = 0\ \sigma_x\sigma_y \qquad \Rightarrow \quad |\mathbf{A}| = 0$

$\Rightarrow$ The area is 0 cm^2. It is not possible to form a parallelogram with the given vectors.

2) d)



As no $\sigma_x\sigma_y$-component A_{4} is shown in the compiler field, this component must be zero: A_{4} &= 0

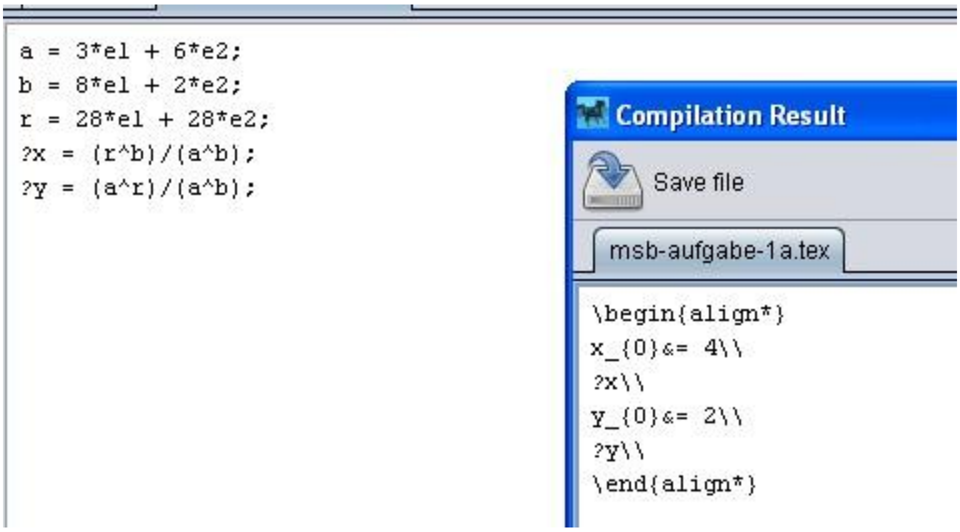
result in Pauli notation:

$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} = 0\ \sigma_x\sigma_y \quad \Rightarrow \quad |\mathbf{A}| = 0$

$\Rightarrow$ The area is 0 cm^2. It is not possible to form a parallelogram with the given vectors.

Problem 3 → Problems 3 & 4 of worksheet 8:

3) a) & 4)



Solution of the system of linear equations:

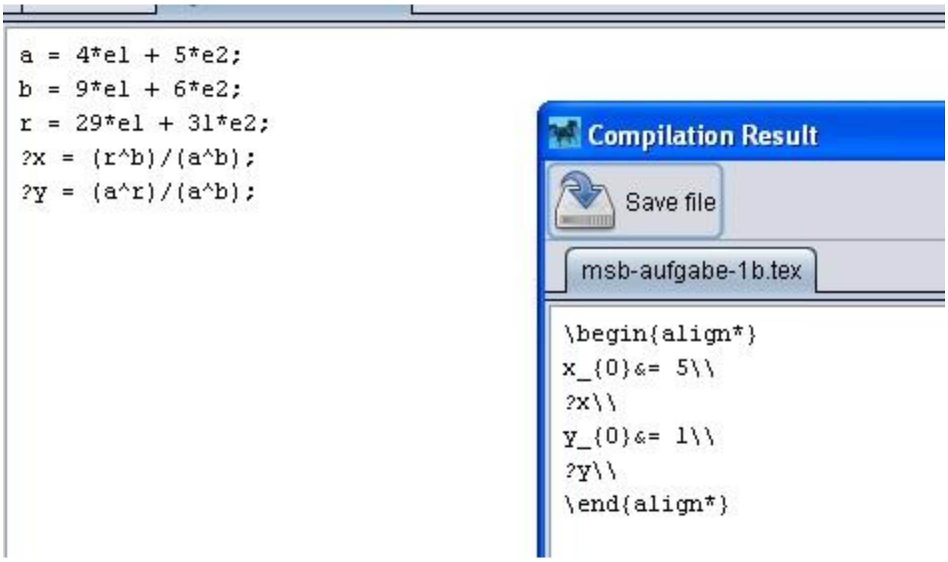
$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r} \wedge \mathbf{b}) = (\mathbf{r} \wedge \mathbf{b}) \ / \ (\mathbf{a} \wedge \mathbf{b}) = 4$

$y = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{a} \wedge \mathbf{r}) = (\mathbf{a} \wedge \mathbf{r}) \ / \ (\mathbf{a} \wedge \mathbf{b}) = 2$

Short remark: If systems of two linear equations are consistent and solvable, the outer products ($\mathbf{a} \wedge \mathbf{b}$), ($\mathbf{a} \wedge \mathbf{r}$), and ($\mathbf{r} \wedge \mathbf{b}$) will represent oriented areas which are parallel to each other and thus have the same orientation in space. Therefore these outer products commute and their order can be changed when multiplied.

to 4) If 28 units of the first raw material R_1 and 28 units of the second raw material R_2 are consummed in the production process, 4 units of the first final product P_1 and 2 units of the second final product P_2 will be produced.

3) b)

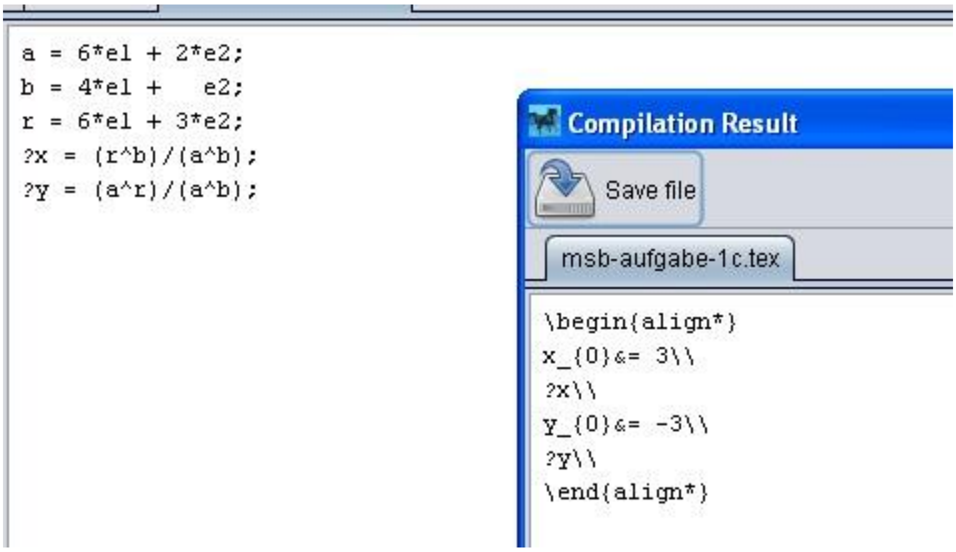


Solution of the system of linear equations:

$x = (\mathbf{r} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 5$

$y = (\mathbf{a} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b}) = 1$

3) c)

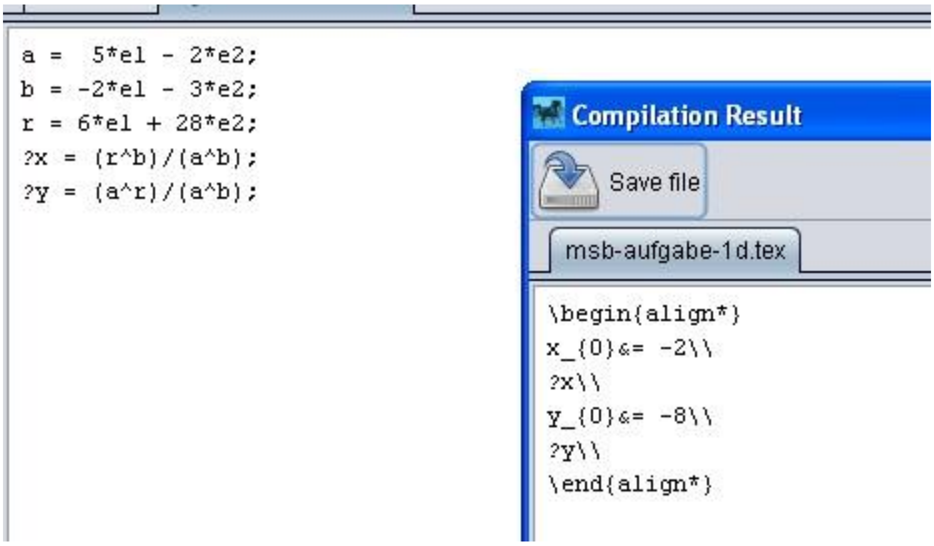


Solution of the system of linear equations:

$x = (\mathbf{r} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 3$

$y = (\mathbf{a} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b}) = -3$

3) d)



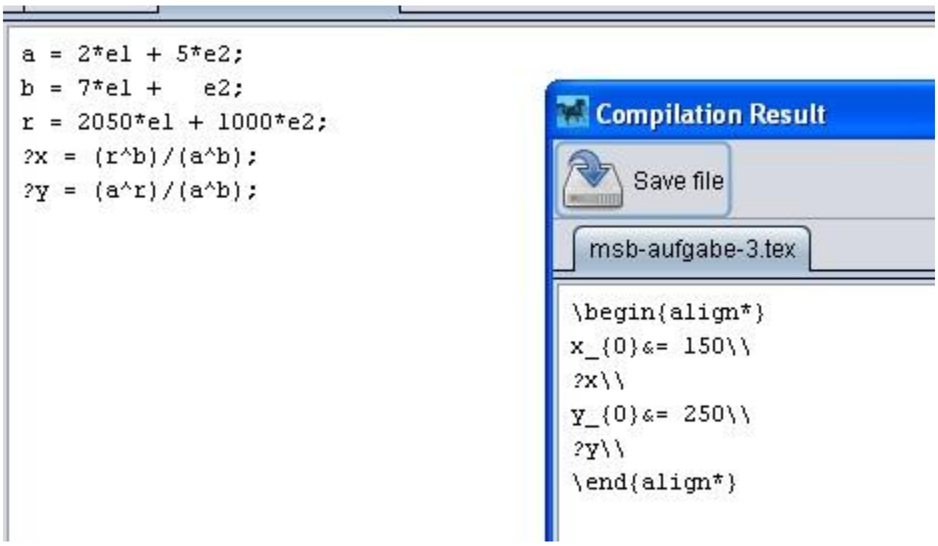
Solution of the system of linear equations:

x = (**r** ∧ **b**) / (**a** ∧ **b**) = – 2

y = (**a** ∧ **r**) / (**a** ∧ **b**) = – 8

Problem 3 → Problem 5 of worksheet 8:

5)



Solution of the system of linear equations:

x = (**r** ∧ **b**) / (**a** ∧ **b**) = 150

y = (**a** ∧ **r**) / (**a** ∧ **b**) = 250

⇒ If 2050 units of the first raw material R_1 and 1000 units of the second raw material R_2 are consumed in the production process, 150 units of the first final product P_1 and 250 units of the second final product P_2 will be produced.

Problem 3 → Problem 6 of worksheet 8:

6)

```
a = 4*e1 +   e2;
b = 3*e1 + 5*e2;
r1 = 33000*e1 + 38000*e2;
r2 = 32000*e1 + 25000*e2;
?Xeins = (r1^b)/(a^b);
?Yeins = (a^r1)/(a^b);
?Xzwei = (r2^b)/(a^b);
?Yzwei = (a^r2)/(a^b);
```

Compilation Result

Save file

msb-aufgabe-4.tex

```
\begin{align*}
Xeins_{0}&= 3000\\
?Xeins\\
Yeins_{0}&= 7000\\
?Yeins\\
Xzwei_{0}&= 5000\\
?Xzwei\\
Yzwei_{0}&= 4000\\
?Yzwei\\
\end{align*}
```

Solutions of the two systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 3000$ $\quad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 5000$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = 7000$ $\quad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = 4000$

⇒ 3000 units of the first final product P_1 and 7000 units of the second final product P_2 will be produced in the first quarter.

5000 units of the first final product P_1 and 4000 units of the second final product P_2 will be produced in the second quarter.

Problem 3 → Problem 7 of worksheet 8:

7)

```
a = 8*e1 + 4*e2;
b = 2*e1 + 3*e2;
r1 = 42*e1 + 23*e2;
r2 = 28*e1 + 26*e2;
?Xeins = (r1^b)/(a^b);
?Yeins = (a^r1)/(a^b);
?Xzwei = (r2^b)/(a^b);
?Yzwei = (a^r2)/(a^b);
```

Compilation Result

Save file

ab8-aufgabe-7.tex

```
\begin{align*}
Xeins_{0}&= 5\\
?Xeins\\
Yeins_{0}&= 1\\
?Yeins\\
Xzwei_{0}&= 2\\
?Xzwei\\
Yzwei_{0}&= 6\\
?Yzwei\\
\end{align*}
```

Solutions of the two systems of linear equations:

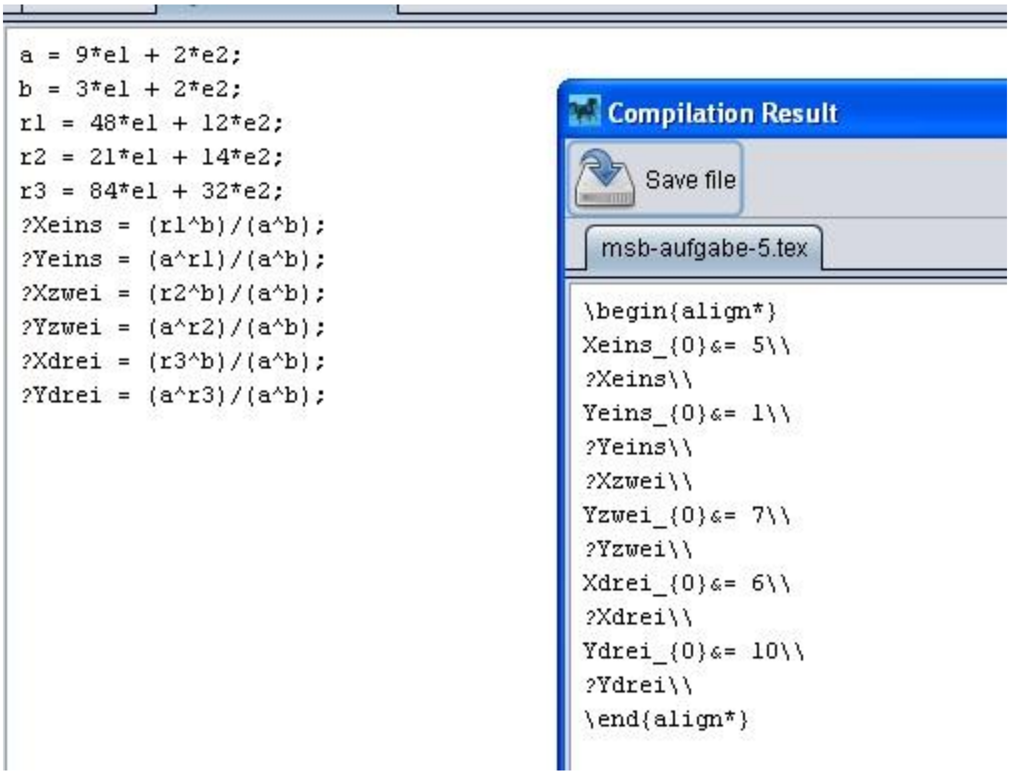
$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 5$ $\qquad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 2$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = 1$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = 6$

$\Rightarrow$ demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix}$

Problem 3 $\rightarrow$ Problem 8 of worksheet 8:

8)



As no scalar component Xzwei_{0} is shown in the compiler field, this component must be zero: Xzwei_{0}&= 0

Solutions of the three systems of linear equations:

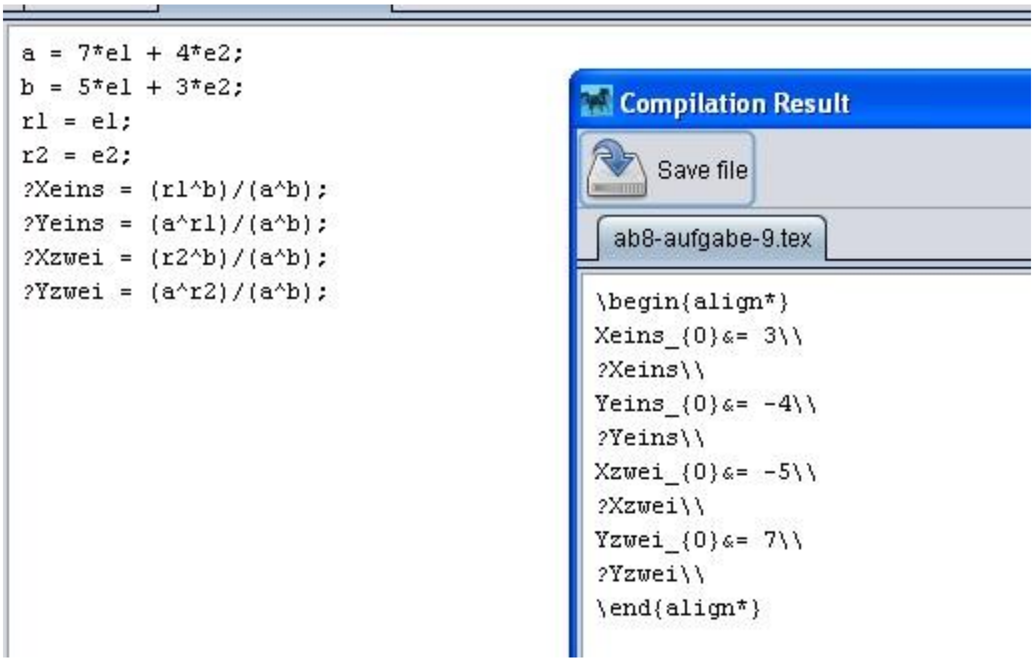
$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 5$ $\qquad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 0$ $\qquad$ $x_3 = (\mathbf{r_3} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 6$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = 1$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = 7$ $\qquad$ $y_3 = (\mathbf{a} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b}) = 10$

$\Rightarrow$ demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 5 & 0 & 6 \\ 1 & 7 & 10 \end{bmatrix}$

Problem 3 $\rightarrow$ Problem 9 of worksheet 8:

9)



Solutions of the two systems of linear equations:

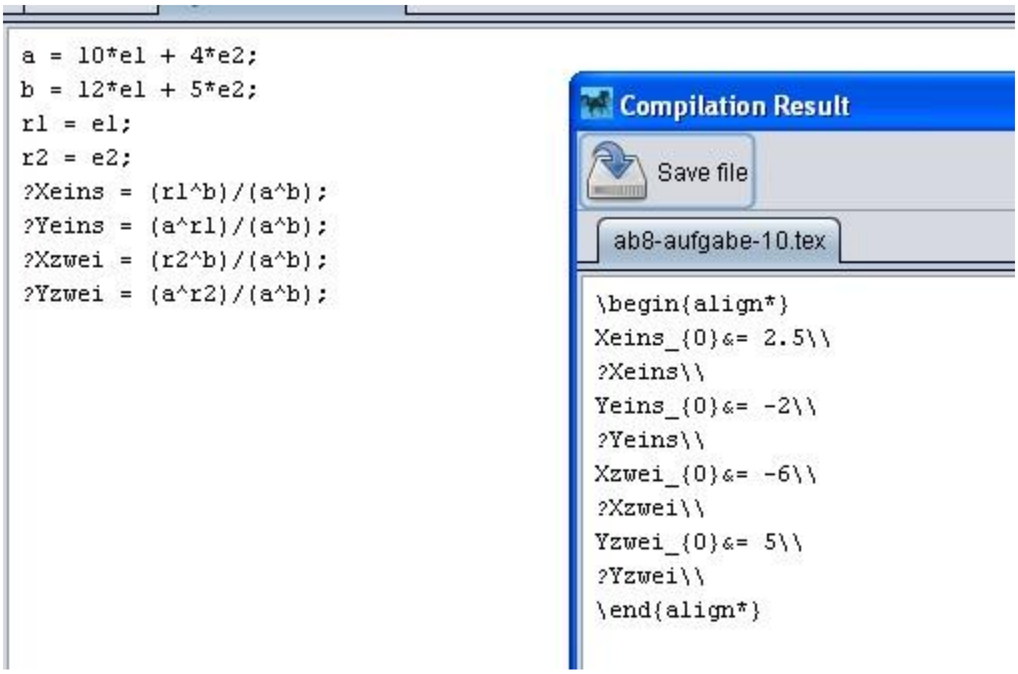
$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \ \ 3$ $\qquad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = -5$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = -4$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \ \ 7$

$\Rightarrow$ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$.

Problem 3 $\rightarrow$ Problem 10 of worksheet 8:

10)



Solutions of the two systems of linear equations:

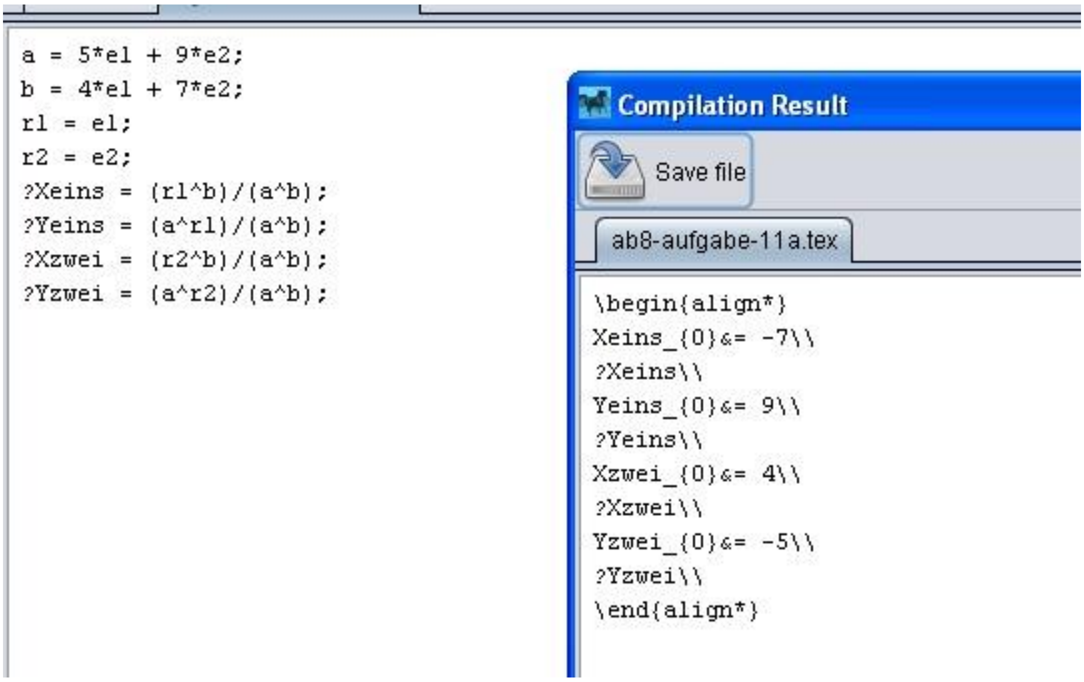
$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{5}{2} = 2.5$ $\qquad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-12}{2} = -6$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-4}{2} = -2$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{10}{2} = 5$

$\Rightarrow$ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 2.5 & -6 \\ -2 & 5 \end{bmatrix}$ is the inverse of demand matrix $\mathbf{A} = \begin{bmatrix} 10 & 12 \\ 4 & 5 \end{bmatrix}$.

Problem 3 $\rightarrow$ Problem 11 of worksheet 8:

11) a)



Solutions of the two systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{7}{-1} = -7$ $\qquad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-4}{-1} = 4$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-9}{-1} = 9$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{5}{-1} = -5$

$\Rightarrow$ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} -7 & 4 \\ 9 & -5 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$.

11) b)

```
a = 10*e1 + 19*e2;
b = 4*e1 + 8*e2;
r1 = e1;
r2 = e2;
?Xeins = (r1^b)/(a^b);
?Yeins = (a^r1)/(a^b);
?Xzwei = (r2^b)/(a^b);
?Yzwei = (a^r2)/(a^b);
```

Compilation Result

Save file

ab8-aufgabe-11b.tex

```
\begin{align*}
Xeins_{0}&= 2\\
?Xeins\\
Yeins_{0}&= -4.75\\
?Yeins\\
Xzwei_{0}&= -1\\
?Xzwei\\
Yzwei_{0}&= 2.5\\
?Yzwei\\
\end{align*}
```

Solutions of the two systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{8}{4} = 2$ $\qquad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-4}{4} = -1$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-19}{4} = -4.75$ $\qquad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{10}{4} = 2.5$

$\Rightarrow$ The resulting matrix $\mathbf{B}^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -19 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4.75 & 2.5 \end{bmatrix}$ is the inverse of matrix $\mathbf{B} = \begin{bmatrix} 10 & 4 \\ 19 & 8 \end{bmatrix}$.

11) c)

```
a = 10*e1 + 20*e2;
b = 6*e1 + 13*e2;
r1 = e1;
r2 = e2;
?Xeins = (r1^b)/(a^b);
?Yeins = (a^r1)/(a^b);
?Xzwei = (r2^b)/(a^b);
?Yzwei = (a^r2)/(a^b);
```

Compilation Result

Save file

ab8-aufgabe-11c.tex

```
\begin{align*}
Xeins_{0}&= 1.3\\
?Xeins\\
Yeins_{0}&= -2\\
?Yeins\\
Xzwei_{0}&= -0.6000000000000001\\
?Xzwei\\
Yzwei_{0}&= 1\\
?Yzwei\\
\end{align*}
```

Solutions of the two systems of linear equations:

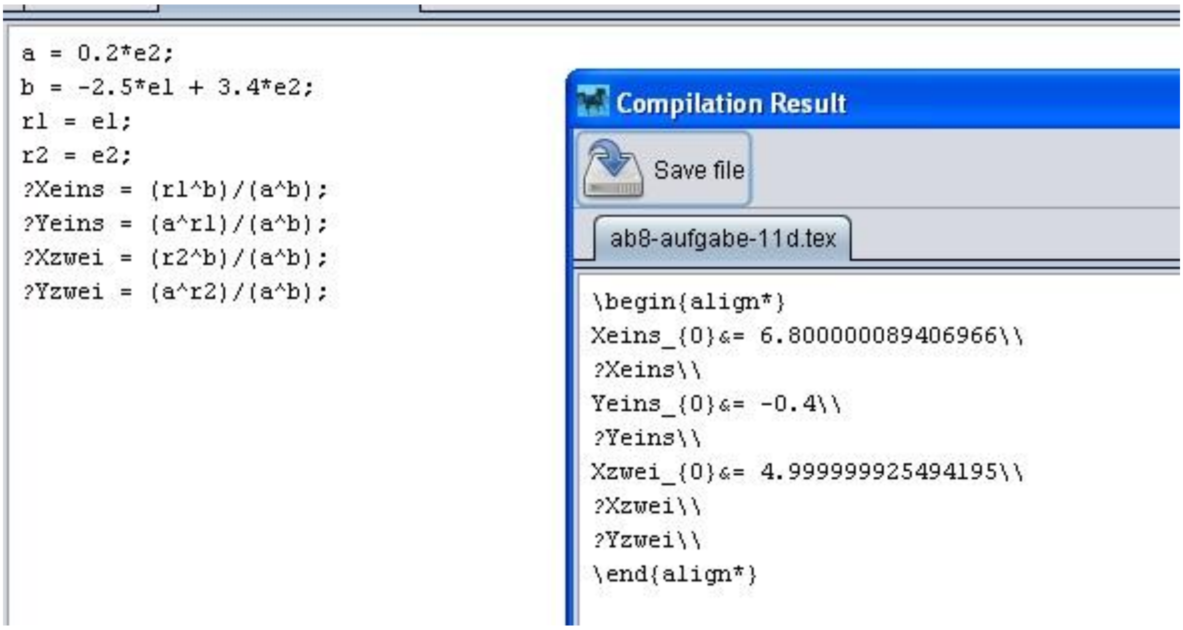
$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \dfrac{13}{10} = 1.3$ $\qquad x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \dfrac{-6}{10} = -0.6$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = \dfrac{-20}{10} = -2.0$ $\qquad y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \dfrac{10}{10} = 1.0$

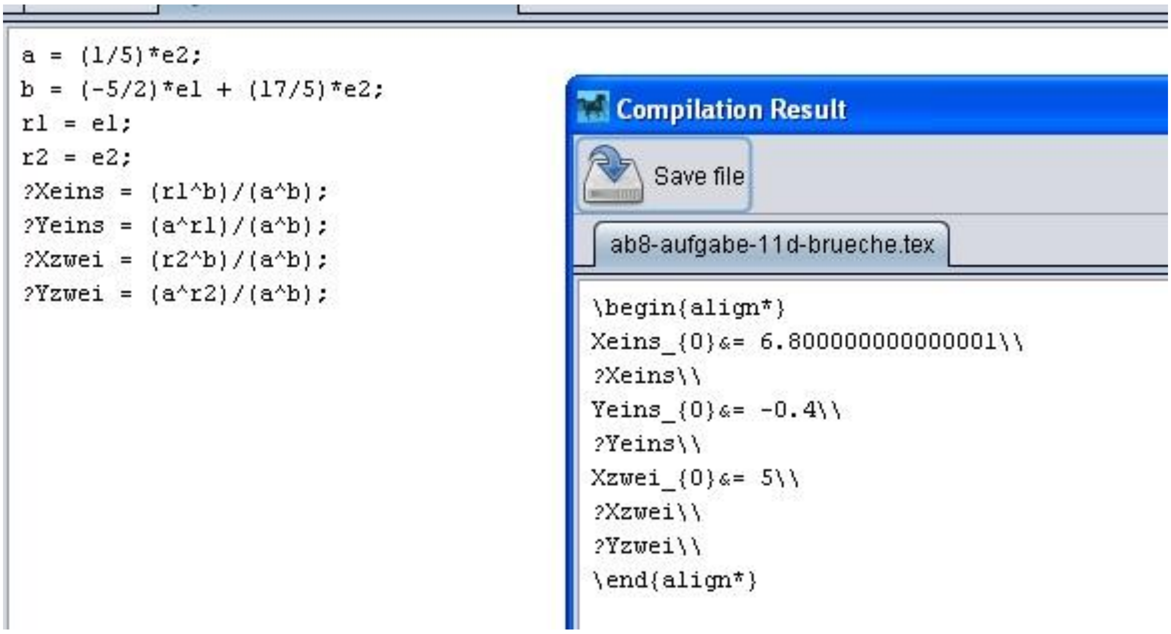
$\Rightarrow$ The resulting matrix $\mathbf{C}^{-1} = \dfrac{1}{10}\begin{bmatrix} 13 & -6 \\ -20 & 10 \end{bmatrix} = \begin{bmatrix} 1.3 & -0.6 \\ -2 & 1 \end{bmatrix}$ is the inverse

of matrix $\mathbf{C} = \begin{bmatrix} 10 & 6 \\ 20 & 13 \end{bmatrix}$.

11) d) Solution with slightly inexact results:



Avoiding decimal numbers will give the following solutions with more accurate results:



As no scalar component Yzwei_{0} is shown in the compiler field, this component must be zero: Yzwei_{0} &= 0

Solutions of the two systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{34}{5} = 6.8$ $\qquad x_2 = (\mathbf{r_2} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{25}{5} = 5$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{-2}{5} = -0.4$ $\qquad y_2 = (\mathbf{a} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b}) = \frac{0}{5} = 0$

$\Rightarrow$ The resulting matrix $\mathbf{D}^{-1} = \frac{1}{5}\begin{bmatrix} 34 & 25 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 6.8 & 5 \\ -0.4 & 0 \end{bmatrix}$ is the inverse of matrix $\mathbf{D} = \begin{bmatrix} 0 & -2.5 \\ 0.2 & 3.4 \end{bmatrix}$.

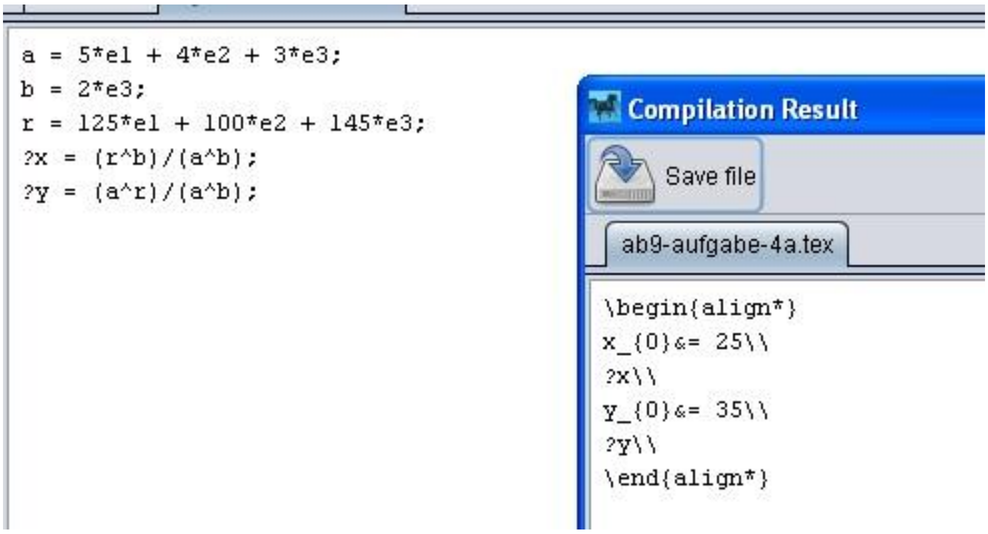
Problem 4:

a) $5x + 0y = 125$ $\qquad \Rightarrow \mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z$

$4x + 0y = 100$ $\qquad \mathbf{b} = 2\,\sigma_z$

$3x + 2y = 145$ $\qquad \mathbf{r} = 125\,\sigma_x + 100\,\sigma_y + 145\,\sigma_z$

Solution found with the help of GAALOP:



Solution of the system of linear equations:

$x = (\mathbf{r} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 25$

$y = (\mathbf{a} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b}) = 35$

Check: $5 \cdot 25 + 0 \cdot 35 = 125 + 0 = 125$

$4 \cdot 25 + 0 \cdot 35 = 100 + 0 = 100$

$3 \cdot 25 + 2 \cdot 35 = 75 + 70 = 145$

Detailed calculation of intermediate steps:

$\mathbf{a}\,\mathbf{b} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(2\,\sigma_z)$

$= 10\,\sigma_x\sigma_z + 8\,\sigma_y\sigma_z + 6\,\sigma_z^2$

$= 6 + 8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 8\ \sigma_y\sigma_z - 10\ \sigma_z\sigma_x$

$\mathbf{r}\ \mathbf{b} = (125\ \sigma_x + 100\ \sigma_y + 145\ \sigma_z)\ (2\ \sigma_z)$

$= 250\ \sigma_x\sigma_z + 200\ \sigma_y\sigma_z + 190\ \sigma_z^2$

$= 190 + 200\ \sigma_y\sigma_z - 250\ \sigma_z\sigma_x$

$\Rightarrow$ $\mathbf{r} \wedge \mathbf{b} = 200\ \sigma_y\sigma_z - 250\ \sigma_z\sigma_x$

$$x = \frac{\mathbf{r} \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}} = \frac{200\,\sigma_y\sigma_z - 250\,\sigma_z\sigma_x}{8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x} = \frac{25\,(8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x)}{8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x} = 25$$

$\mathbf{a}\ \mathbf{r} = (5\ \sigma_x + 4\ \sigma_y + 3\ \sigma_z)\ (125\ \sigma_x + 100\ \sigma_y + 145\ \sigma_z)$

$= 625\ \sigma_x^2 + 500\ \sigma_x\sigma_y + 725\ \sigma_x\sigma_z + 500\ \sigma_y\sigma_x + 400\ \sigma_y^2 + 580\ \sigma_y\sigma_z + 375\ \sigma_z\sigma_x + 300\ \sigma_z\sigma_y + 435\ \sigma_z^2$

$= 1460 + 0\ \sigma_x\sigma_y + 280\ \sigma_y\sigma_z - 350\ \sigma_z\sigma_x$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{r} = 280\ \sigma_y\sigma_z - 350\ \sigma_z\sigma_x$

$$y = \frac{\mathbf{a} \wedge \mathbf{r}}{\mathbf{a} \wedge \mathbf{b}} = \frac{280\,\sigma_y\sigma_z - 350\,\sigma_z\sigma_x}{8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x} = \frac{35\,(8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x)}{8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x} = 35$$

Conventional solution:

$5\ x + 0\ y = 125 \quad \Rightarrow \quad 5\ x = 125 \quad \Rightarrow \quad x = \frac{125}{5} = 25$

$4\ x + 0\ y = 100 \quad \Rightarrow \quad 4\ x = 100 \quad \Rightarrow \quad x = \frac{100}{4} = 25$

$3\ x + 2\ y = 145 \quad \Rightarrow \quad 75 + 2\ y = 145 \quad \Rightarrow \quad 2\ y = 70 \quad \Rightarrow \quad y = \frac{70}{2} = 35$

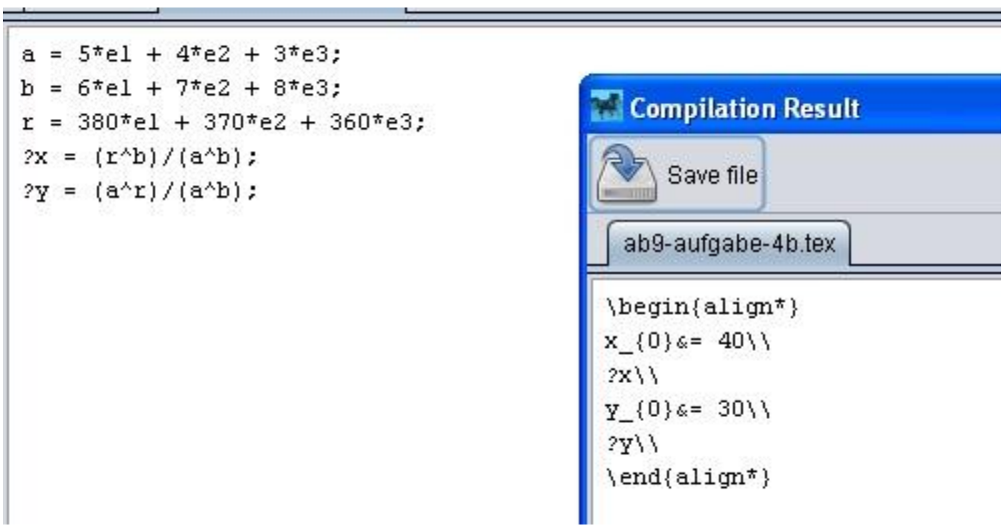
$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

b) $5\ x + 6\ y = 380 \quad \Rightarrow \quad \mathbf{a} = 5\ \sigma_x + 4\ \sigma_y + 3\ \sigma_z$

$4\ x + 7\ y = 370 \quad\quad \mathbf{b} = 6\ \sigma_x + 7\ \sigma_y + 8\ \sigma_z$

$3\ x + 8\ y = 360 \quad\quad \mathbf{r} = 380\ \sigma_x + 370\ \sigma_y + 360\ \sigma_z$

Solution found with the help of GAALOP:



Solution of the system of linear equations:

$x = (\mathbf{r} \wedge \mathbf{b}) / (\mathbf{a} \wedge \mathbf{b}) = 40$

$y = (\mathbf{a} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b}) = 30$

Check:
$$5 \cdot 40 + 6 \cdot 30 = 125 + 0 = 380$$
$$4 \cdot 40 + 7 \cdot 30 = 100 + 0 = 370$$
$$3 \cdot 40 + 8 \cdot 30 = 75 + 70 = 360$$

Detailed calculation of intermediate steps:

$$\mathbf{a}\,\mathbf{b} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z)$$
$$= 30\,\sigma_x^2 + 35\,\sigma_x\sigma_y + 40\,\sigma_x\sigma_z + 24\,\sigma_y\sigma_x + 28\,\sigma_y^2 + 32\,\sigma_y\sigma_z + 18\,\sigma_z\sigma_x + 21\,\sigma_z\sigma_y + 24\,\sigma_z^2$$
$$= 82 + 11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x$$

$$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x$$

$$\mathbf{r}\,\mathbf{b} = (380\,\sigma_x + 370\,\sigma_y + 360\,\sigma_z)\,(6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z)$$
$$= 2280\,\sigma_x^2 + 2660\,\sigma_x\sigma_y + 3040\,\sigma_x\sigma_z + 2220\,\sigma_y\sigma_x + 2590\,\sigma_y^2 + 2960\,\sigma_y\sigma_z + 2160\,\sigma_z\sigma_x + 2520\,\sigma_z\sigma_y + 2880\,\sigma_z^2$$
$$= 7750 + 440\,\sigma_x\sigma_y + 440\,\sigma_y\sigma_z - 880\,\sigma_z\sigma_x$$

$$\Rightarrow\ \mathbf{r} \wedge \mathbf{b} = 440\,\sigma_x\sigma_y + 440\,\sigma_y\sigma_z - 880\,\sigma_z\sigma_x$$

$$x = \frac{\mathbf{r} \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}} = \frac{440\,\sigma_x\sigma_y + 440\,\sigma_y\sigma_z - 880\,\sigma_z\sigma_x}{11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x} = \frac{40\,(11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)}{11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x} = 40$$

$$\mathbf{a}\,\mathbf{r} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(380\,\sigma_x + 370\,\sigma_y + 360\,\sigma_z)$$
$$= 1900\,\sigma_x^2 + 1850\,\sigma_x\sigma_y + 1800\,\sigma_x\sigma_z + 1520\,\sigma_y\sigma_x + 1480\,\sigma_y^2 + 1440\,\sigma_y\sigma_z + 1140\,\sigma_z\sigma_x + 1110\,\sigma_z\sigma_y + 1080\,\sigma_z^2$$
$$= 4460 + 330\,\sigma_x\sigma_y + 330\,\sigma_y\sigma_z - 660\,\sigma_z\sigma_x$$

$$\Rightarrow\ \mathbf{a} \wedge \mathbf{r} = 330\,\sigma_x\sigma_y + 330\,\sigma_y\sigma_z - 660\,\sigma_z\sigma_x$$

$$y = \frac{\mathbf{a} \wedge \mathbf{r}}{\mathbf{a} \wedge \mathbf{b}} = \frac{330\,\sigma_x\sigma_y + 330\,\sigma_y\sigma_z - 660\,\sigma_z\sigma_x}{11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x} = \frac{30\,(11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)}{11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x} = 30$$

Conventional solution:

$$\left.\begin{aligned} 5x + 6y &= 380 \\ 4x + 7y &= 370 \end{aligned}\right\} \ 9x + 13y = 750$$
$$3x + 8y = 360 \ \Rightarrow\ 9x + 24y = 1080$$
$$\left.\begin{aligned} 9x + 13y &= 750 \\ 9x + 24y &= 1080 \end{aligned}\right\} \ 11y = 330 \ \Rightarrow\ y = \frac{330}{11} = 30$$
$$\Rightarrow\ 5x + 180 = 380 \ \Rightarrow\ 5x = 200 \qquad x = \frac{200}{5} = 40$$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

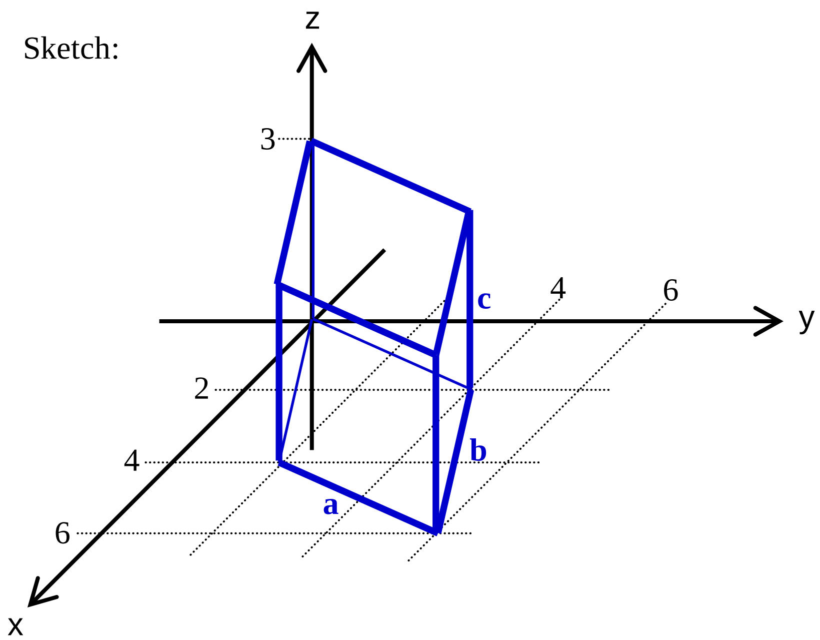
Problem 5:

a) $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$

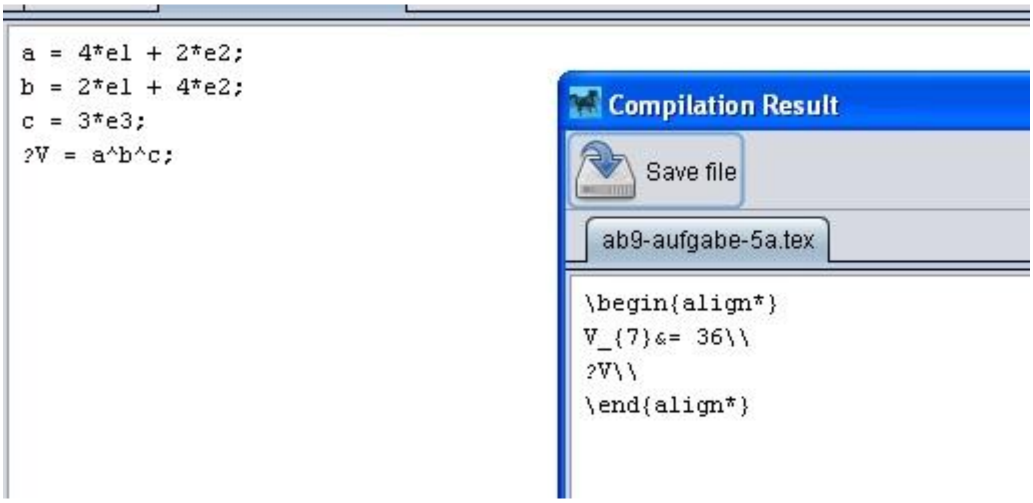
$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y$

$\mathbf{c} = 3\,\sigma_z$

Sketch:



GAALOP program and compilation result:



Detailed calculation:

$\mathbf{a}\,\mathbf{b} = (4\,\sigma_x + 2\,\sigma_y)\,(2\,\sigma_x + 4\,\sigma_y) = 16 + 12\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 12\,\sigma_x\sigma_y$

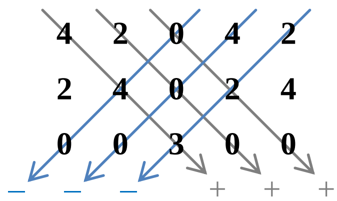
$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (16 + 12\,\sigma_x\sigma_y)\,(3\,\sigma_z) = 48\,\sigma_z + 36\,\sigma_x\sigma_y\sigma_z \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 36\,\sigma_x\sigma_y\sigma_z$

$\Rightarrow \quad |\mathbf{V}| = 36$

$\Rightarrow$ The volume of the parallelepiped is 36 cm^3.

Check by applying the rule of Sarrus:

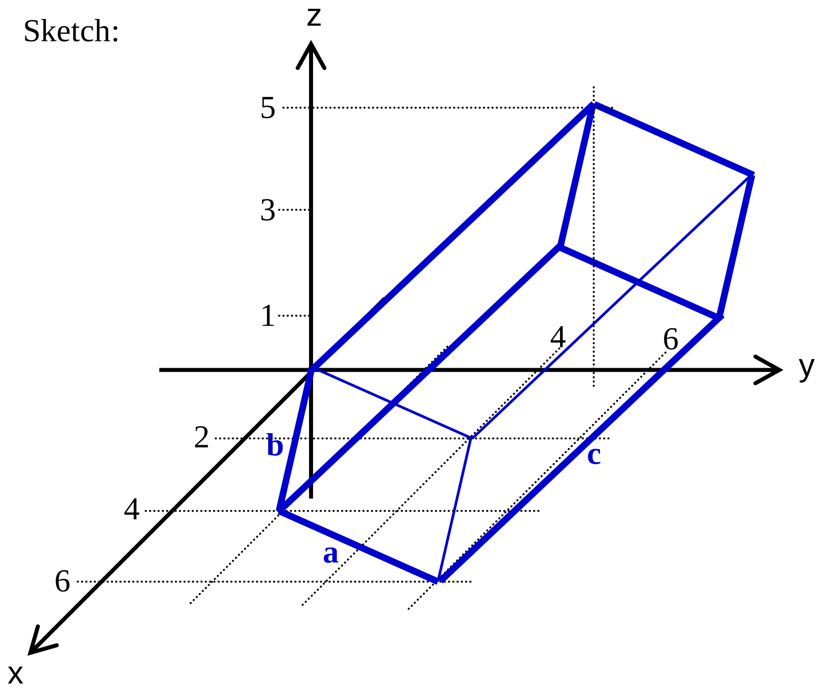
$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$



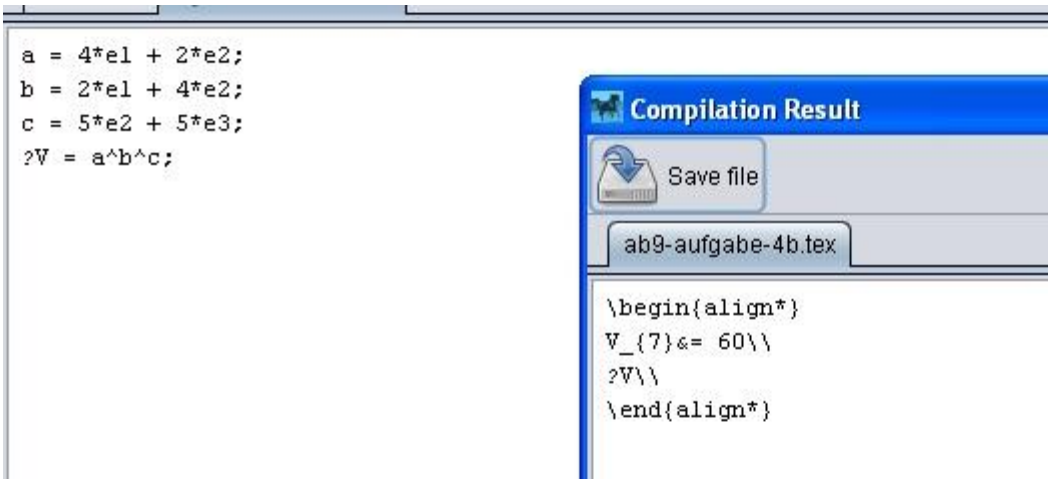
$\det \mathbf{A} = 4 \cdot 4 \cdot 3 + 2 \cdot 0 \cdot 0 + 0 \cdot 2 \cdot 0 - 0 \cdot 4 \cdot 0 - 4 \cdot 0 \cdot 0 - 2 \cdot 2 \cdot 3 = 48 - 12 = 36$

b) $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$
$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y$
$\mathbf{c} = 5\,\sigma_y + 5\,\sigma_z$

Sketch:



GAALOP program and compilation result:



Detailed calculation:

$\mathbf{a}\,\mathbf{b} = (4\,\sigma_x + 2\,\sigma_y)\,(2\,\sigma_x + 4\,\sigma_y) = 16 + 12\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 12\,\sigma_x\sigma_y$

$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (16 + 12\,\sigma_x\sigma_y)\,(5\,\sigma_y + 5\,\sigma_z) = 60\,\sigma_x + 80\,\sigma_y + 80\,\sigma_z + 60\,\sigma_x\sigma_y\sigma_z$

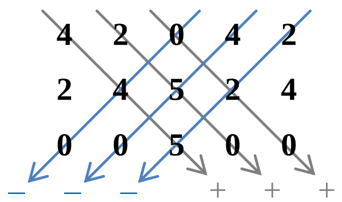
$\Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 60\,\sigma_x\sigma_y\sigma_z$

$\Rightarrow \quad |\mathbf{V}| = 60$

$\Rightarrow$ The volume of the parallelepiped is 60 cm^3.

Check by applying the rule of Sarrus:

$$\mathbf{B} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 0 & 5 \end{pmatrix}$$



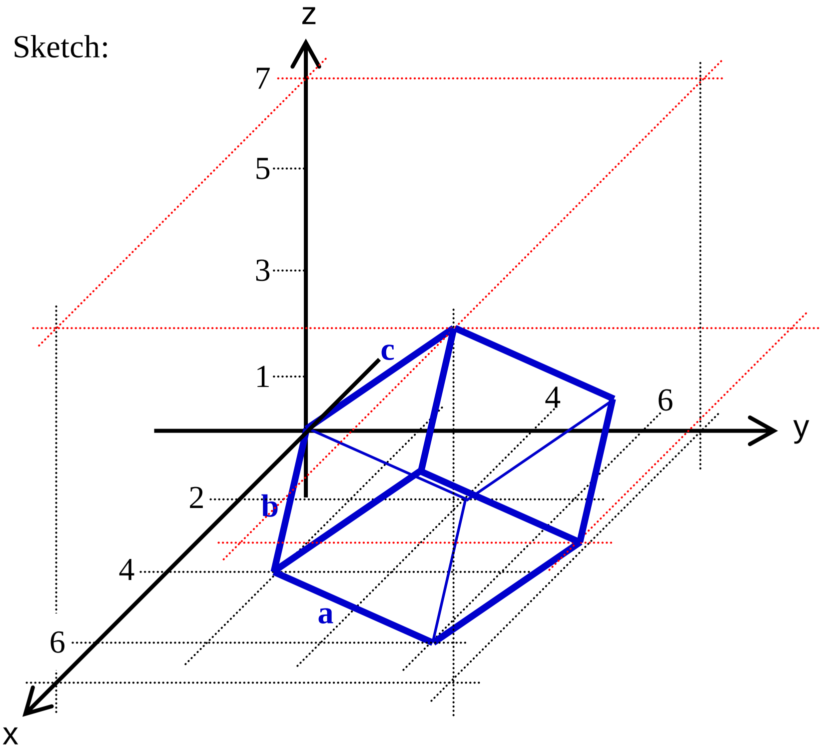
$\det \mathbf{B} = 4 \cdot 4 \cdot 5 + 2 \cdot 5 \cdot 0 + 0 \cdot 2 \cdot 0 - 0 \cdot 4 \cdot 0 - 4 \cdot 5 \cdot 0 - 2 \cdot 2 \cdot 5 = 80 - 20 = 60$

c) $\mathbf{a} = 4\,\sigma_x + 2\,\sigma_y$

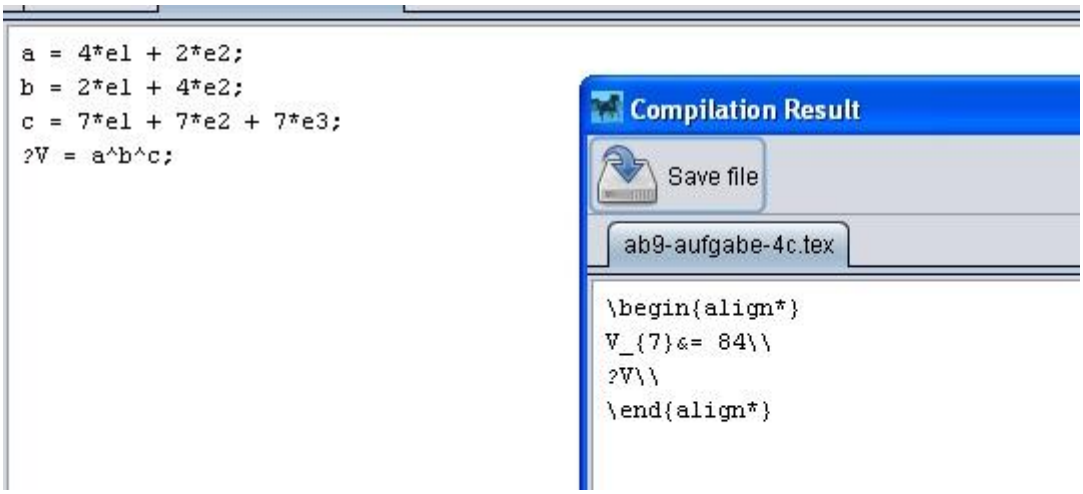
$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y$

$\mathbf{c} = 7\,\sigma_x + 7\,\sigma_y + 7\,\sigma_z$

Sketch:



GAALOP program and compilation result:



```
a = 4*e1 + 2*e2;
b = 2*e1 + 4*e2;
c = 7*e1 + 7*e2 + 7*e3;
?V = a^b^c;
```

Compilation Result

Save file

ab9-aufgabe-4c.tex

```
\begin{align*}
V_{7}&= 84\\
?V\\
\end{align*}
```

Detailed calculation:

$\mathbf{a}\,\mathbf{b} = (4\,\sigma_x + 2\,\sigma_y)\,(2\,\sigma_x + 4\,\sigma_y) = 16 + 12\,\sigma_x\sigma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 12\,\sigma_x\sigma_y$

$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (16 + 12\,\sigma_x\sigma_y)\,(7\,\sigma_x + 7\,\sigma_y + 7\,\sigma_z) = 112\,\sigma_x + 28\,\sigma_y + 112\,\sigma_z + 84\,\sigma_x\sigma_y\sigma_z$

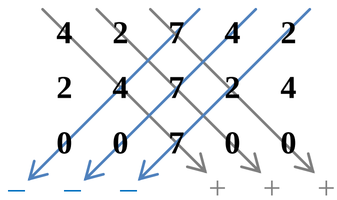
$\Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 84\,\sigma_x\sigma_y\sigma_z$

$\Rightarrow \quad |\mathbf{V}| = 84$

$\Rightarrow$ The volume of the parallelepiped is 84 cm^3.

Check by applying the rule of Sarrus:

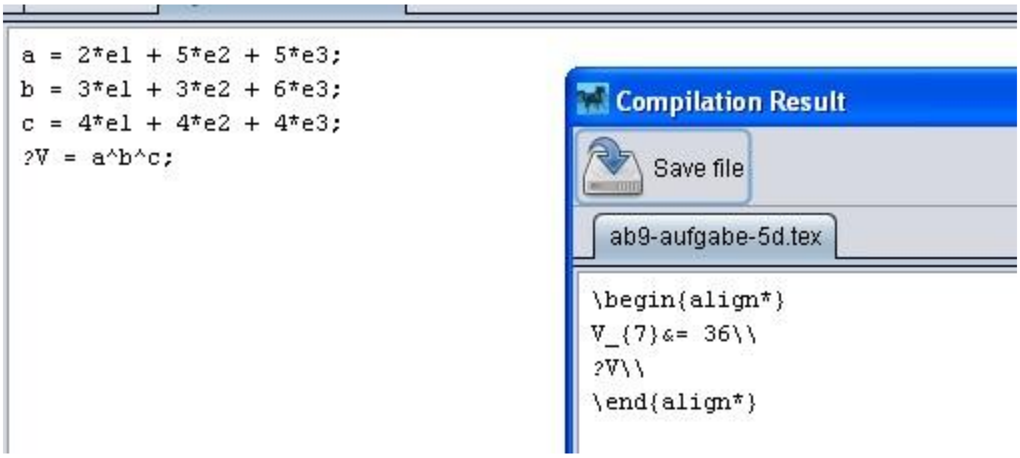
$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 7 \\ 2 & 4 & 7 \\ 0 & 0 & 7 \end{pmatrix}$$



$\det \mathbf{C} = 4 \cdot 4 \cdot 7 + 2 \cdot 7 \cdot 0 + 7 \cdot 2 \cdot 0 - 7 \cdot 4 \cdot 0 - 4 \cdot 7 \cdot 0 - 2 \cdot 2 \cdot 7 = 112 - 28 = 84$

d) $\mathbf{a} = 2\,\sigma_x + 5\,\sigma_y + 5\,\sigma_z$
$\mathbf{b} = 3\,\sigma_x + 3\,\sigma_y + 6\,\sigma_z$
$\mathbf{c} = 4\,\sigma_x + 4\,\sigma_y + 4\,\sigma_z$

GAALOP program and compilation result:



Detailed calculation:

$\mathbf{a}\,\mathbf{b} = (2\,\sigma_x + 5\,\sigma_y + 5\,\sigma_z)\,(3\,\sigma_x + 3\,\sigma_y + 6\,\sigma_z) = 51 - 9\,\sigma_x\sigma_y + 15\,\sigma_y\sigma_z + 3\,\sigma_z\sigma_x$

$\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} = -\,9\,\sigma_x\sigma_y + 15\,\sigma_y\sigma_z + 3\,\sigma_z\sigma_x$

$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (51 - 9\,\sigma_x\sigma_y + 15\,\sigma_y\sigma_z + 3\,\sigma_z\sigma_x)\,(4\,\sigma_x + 4\,\sigma_y + 4\,\sigma_z) = 156\,\sigma_x + 300\,\sigma_y + 156\,\sigma_z + 36\,\sigma_x\sigma_y\sigma_z$

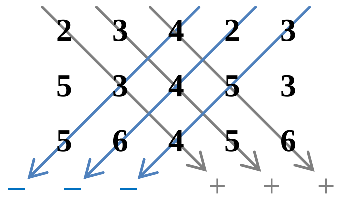
$\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 36\,\sigma_x\sigma_y\sigma_z$

$\Rightarrow\quad |\mathbf{V}| = 36$

$\Rightarrow$ The volume of the parallelepiped is 36 cm^3.

Check by applying the rule of Sarrus:

$$\mathbf{D} = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 3 & 4 \\ 5 & 6 & 4 \end{pmatrix}$$



$$\det \mathbf{D} = 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + 4\cdot 5\cdot 6 - 4\cdot 3\cdot 5 - 2\cdot 4\cdot 6 - 3\cdot 5\cdot 4$$
$$= 24 + 60 + 120 - 60 - 48 - 60 = 36$$

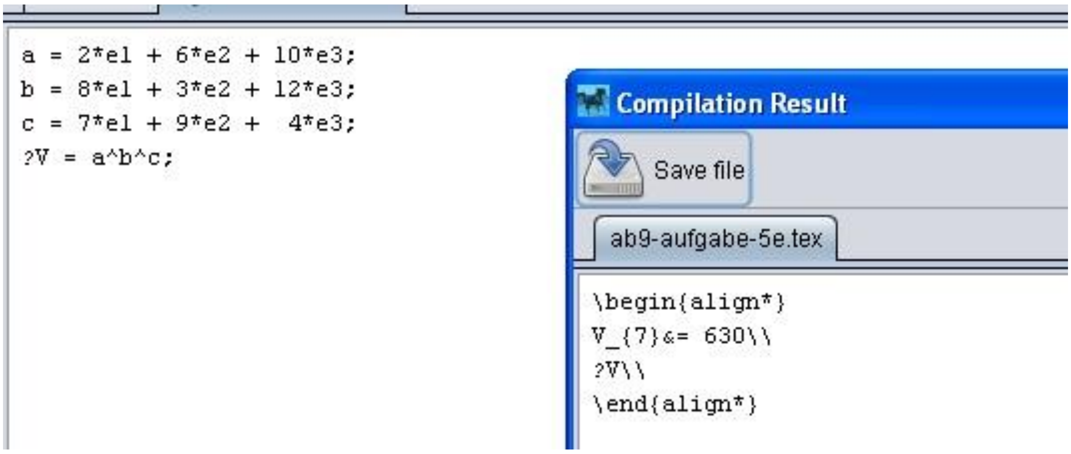
Another check: $(2^2 + 5^2 + 5^2)\,(3^2 + 3^2 + 6^2) = 51^2 + (-9)^2 + 15^2 + 3^2 = 2916$

$(51^2 + (-9)^2 + 15^2 + 3^2)\,(4^2 + 4^2 + 4^2) = 156^2 + 300^2 + 156^2 + 36^2 = 139968$

$\Rightarrow$ trigonometric Pythagoras: $\sin^2\alpha + \cos^2\alpha = 1$

e) $\mathbf{a} = 2\,\sigma_x + 6\,\sigma_y + 10\,\sigma_z$
$\mathbf{b} = 8\,\sigma_x + 3\,\sigma_y + 12\,\sigma_z$
$\mathbf{c} = 7\,\sigma_x + 9\,\sigma_y + 4\,\sigma_z$

GAALOP program and compilation result:



Detailed calculation:

$\mathbf{a}\,\mathbf{b} = (2\,\sigma_x + 6\,\sigma_y + 10\,\sigma_z)\,(8\,\sigma_x + 3\,\sigma_y + 12\,\sigma_z) = 154 - 42\,\sigma_x\sigma_y + 42\,\sigma_y\sigma_z + 56\,\sigma_z\sigma_x$

$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = -42\,\sigma_x\sigma_y + 42\,\sigma_y\sigma_z + 56\,\sigma_z\sigma_x$

$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (154 - 42\,\sigma_x\sigma_y + 42\,\sigma_y\sigma_z + 56\,\sigma_z\sigma_x)\,(7\,\sigma_x + 9\,\sigma_y + 4\,\sigma_z)$

$= 476\,\sigma_x + 1848\,\sigma_y + 630\,\sigma_z + 630\,\sigma_x\sigma_y\sigma_z$

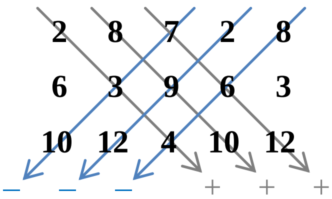
$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 630\,\sigma_x\sigma_y\sigma_z$

$\Rightarrow\ |\mathbf{V}| = 630$

$\Rightarrow$ The volume of the parallelepiped is 630 cm^3.

Check by applying the rule of Sarrus:

$$\mathbf{T} = \begin{pmatrix} 2 & 8 & 7 \\ 6 & 3 & 9 \\ 10 & 12 & 4 \end{pmatrix}$$



$\det \mathbf{T} = 2 \cdot 3 \cdot 4 + 8 \cdot 9 \cdot 10 + 7 \cdot 6 \cdot 12 - 7 \cdot 3 \cdot 10 - 2 \cdot 9 \cdot 12 - 8 \cdot 6 \cdot 4$

$= 24 + 720 + 504 - 210 - 216 - 192 = 630$

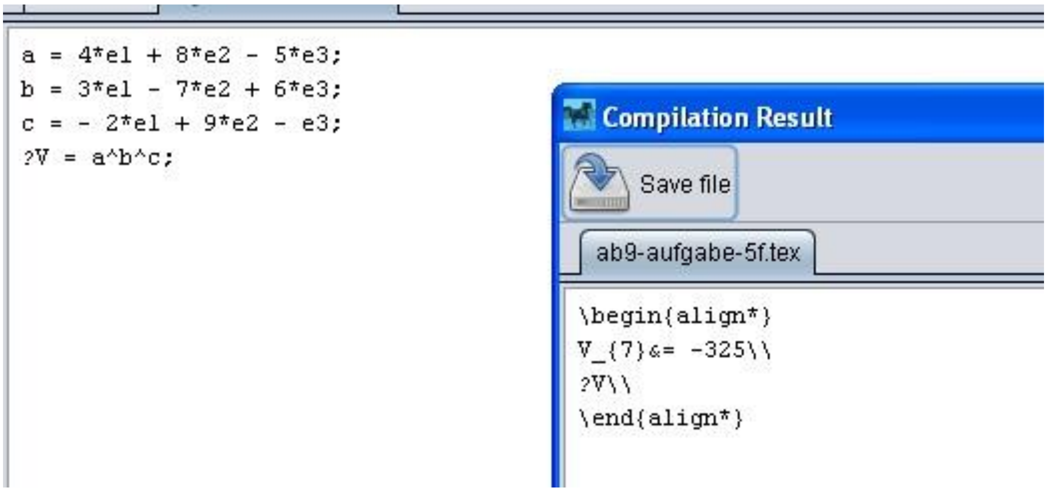
Another check: $(2^2 + 6^2 + 10^2)\,(8^2 + 3^2 + 12^2) = 154^2 + (-42)^2 + 42^2 + 56^2 = 30\,380$

$(154^2 + (-42)^2 + 42^2 + 56^2)\,(7^2 + 9^2 + 4^2) = 476^2 + 1848^2 + 630^2 + 630^2 = 4\,435\,480$

$\Rightarrow$ trigonometric Pythagoras: $\sin^2\alpha + \cos^2\alpha = 1$

f) $\mathbf{a} = 4\,\sigma_x + 8\,\sigma_y - 5\,\sigma_z$
$\mathbf{b} = 3\,\sigma_x - 7\,\sigma_y + 6\,\sigma_z$
$\mathbf{c} = -\,2\,\sigma_x + 9\,\sigma_y - \sigma_z$

GAALOP program and compilation result:



Detailed calculation:

$\mathbf{a\,b} = (4\,\sigma_x + 8\,\sigma_y - 5\,\sigma_z)\,(3\,\sigma_x - 7\,\sigma_y + 6\,\sigma_z) = -\,74 - 52\,\sigma_x\sigma_y + 13\,\sigma_y\sigma_z - 39\,\sigma_z\sigma_x$

$\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} = -\,52\,\sigma_x\sigma_y + 13\,\sigma_y\sigma_z - 39\,\sigma_z\sigma_x$

$\mathbf{a\,b\,c} = (-\,74 - 52\,\sigma_x\sigma_y + 13\,\sigma_y\sigma_z - 39\,\sigma_z\sigma_x)\,(-\,2\,\sigma_x + 9\,\sigma_y - \sigma_z)$

$= -\,359\,\sigma_x - 783\,\sigma_y + 35\,\sigma_z - 325\,\sigma_x\sigma_y\sigma_z$

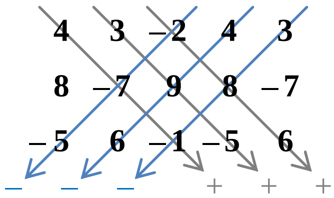
$\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -\,325\,\sigma_x\sigma_y\sigma_z$

$\Rightarrow\quad |\mathbf{V}| = 325$

$\Rightarrow$ The volume of the parallelepiped is 325 cm^3.

Check by applying the rule of Sarrus:

$$\mathbf{F} = \begin{pmatrix} 4 & 3 & -2 \\ 8 & -7 & 9 \\ -5 & 6 & -1 \end{pmatrix}$$



$\det \mathbf{F} = 4 \cdot (-\,7) \cdot (-\,1) + 3 \cdot 9 \cdot (-\,5) + (-\,2) \cdot 8 \cdot 6 - (-\,2) \cdot (-\,7) \cdot (-\,5) - 4 \cdot 9 \cdot 6 - 3 \cdot 8 \cdot (-\,1)$
$= 28 - 135 - 96 + 70 - 216 + 24 = -\,325$

Another check: $(4^2 + 8^2 + (-\,5)^2)\,(3^2 + (-\,7)^2 + 6^2) = (-\,74)^2 + (-\,52)^2 + 13^2 + (-\,39)^2 = 9870$

$((-\,74)^2 + (-\,52)^2 + 13^2 + (-\,39)^2)\,((-\,2)^2 + 9^2 + (-\,1)^2)$
$= (-\,359)^2 + (-\,783)^2 + 35^2 + (-\,325)^2 = 848820$

$\Rightarrow$ trigonometric Pythagoras: $\sin^2\alpha + \cos^2\alpha = 1$

Problem 6:

a) $$\begin{aligned} 3x + 8y &= 28 \\ 6x + 2y &= 28 \\ 2x + 4y + 2z &= 28 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 3\,\sigma_x + 6\,\sigma_y + 2\,\sigma_z \\ \mathbf{b} &= 8\,\sigma_x + 2\,\sigma_y + 4\,\sigma_z \\ \mathbf{c} &= 2\,\sigma_z \end{aligned} \qquad \mathbf{r} = 28\,\sigma_x + 28\,\sigma_y + 28\,\sigma_z$$

Short GAALOP program without intermediate steps:

```
a = 3*e1 + 6*e2 + 2*e3;
b = 8*e1 + 2*e2 + 4*e3;
c = 2*e3;
r = 28*e1 + 28*e2 + 28*e3;
?x = (r^b^c)/(a^b^c);
?y = (a^r^c)/(a^b^c);
?z = (a^b^r)/(a^b^c);
```

Compilation Result

Save file

ab9-aufgabe-6a.tex

```
\begin{align*}
x_{0}&= 4\\
?x\\
y_{0}&= 2\\
?y\\
z_{0}&= 6\\
?z\\
\end{align*}
```

Extended GAALOP program with intermediate steps (in German: Zwischenschritte) and solutions (in German: Loesungen):

```
a = 3*e1 + 6*e2 + 2*e3;
b = 8*e1 + 2*e2 + 4*e3;
c = 2*e3;
r = 28*e1 + 28*e2 + 28*e3;
#(Zwischenschritte);
?VOLUMENabc = a^b^c;
?VOLUMENrbc = r^b^c;
?VOLUMENarc = a^r^c;
?VOLUMENabr = a^b^r;
#(Loesungen);
?x = (r^b^c)/(a^b^c);
?y = (a^r^c)/(a^b^c);
?z = (a^b^r)/(a^b^c);
```

Compilation Result

Save file

ab9-aufgabe-6a-zwischenschritte.tex

```
\begin{align*}
Zwischenschritte\\
VOLUMENabc_{7}&= -84\\
?VOLUMENabc\\
VOLUMENrbc_{7}&= -336\\
?VOLUMENrbc\\
VOLUMENarc_{7}&= -168\\
?VOLUMENarc\\
VOLUMENabr_{7}&= -504\\
?VOLUMENabr\\
Loesungen\\
x_{0}&= 4\\
?x\\
y_{0}&= 2\\
?y\\
z_{0}&= 6\\
?z\\
\end{align*}
```

Intermediate steps:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -84\ \sigma_x\sigma_y\sigma_z$

$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = -336\ \sigma_x\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = -168\ \sigma_x\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = -504\ \sigma_x\sigma_y\sigma_z$

Solution of the system of linear equations:

$$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-336}{-84} = 4$$

$$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-168}{-84} = 2$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-504}{-84} = 6$$

Check: $3 \cdot 4 + 8 \cdot 2 = 12 + 16 = 28$

$6 \cdot 4 + 2 \cdot 2 = 24 + 4 = 28$

$2 \cdot 4 + 4 \cdot 2 + 2 \cdot 6 = 8 + 8 + 12 = 28$

b) $8x + 5y + 10z = 396$

$3x + 7y + 12z = 375$

$2x + 6y + 14z = 386$

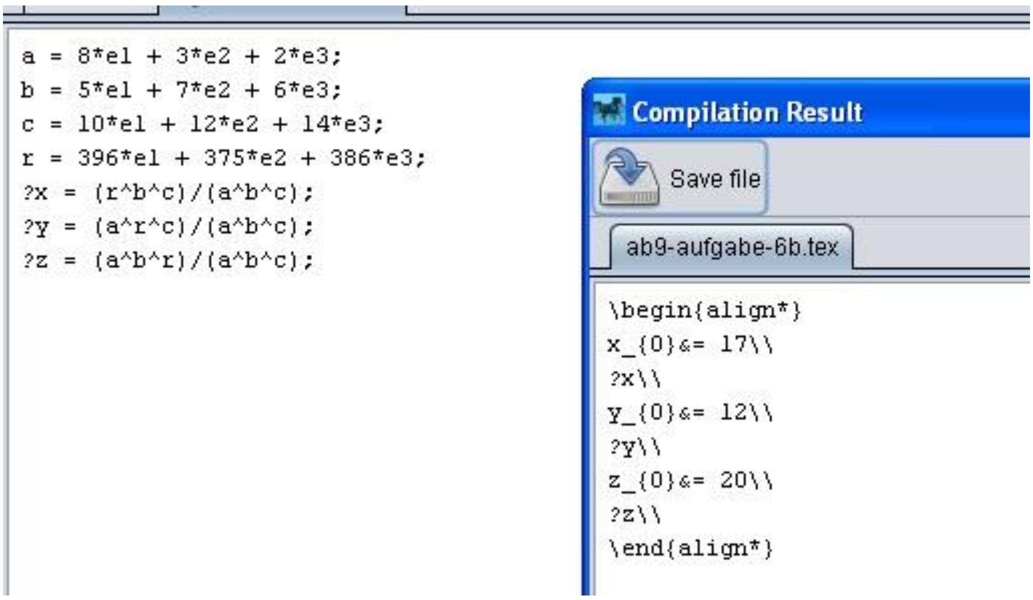
$\Rightarrow$ $\mathbf{a} = 8\,\sigma_x + 3\,\sigma_y + 2\,\sigma_z$

$\mathbf{b} = 5\,\sigma_x + 7\,\sigma_y + 6\,\sigma_z$

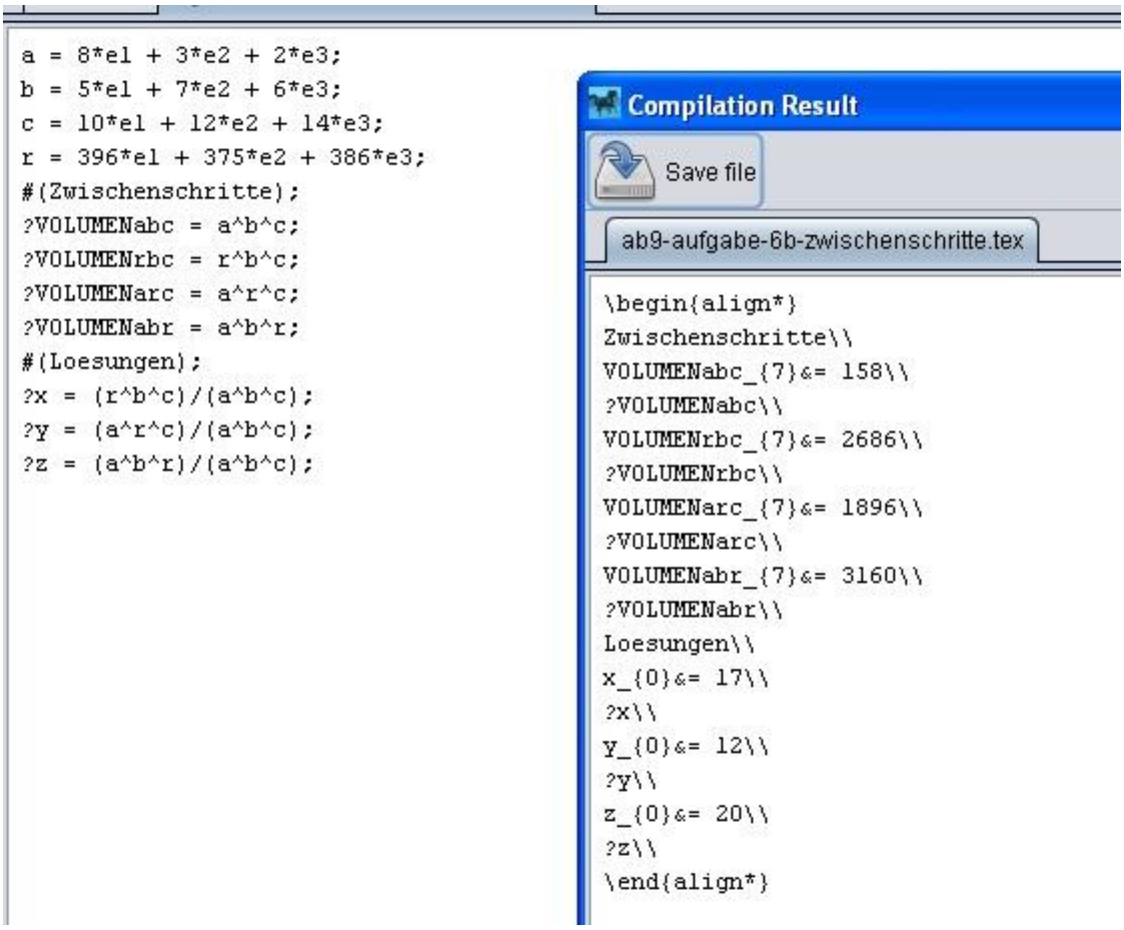
$\mathbf{c} = 10\,\sigma_x + 12\,\sigma_y + 14\,\sigma_z$

$\mathbf{r} = 396\,\sigma_x + 375\,\sigma_y + 386\,\sigma_z$

Short GAALOP program without intermediate steps:



Extended GAALOP program with intermediate steps (in German: Zwischenschritte) and solutions (in German: Loesungen):



Intermediate steps:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 158\, \sigma_x\sigma_y\sigma_z$

$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = 2686\, \sigma_x\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = 1896\, \sigma_x\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = 3160\, \sigma_x\sigma_y\sigma_z$

Solution of the system of linear equations:

$$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{2686}{158} = 17$$

$$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{1896}{158} = 12$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{3160}{158} = 20$$

Check: $8 \cdot 17 + 5 \cdot 12 + 10 \cdot 20 = 136 + 60 + 200 = 396$

$3 \cdot 17 + 7 \cdot 12 + 12 \cdot 20 = 51 + 84 + 240 = 375$

$2 \cdot 17 + 6 \cdot 12 + 14 \cdot 20 = 34 + 72 + 280 = 386$

c) $3x - 5y + 6z = 41$ $\Rightarrow$ $\mathbf{a} = 3\,\sigma_x - 2\,\sigma_y + 7\,\sigma_z$ $\mathbf{r} = 41\,\sigma_x + 111\,\sigma_y + 185\,\sigma_z$

$-2x + 5y + 8z = 111$ $\mathbf{b} = -5\,\sigma_x + 5\,\sigma_y + \sigma_z$

$7x + y + 9z = 185$ $\mathbf{c} = 6\,\sigma_x + 8\,\sigma_y + 9\,\sigma_z$

Short GAALOP program without intermediate steps:

```
a = 3*e1 - 2*e2 + 7*e3;
b = -5*e1 + 5*e2 + e3;
c = 6*e1 + 8*e2 + 9*e3;
r = 41*e1 + 111*e2 + 185*e3;
?x = (r^b^c)/(a^b^c);
?y = (a^r^c)/(a^b^c);
?z = (a^b^r)/(a^b^c);
```

Compilation Result

Save file

ab9-aufgabe-6c.tex

```
\begin{align*}
x_{0}&= 12\\
?x\\
y_{0}&= 11\\
?y\\
z_{0}&= 10\\
?z\\
\end{align*}
```

Extended GAALOP program with intermediate steps (in German: Zwischenschritte) and solutions (in German: Loesungen):

```
a = 3*e1 - 2*e2 + 7*e3;
b = -5*e1 + 5*e2 + e3;
c = 6*e1 + 8*e2 + 9*e3;
r = 41*e1 + 111*e2 + 185*e3;
#(Zwischenschritte);
?VOLUMENabc = a^b^c;
?VOLUMENrbc = r^b^c;
?VOLUMENarc = a^r^c;
?VOLUMENabr = a^b^r;
#(Loesungen);
?x = (r^b^c)/(a^b^c);
?y = (a^r^c)/(a^b^c);
?z = (a^b^r)/(a^b^c);
```

Compilation Result

Save file

ab9-aufgabe-6c-zwischenschritte.tex

```
\begin{align*}
Zwischenschritte\\
VOLUMENabc_{7}&= -481\\
?VOLUMENabc\\
VOLUMENrbc_{7}&= -5772\\
?VOLUMENrbc\\
VOLUMENarc_{7}&= -5291\\
?VOLUMENarc\\
VOLUMENabr_{7}&= -4810\\
?VOLUMENabr\\
Loesungen\\
x_{0}&= 12\\
?x\\
y_{0}&= 11\\
?y\\
z_{0}&= 10\\
?z\\
\end{align*}
```

Intermediate steps:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -481\ \sigma_x\sigma_y\sigma_z$

$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = -5772\ \sigma_x\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = -5291\ \sigma_x\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = -4810\ \sigma_x\sigma_y\sigma_z$

Solution of the system of linear equations:

$$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-5772}{-481} = 12$$

$$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-5291}{-481} = 11$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-4810}{-481} = 10$$

Check: $3 \cdot 12 - 5 \cdot 11 + 6 \cdot 10 = 36 - 55 + 60 = 41$

$-2 \cdot 12 + 5 \cdot 11 + 8 \cdot 10 = -24 + 55 + 80 = 111$

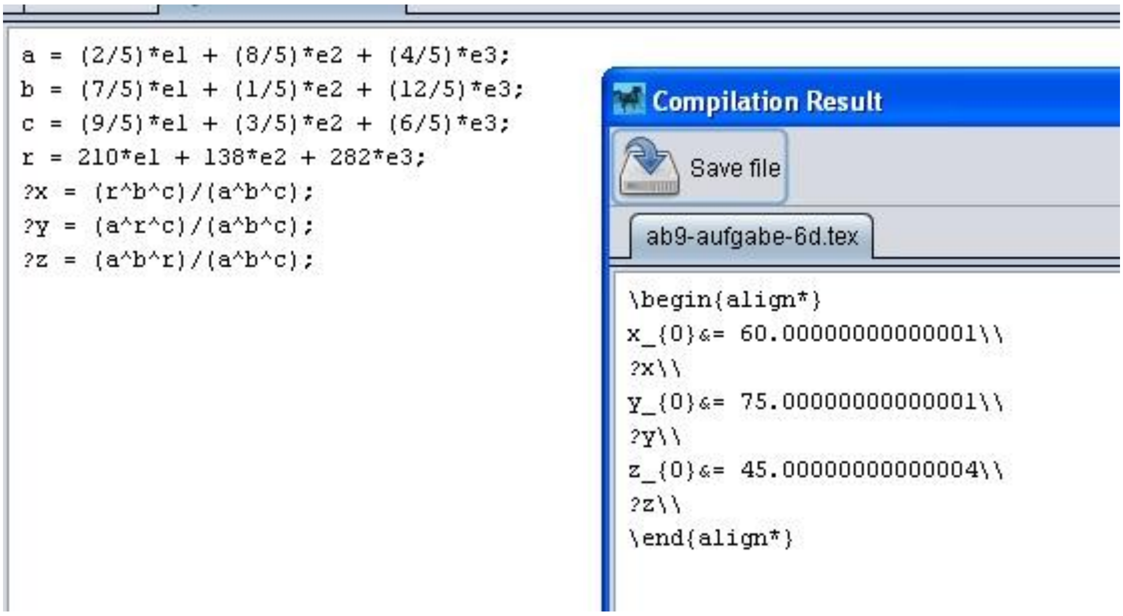
$7 \cdot 12 + 11 + 9 \cdot 10 = 84 + 11 + 90 = 185$

d) $^2/_5\, x + {}^7/_5\, y + {}^9/_5\, z = 210$ $\Rightarrow$ $\mathbf{a} = {}^2/_5\, \sigma_x + {}^8/_5\, \sigma_y + {}^4/_5\, \sigma_z$ $\mathbf{r} = 210\, \sigma_x + 138\, \sigma_y + 282\, \sigma_z$

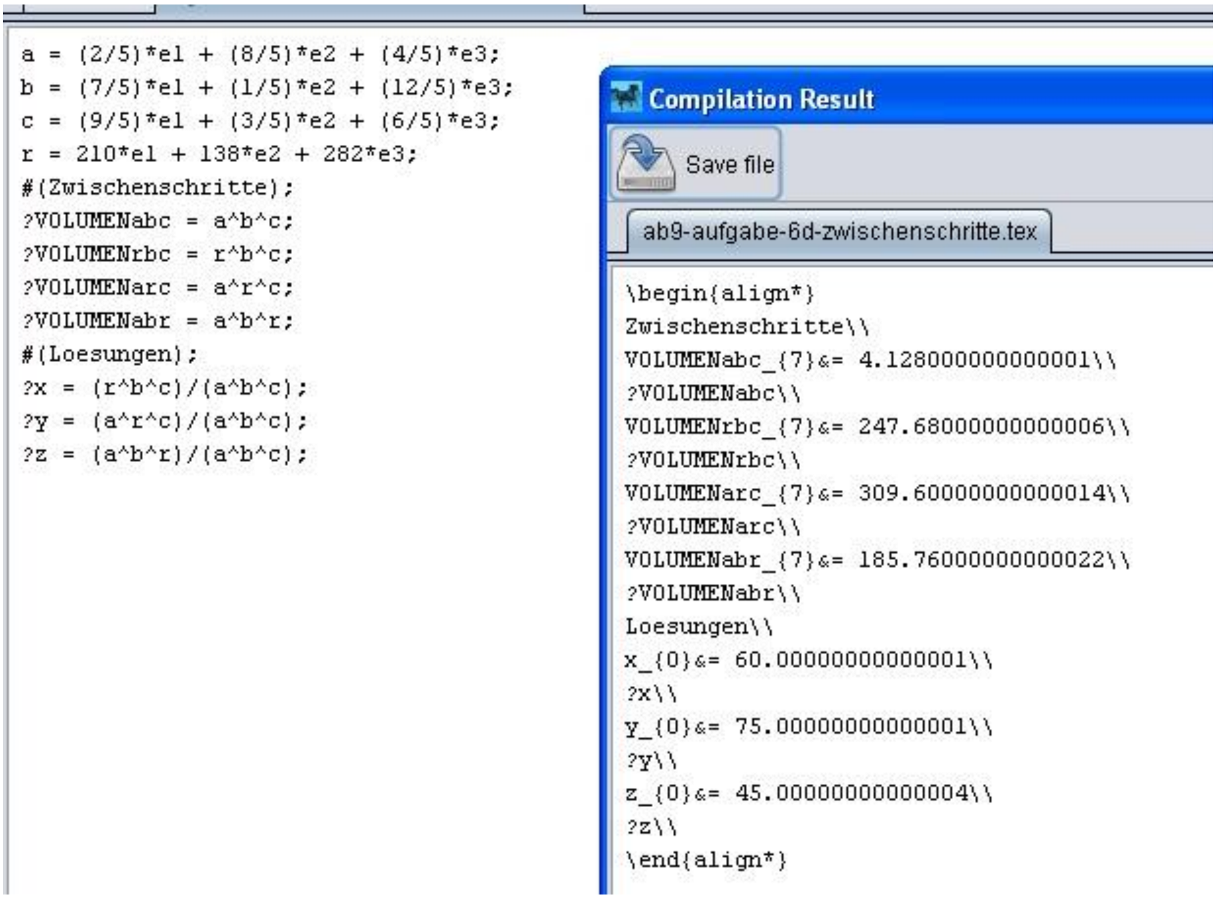
$^8/_5\, x + {}^1/_5\, y + {}^3/_5\, z = 138$ $\mathbf{b} = {}^7/_5\, \sigma_x + {}^1/_5\, \sigma_y + {}^{12}/_5\, \sigma_z$

$^4/_5\, x + {}^{12}/_5\, y + {}^6/_5\, z = 282$ $\mathbf{c} = {}^9/_5\, \sigma_x + {}^3/_5\, \sigma_y + {}^6/_5\, \sigma_z$

Short GAALOP program without intermediate steps:



Extended GAALOP program with intermediate steps (in German: Zwischenschritte) and solutions (in German: Loesungen):



Intermediate steps:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 4.128\ \sigma_x\sigma_y\sigma_z = \frac{516}{125}\ \sigma_x\sigma_y\sigma_z$$

$$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = 247.680\ \sigma_x\sigma_y\sigma_z = \frac{30960}{125}\ \sigma_x\sigma_y\sigma_z$$

$$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = 309.600\ \sigma_x\sigma_y\sigma_z = \frac{38700}{125}\ \sigma_x\sigma_y\sigma_z$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = 185.760\ \sigma_x\sigma_y\sigma_z = \frac{23220}{125}\ \sigma_x\sigma_y\sigma_z$$

Solution of the system of linear equations:

$$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{30960}{516} = 60$$

$$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{38700}{516} = 75$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{23220}{516} = 45$$

Check: $^2/_5 \cdot 60 + {}^7/_5 \cdot 75 + {}^9/_5 \cdot 45 = 24 + 105 + 81 = 210$

$^8/_5 \cdot 60 + {}^1/_5 \cdot 75 + {}^3/_5 \cdot 45 = 96 + 15 + 27 = 138$

$^4/_5 \cdot 60 + {}^{12}/_5 \cdot 75 + {}^6/_5 \cdot 45 = 48 + 180 + 54 = 282$

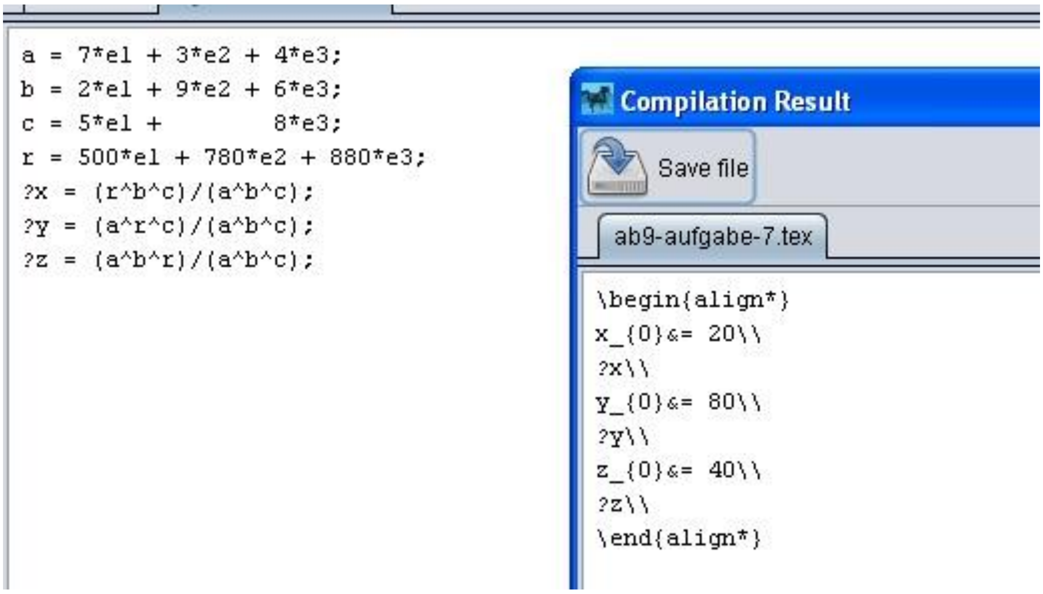
Problem 7:

$7\,x + 2\,y + 5\,z = 500$ $\quad\Rightarrow\quad$ $\mathbf{a} = 7\,\sigma_x + 3\,\sigma_y + 4\,\sigma_z$ $\qquad$ $\mathbf{r} = 500\,\sigma_x + 780\,\sigma_y + 880\,\sigma_z$

$3\,x + 9\,y + \quad = 780$ $\qquad\qquad$ $\mathbf{b} = 2\,\sigma_x + 9\,\sigma_y + 6\,\sigma_z$

$4\,x + 6\,y + 8\,z = 880$ $\qquad\qquad$ $\mathbf{c} = 5\,\sigma_x \quad + 8\,\sigma_z$

GAALOP program and compilation result:



Solution of the system of linear equations:

$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) \,/\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 20$

$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) \,/\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 80$

$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) \,/\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 40$

Check: $7 \cdot 20 + 2 \cdot 80 + 5 \cdot 40 = 140 + 160 + 200 = 500$

$3 \cdot 20 + 9 \cdot 80 = 60 + 720 = 780$

$4 \cdot 20 + 6 \cdot 80 + 8 \cdot 40 = 80 + 480 + 320 = 880$

$\Rightarrow$ If 500 units of the first raw material R_1, 780 units of the second raw material R_2, and 880 units of the third raw material R_3 are consumed in the production process, 20 units of the first final product P_1, 80 units of the second final product P_2, and 40 units of the third final product P_3 will be produced.

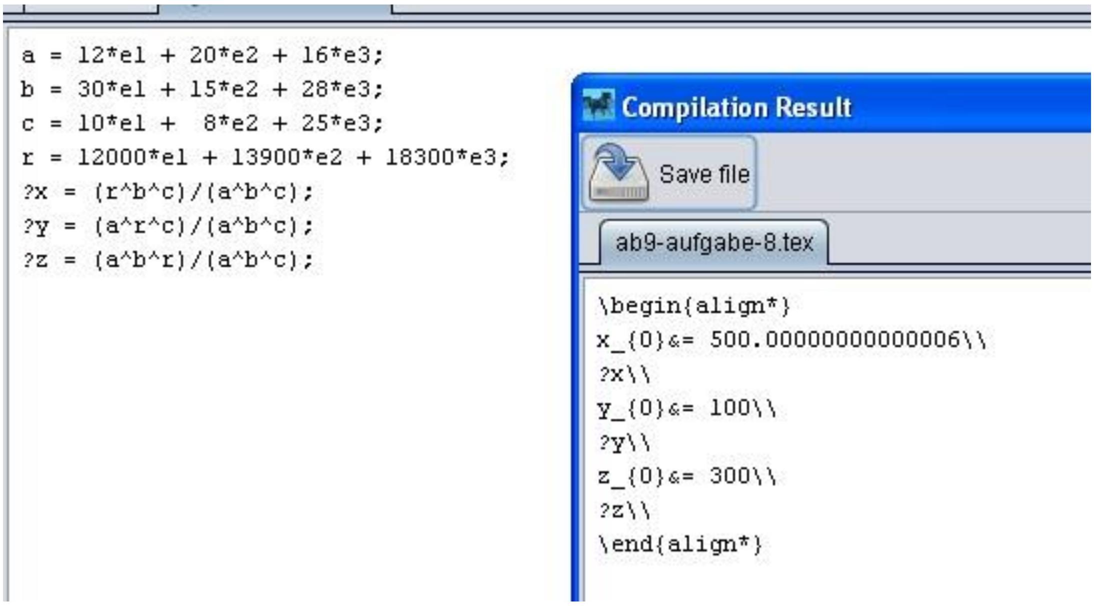
Problem 8:

$12\,x + 30\,y + 10\,z = 12000$ $\quad\Rightarrow\quad$ $\mathbf{a} = 12\,\sigma_x + 20\,\sigma_y + 16\,\sigma_z$ $\qquad$ $\mathbf{r} = 12000\,\sigma_x + 13900\,\sigma_y + 18300\,\sigma_z$

$20\,x + 15\,y + \;\,8\,z = 13900$ $\qquad\qquad$ $\mathbf{b} = 30\,\sigma_x + 15\,\sigma_y + 28\,\sigma_z$

$16\,x + 28\,y + 25\,z = 18300$ $\qquad\qquad$ $\mathbf{c} = 10\,\sigma_x + \;\,8\,\sigma_y + 25\,\sigma_z$

GAALOP program and compilation result:



Solution of the system of linear equations:

$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 500$

$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 100$

$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 300$

Check: $12 \cdot 500 + 30 \cdot 100 + 10 \cdot 300 = 6000 + 3000 + 3000 = 12000$

$20 \cdot 500 + 15 \cdot 100 + 8 \cdot 300 = 10000 + 1500 + 2400 = 13900$

$16 \cdot 500 + 28 \cdot 100 + 25 \cdot 300 = 8000 + 2800 + 7500 = 18300$

$\Rightarrow$ If 12000 units of the first raw material R_1, 13900 units of the second raw material R_2, and 18300 units of the third raw material R_3 are consumed in the production process, 500 units of the first final product P_1, 100 units of the second final product P_2, and 300 units of the third final product P_3 will be produced.

Problem 9:

first quarter second quarter

$$\underbrace{\begin{bmatrix} 9 & 3 & 4 \\ 2 & 2 & 3 \\ 7 & 5 & 2 \end{bmatrix}}\underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}}_{\mathbf{P}} = \underbrace{\begin{bmatrix} 98 & 61 \\ 35 & 30 \\ 76 & 59 \end{bmatrix}}_{\mathbf{R}}$$

R .... matrix of quarterly consumption of raw materials (consumption matrix)

P .... matrix of quarterly production (production matrix)

$\Rightarrow$ Two systems of linear equations:

$9x_1 + 3y_1 + 4z_1 = 98$		$9x_2 + 3y_2 + 4z_2 = 61$
$2x_1 + 2y_1 + 3z_1 = 35$	and	$2x_2 + 2y_2 + 3z_2 = 30$
$7x_1 + 5y_1 + 2z_1 = 76$		$7x_2 + 5y_2 + 2z_2 = 59$

⇒ $\mathbf{a} = 9\,\sigma_x + 2\,\sigma_y + 7\,\sigma_z$ $\quad$ $\mathbf{r} = 98\,\sigma_x + 35\,\sigma_y + 76\,\sigma_z$

$\mathbf{b} = 3\,\sigma_x + 2\,\sigma_y + 2\,\sigma_z$ $\quad$ $\mathbf{q} = 61\,\sigma_x + 30\,\sigma_y + 59\,\sigma_z$

$\mathbf{c} = 4\,\sigma_x + 3\,\sigma_y + 2\,\sigma_z$

GAALOP program and compilation result:

```
a = 9*e1 + 2*e2 + 7*e3;
b = 3*e1 + 2*e2 + 5*e3;
c = 4*e1 + 3*e2 + 2*e3;
r = 98*e1 + 35*e2 + 76*e3;
q = 61*e1 + 30*e2 + 59*e3;
?Xeins = (r^b^c)/(a^b^c);
?Yeins = (a^r^c)/(a^b^c);
?Zeins = (a^b^r)/(a^b^c);
?Xzwei = (q^b^c)/(a^b^c);
?Yzwei = (a^q^c)/(a^b^c);
?Zzwei = (a^b^q)/(a^b^c);
```

Compilation Result

Save file

ab9-aufgabe-9.tex

```
\begin{align*}
Xeins_{0}&= 8\\
?Xeins\\
Yeins_{0}&= 2\\
?Yeins\\
Zeins_{0}&= 5\\
?Zeins\\
Xzwei_{0}&= 3\\
?Xzwei\\
Yzwei_{0}&= 6\\
?Yzwei\\
Zzwei_{0}&= 4\\
?Zzwei\\
\end{align*}
```

Solutions of the two systems of linear equations:

$x_1 = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 8$ $\quad$ $x_2 = (\mathbf{q} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 3$

$y_1 = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 2$ $\quad$ $y_2 = (\mathbf{a} \wedge \mathbf{q} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 6$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 5$ $\quad$ $z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{q}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 4$

Check:				
			8	3
			2	6
			5	4
9	3	4	98	61
2	2	3	35	30
7	5	2	76	59

⇒ 8 units of the first final product P_1, 2 units of the second final product P_2, and 4 units of the third final product P_3 will be produced in the first quarter.

3 units of the first final product P_1, 6 units of the second final product P_2, and 4 units of the third final product P_3 will be produced in the second quarter.

Problem 10:

$$\begin{bmatrix} 10 & 15 & 11 \\ 17 & 20 & 16 \\ 12 & 14 & 25 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} 964 & 814 \\ 1409 & 1184 \\ 1320 & 1093 \end{bmatrix}$$

A B = **D**

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

⇒ Two systems of linear equations:

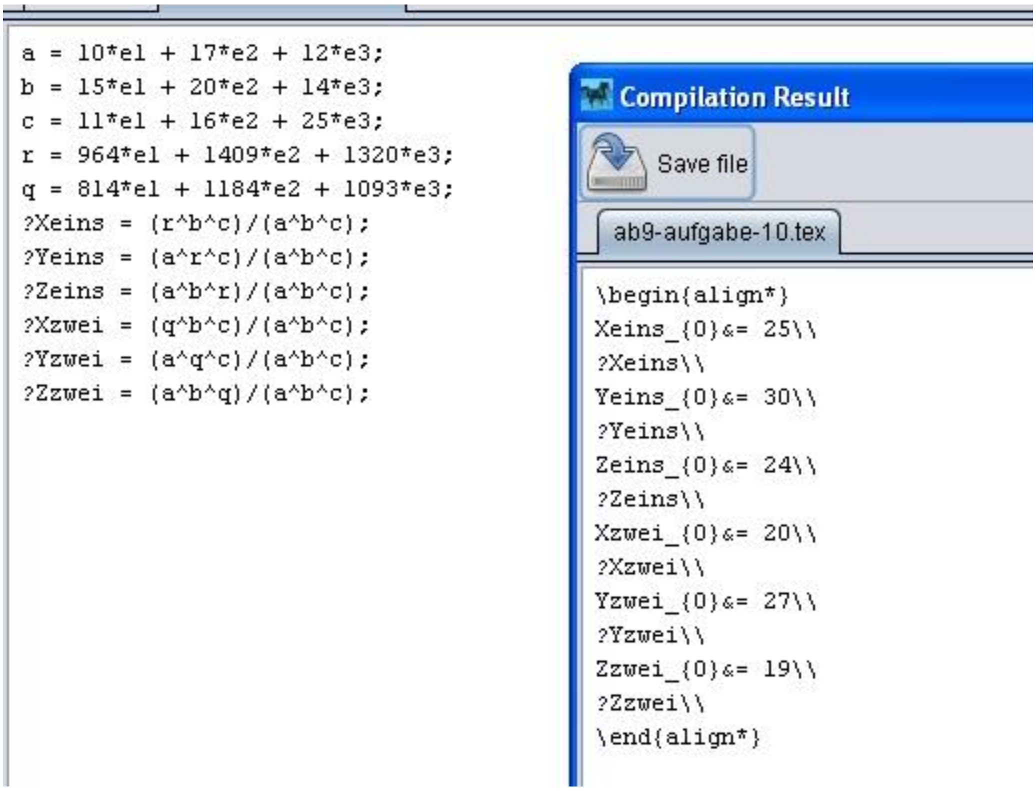
$$\begin{aligned} 10\,x_1 + 15\,y_1 + 11\,z_1 &= 964 \\ 17\,x_1 + 20\,y_1 + 16\,z_1 &= 1409 \\ 12\,x_1 + 14\,y_1 + 25\,z_1 &= 1320 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 10\,x_2 + 15\,y_2 + 11\,z_2 &= 814 \\ 17\,x_2 + 20\,y_2 + 16\,z_2 &= 1184 \\ 12\,x_2 + 14\,y_2 + 25\,z_2 &= 1093 \end{aligned}$$

⇒ $\mathbf{a} = 10\,\sigma_x + 17\,\sigma_y + 12\,\sigma_z$ $\qquad \mathbf{r} = 964\,\sigma_x + 1409\,\sigma_y + 1320\,\sigma_z$

$\mathbf{b} = 15\,\sigma_x + 20\,\sigma_y + 14\,\sigma_z$ $\qquad \mathbf{q} = 814\,\sigma_x + 1184\,\sigma_y + 1093\,\sigma_z$

$\mathbf{c} = 11\,\sigma_x + 16\,\sigma_y + 25\,\sigma_z$

GAALOP program and compilation result:



Solutions of the two systems of linear equations:

$x_1 = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 25$ $\qquad x_2 = (\mathbf{q} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 20$

$y_1 = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 30$ $\qquad y_2 = (\mathbf{a} \wedge \mathbf{q} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 27$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 24$ $\qquad z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{q}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 19$

Check:

			25	20
			30	27
			24	19
10	15	11	964	814
17	20	16	1409	1184
12	14	25	1320	1093

$\Rightarrow$ Demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 25 & 20 \\ 30 & 27 \\ 24 & 19 \end{bmatrix}$

Problem 11:

$$\underbrace{\begin{bmatrix} 8 & 6 & 6 \\ 7 & 5 & 7 \\ 5 & 4 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 228 & 186 & 308 \\ 186 & 166 & 282 \\ 108 & 107 & 160 \end{bmatrix}}_{\mathbf{D}}$$

A B = **D**

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$\Rightarrow$ Three systems of linear equations:

$8\,x_1 + 6\,y_1 + 6\,z_1 = 228$ $\quad$ $8\,x_2 + 6\,y_2 + 6\,z_2 = 308$ $\quad$ $8\,x_3 + 6\,y_3 + 6\,z_3 = 308$

$7\,x_1 + 5\,y_1 + 7\,z_1 = 186$ $\quad$ and $\quad$ $7\,x_2 + 5\,y_2 + 7\,z_2 = 166$ $\quad$ and $\quad$ $7\,x_3 + 5\,y_3 + 7\,z_3 = 282$

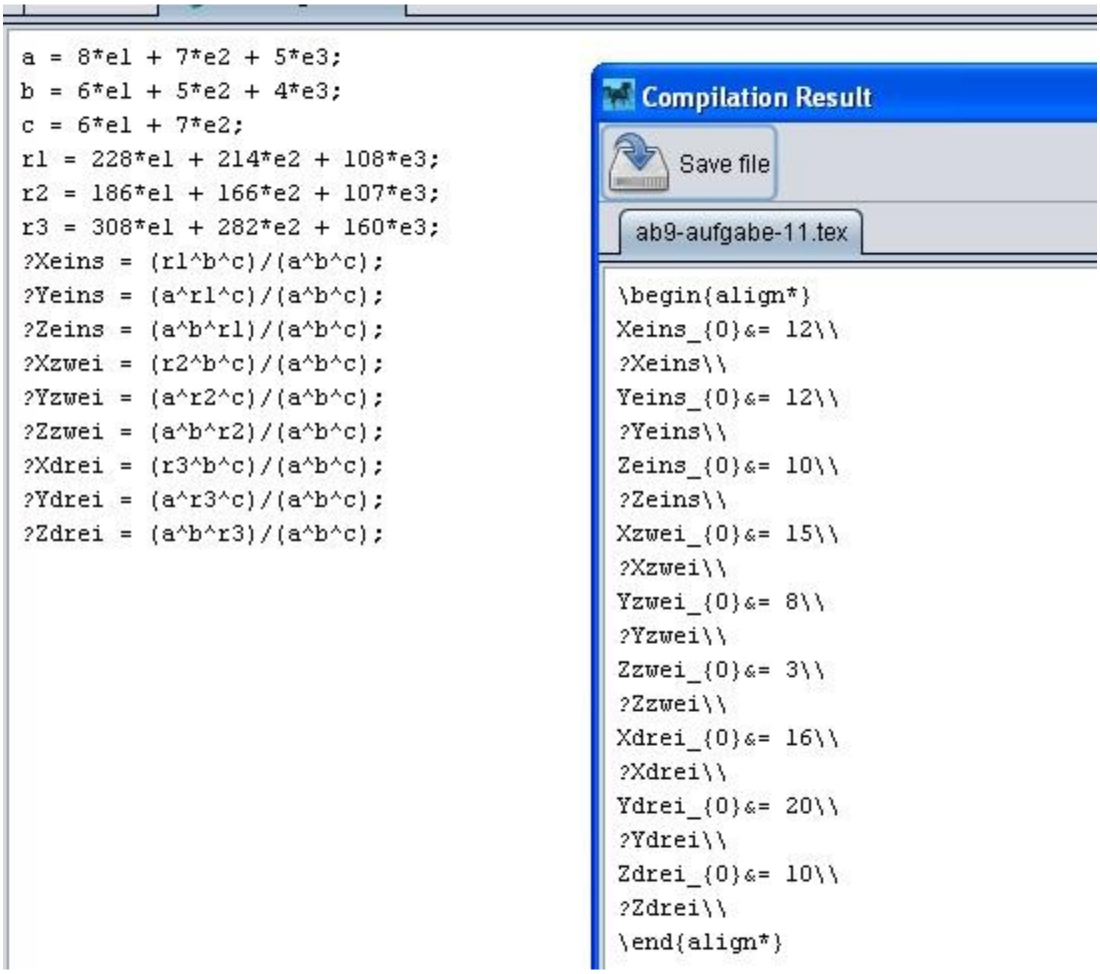
$5\,x_1 + 4\,y_1 = 108$ $\quad$ $5\,x_2 + 4\,y_2 = 107$ $\quad$ $5\,x_3 + 4\,y_3 = 160$

$\Rightarrow$ $\mathbf{a} = 8\,\sigma_x + 7\,\sigma_y + 5\,\sigma_z$ $\qquad$ $\mathbf{r_1} = 228\,\sigma_x + 186\,\sigma_y + 108\,\sigma_z$

$\mathbf{b} = 6\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z$ $\qquad$ $\mathbf{r_2} = 186\,\sigma_x + 166\,\sigma_y + 107\,\sigma_z$

$\mathbf{c} = 6\,\sigma_x + 7\,\sigma_y$ $\qquad$ $\mathbf{r_3} = 308\,\sigma_x + 282\,\sigma_y + 160\,\sigma_z$

GAALOP program and compilation result:



Solutions of the three systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 12$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 12$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 10$

$x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 15$

$y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 8$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 3$

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 16$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 20$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 10$

Check:					
			12	15	16
			12	8	20
			10	3	10
8	6	6	228	186	308
7	5	7	214	166	282
5	4	0	108	107	160

$\Rightarrow$ Demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 12 & 15 & 16 \\ 12 & 8 & 20 \\ 10 & 3 & 10 \end{bmatrix}$

Problem 12:

$$\begin{bmatrix} 82 & 63 & 20 \\ 44 & 19 & 37 \\ 10 & 52 & 92 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 4496 & 5462 & 4815 \\ 2530 & 3482 & 2801 \\ 3224 & 4062 & 4646 \end{bmatrix}$$

A B = **D**

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$\Rightarrow$ Three systems of linear equations:

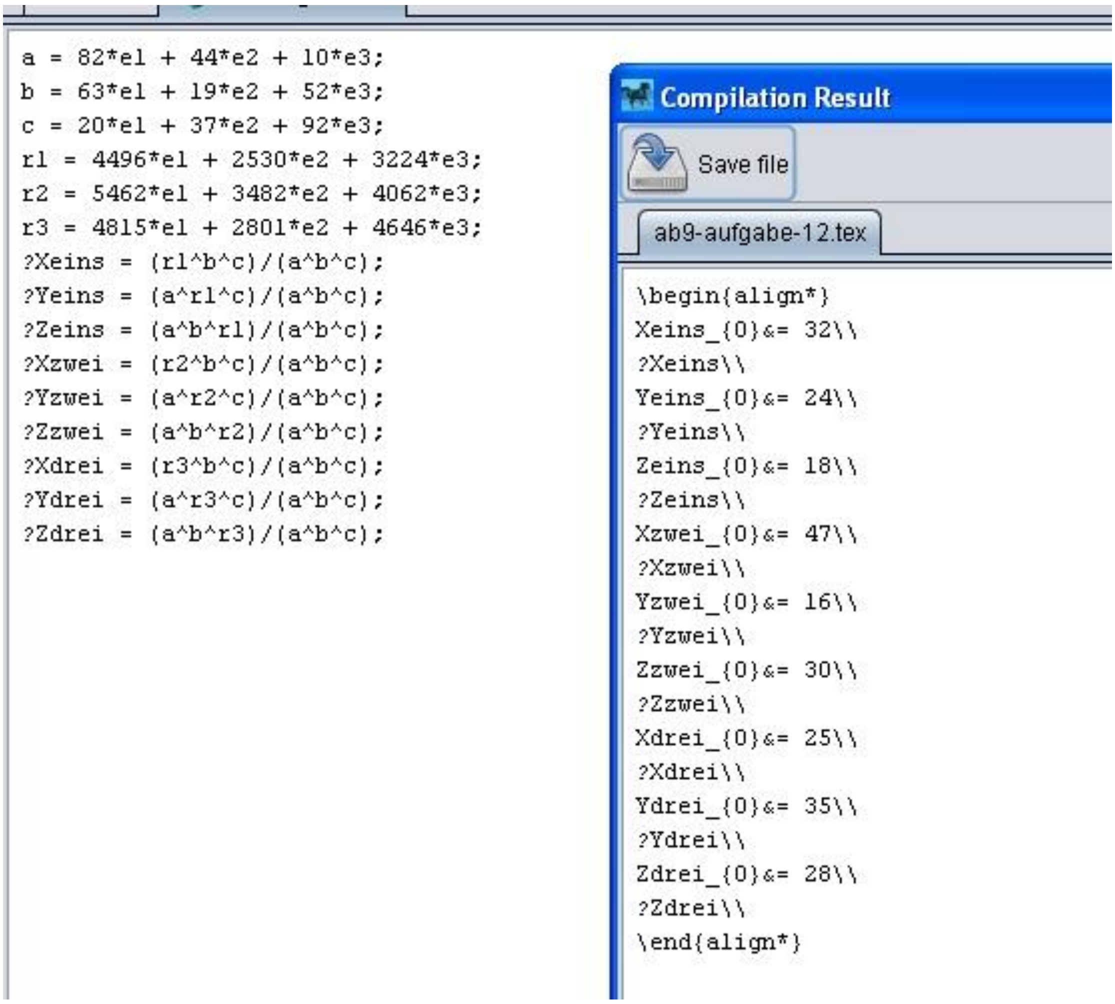
$$\begin{aligned} 82\,x_1 + 63\,y_1 + 20\,z_1 &= 4496 \\ 44\,x_1 + 19\,y_1 + 37\,z_1 &= 2530 \\ 10\,x_1 + 52\,y_1 + 92\,z_1 &= 3224 \end{aligned} \quad \text{and} \quad \begin{aligned} 82\,x_2 + 63\,y_2 + 20\,z_2 &= 5462 \\ 44\,x_2 + 19\,y_2 + 37\,z_2 &= 3482 \\ 10\,x_2 + 52\,y_2 + 92\,z_2 &= 4062 \end{aligned} \quad \text{and} \quad \begin{aligned} 82\,x_3 + 63\,y_3 + 20\,z_3 &= 4815 \\ 44\,x_3 + 19\,y_3 + 37\,z_3 &= 2801 \\ 10\,x_3 + 52\,y_3 + 92\,z_3 &= 4646 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 82\,\sigma_x + 44\,\sigma_y + 10\,\sigma_z$ $\qquad$ $\mathbf{r_1} = 4496\,\sigma_x + 2530\,\sigma_y + 3224\,\sigma_z$

$\mathbf{b} = 63\,\sigma_x + 19\,\sigma_y + 52\,\sigma_z$ $\qquad$ $\mathbf{r_2} = 5462\,\sigma_x + 3482\,\sigma_y + 4062\,\sigma_z$

$\mathbf{c} = 20\,\sigma_x + 37\,\sigma_y + 92\,\sigma_z$ $\qquad$ $\mathbf{r_3} = 4815\,\sigma_x + 2801\,\sigma_y + 4646\,\sigma_z$

GAALOP program and compilation result:



Solutions of the three systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 32$ $\quad$ $x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 47$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 24$ $\quad$ $y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 16$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 18$ $\quad$ $z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 30$

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 25$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 35$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 28$

Check:					
			32	47	25
			24	16	35
			18	30	28
82	63	20	4496	5462	4815
44	19	37	2530	3482	2801
10	52	92	3224	4062	4646

$\Rightarrow$ Demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 32 & 47 & 25 \\ 24 & 16 & 35 \\ 18 & 30 & 28 \end{bmatrix}$

Problem 13:

$$\underbrace{\begin{bmatrix} 3 & 5 & 4 \\ 2 & 6 & 3 \\ 8 & 7 & 10 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{A}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}} \qquad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

I .... identity matrix

$\Rightarrow$ Three systems of linear equations:

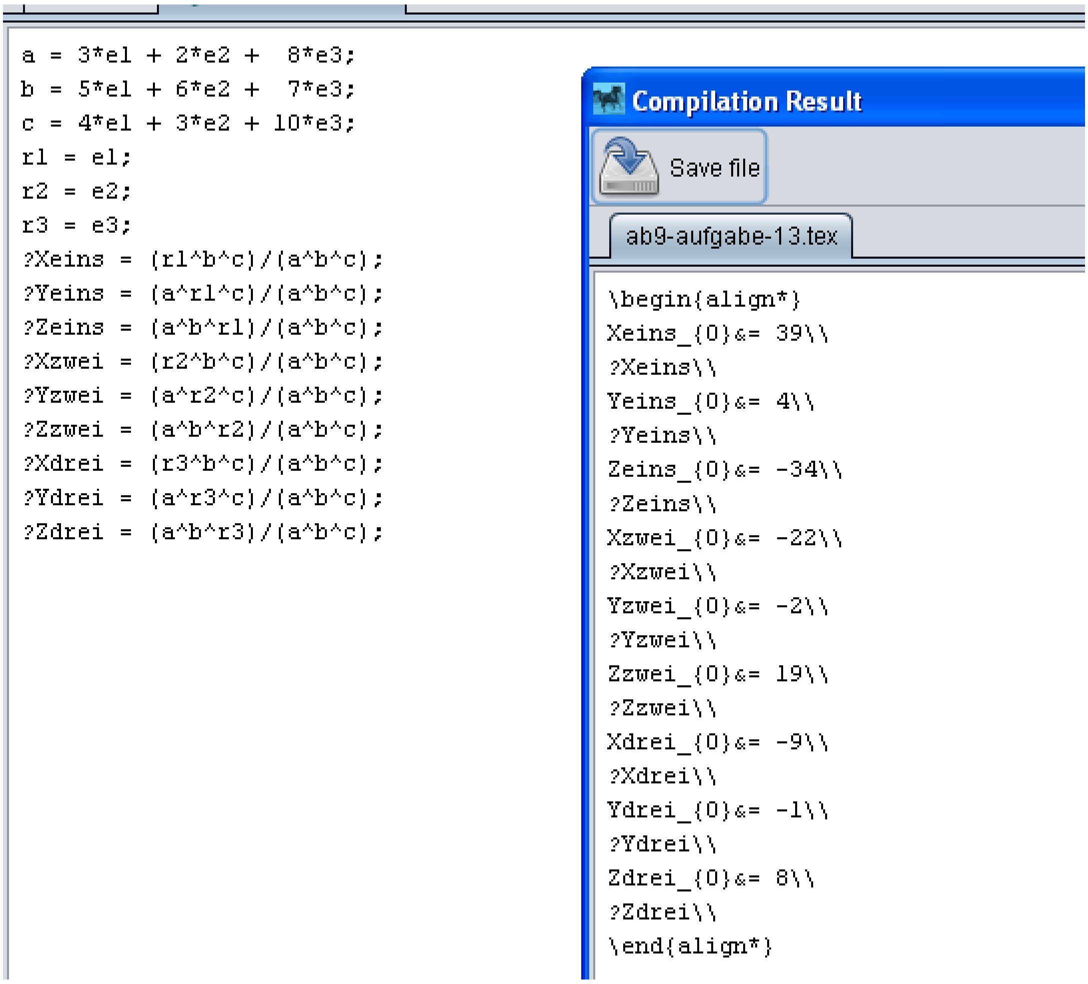
$$\begin{aligned} 3x_1 + 5y_1 + 4z_1 &= 1 \\ 2x_1 + 6y_1 + 3z_1 &= 0 \\ 8x_1 + 7y_1 + 10z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 3x_2 + 5y_2 + 4z_2 &= 0 \\ 2x_2 + 6y_2 + 3z_2 &= 1 \\ 8x_2 + 7y_2 + 10z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 3x_3 + 5y_3 + 4z_3 &= 0 \\ 2x_3 + 6y_3 + 3z_3 &= 0 \\ 8x_3 + 7y_3 + 10z_3 &= 1 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 3\sigma_x + 2\sigma_y + 8\sigma_z$ $\quad$ $\mathbf{r_1} = \sigma_x$

$\mathbf{b} = 5\sigma_x + 6\sigma_y + 7\sigma_z$ $\quad$ $\mathbf{r_2} = \sigma_y$

$\mathbf{c} = 4\sigma_x + 3\sigma_y + 10\sigma_z$ $\quad$ $\mathbf{r_3} = \sigma_z$

GAALOP program and compilation result:



Solution of the first system of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 39$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 4$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -34$

$\Rightarrow$ If exactly one unit of the first raw material R_1 had been consumed in the production process, 39 units of the first final product P_1 and 4 units of the second final product P_2 would have been produced and additionally 34 units of the third final product P_3 would have been consumed (and split again completely into the raw materials).

Or more realistic:

If it just happened that one **more** unit of the first raw material R_1 had been delivered accidentally and has had to be consumed in addition in the production process, the output of the first final product P_1 would have been increased by 39 units, the output of the second final product P_2 would have been increased by 4 units, and the output of the third final product P_3 would have been reduced by 34 units.

(You can imagine the economy of the GDR as it existed in reality somehow working in that way, as production plans didn't depend on the demand of customers, but they depended strongly on the erratic supply of raw materials.)

Solution of the second system of linear equations:

$x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -22$

$y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -2$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 19$

⇒ If exactly one unit of the second raw material R_2 had been consumed in the production process, 19 units of the third final product P_3 would have been produced and additionally 22 units of the first final product P_1 and 2 units of the second final product P_2 would have been consumed (and split again completely into the raw materials).

Or more realistic:

If it just happened that one **more** unit of the second raw material R_2 had been delivered accidentally and has had to be consumed in addition in the production process, the output of the first final product P_1 would have been reduced by 22 units, the output of the second final product P_2 would have been reduced by 2 units, and the output of the third final product P_3 would have been increased by 19 units.

Solution of the third system of linear equations:

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -9$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -1$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 8$

⇒ If exactly one unit of the third raw material R_3 had been consumed in the production process, 8 units of the third final product P_3 would have been produced and additionally 9 units of the first final product P_1 and one unit of the second final product P_2 would have been consumed (and split again completely into the raw materials).

Or more realistic:

If it just happened that one **more** unit of the third raw material R_3 had been delivered accidentally and has had to be consumed in addition in the production process, the output of the first final product P_1 would have been reduced by 9 units, the output of the second final product P_2 would have been reduced by one unit, and the output of the third final product P_3 would have been increased by 8 units.

Check:			39	−22	−9
			4	−2	−1
			−34	19	8
3	5	4	1	0	0
2	6	3	0	1	0
8	7	10	0	0	1

⇒ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 39 & -22 & -9 \\ 4 & -2 & -1 \\ -34 & 19 & 8 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 6 & 3 \\ 8 & 7 & 10 \end{bmatrix}$.

Problem 14:

a) $$\underbrace{\begin{bmatrix} 1 & 4 & 9 \\ 7 & 2 & 6 \\ 6 & 3 & 8 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{A}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}} \qquad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

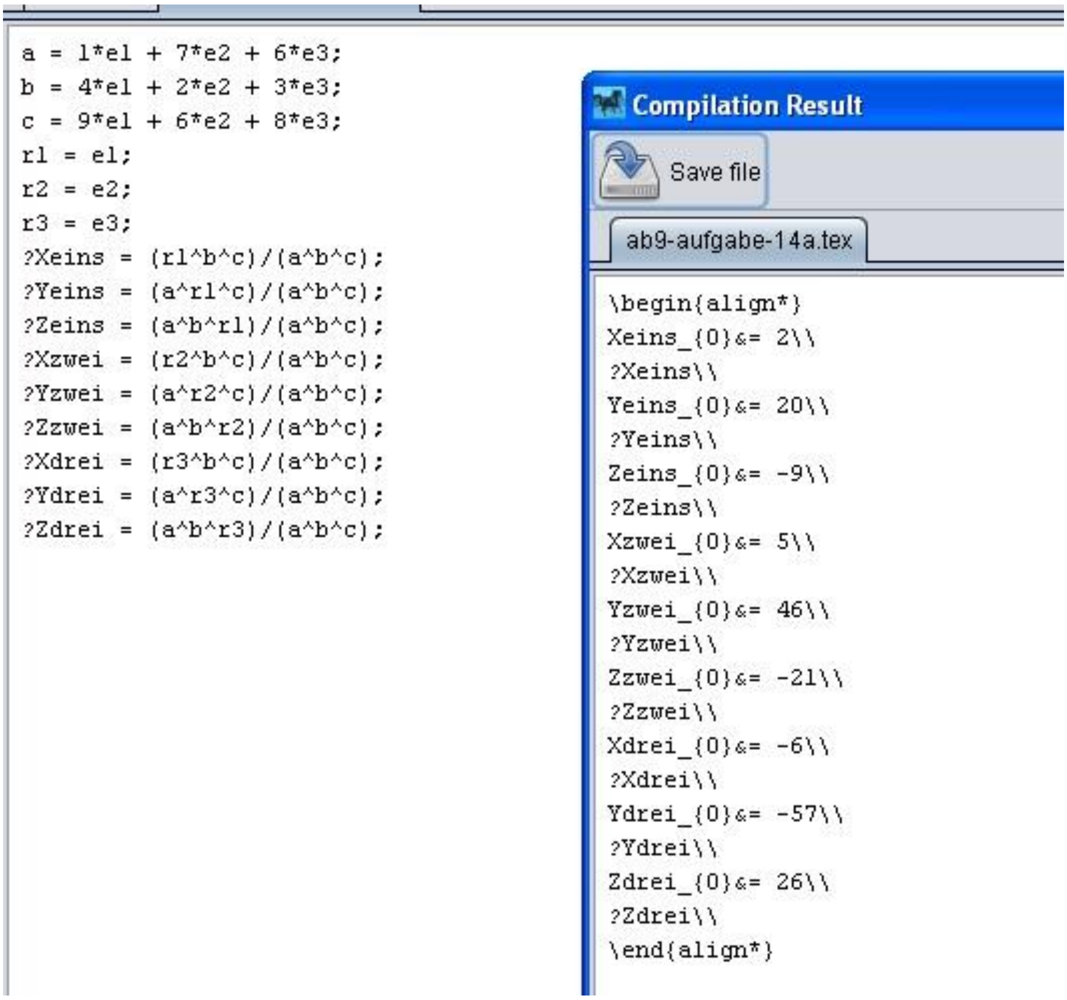
I .... identity matrix

$\Rightarrow$ Three systems of linear equations:

$$\begin{aligned} x_1 + 4\,y_1 + 9\,z_1 &= 1 \\ 7\,x_1 + 2\,y_1 + 6\,z_1 &= 0 \\ 6\,x_1 + 3\,y_1 + 8\,z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_2 + 4\,y_2 + 9\,z_2 &= 0 \\ 7\,x_2 + 2\,y_2 + 6\,z_2 &= 1 \\ 6\,x_2 + 3\,y_2 + 8\,z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_3 + 4\,y_3 + 9\,z_3 &= 0 \\ 7\,x_3 + 2\,y_3 + 6\,z_3 &= 0 \\ 6\,x_3 + 3\,y_3 + 8\,z_3 &= 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} \mathbf{a} &= \sigma_x + 7\,\sigma_y + 6\,\sigma_z \\ \mathbf{b} &= 4\,\sigma_x + 2\,\sigma_y + 3\,\sigma_z \\ \mathbf{c} &= 9\,\sigma_x + 6\,\sigma_y + 8\,\sigma_z \end{aligned} \qquad \begin{aligned} \mathbf{r_1} &= \sigma_x \\ \mathbf{r_2} &= \sigma_y \\ \mathbf{r_3} &= \sigma_z \end{aligned}$$

GAALOP program and compilation result:



```
a = 1*e1 + 7*e2 + 6*e3;
b = 4*e1 + 2*e2 + 3*e3;
c = 9*e1 + 6*e2 + 8*e3;
r1 = e1;
r2 = e2;
r3 = e3;
?Xeins = (r1^b^c)/(a^b^c);
?Yeins = (a^r1^c)/(a^b^c);
?Zeins = (a^b^r1)/(a^b^c);
?Xzwei = (r2^b^c)/(a^b^c);
?Yzwei = (a^r2^c)/(a^b^c);
?Zzwei = (a^b^r2)/(a^b^c);
?Xdrei = (r3^b^c)/(a^b^c);
?Ydrei = (a^r3^c)/(a^b^c);
?Zdrei = (a^b^r3)/(a^b^c);
```



```
\begin{align*}
Xeins_{0}&= 2\\
?Xeins\\
Yeins_{0}&= 20\\
?Yeins\\
Zeins_{0}&= -9\\
?Zeins\\
Xzwei_{0}&= 5\\
?Xzwei\\
Yzwei_{0}&= 46\\
?Yzwei\\
Zzwei_{0}&= -21\\
?Zzwei\\
Xdrei_{0}&= -6\\
?Xdrei\\
Ydrei_{0}&= -57\\
?Ydrei\\
Zdrei_{0}&= 26\\
?Zdrei\\
\end{align*}
```

Solution of the three systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 2$ $\quad x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 5$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 20$ $\quad y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 46$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -9$ $\quad z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -21$

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -6$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -57$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 26$

Check:					
			2	5	– 6
			20	46	– 57
			– 9	– 21	26
1	4	9	1	0	0
7	2	6	0	1	0
6	3	8	0	0	1

$\Rightarrow$ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 5 & -6 \\ 20 & 46 & -57 \\ -9 & -21 & 26 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 & 9 \\ 7 & 2 & 6 \\ 6 & 3 & 8 \end{bmatrix}$.

b) $$\underbrace{\begin{bmatrix} 0 & 4 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{B}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}\text{ .... identity matrix}} \qquad \mathbf{B}\,\mathbf{B}^{-1} = \mathbf{I}$$

$\Rightarrow$ Three systems of linear equations:

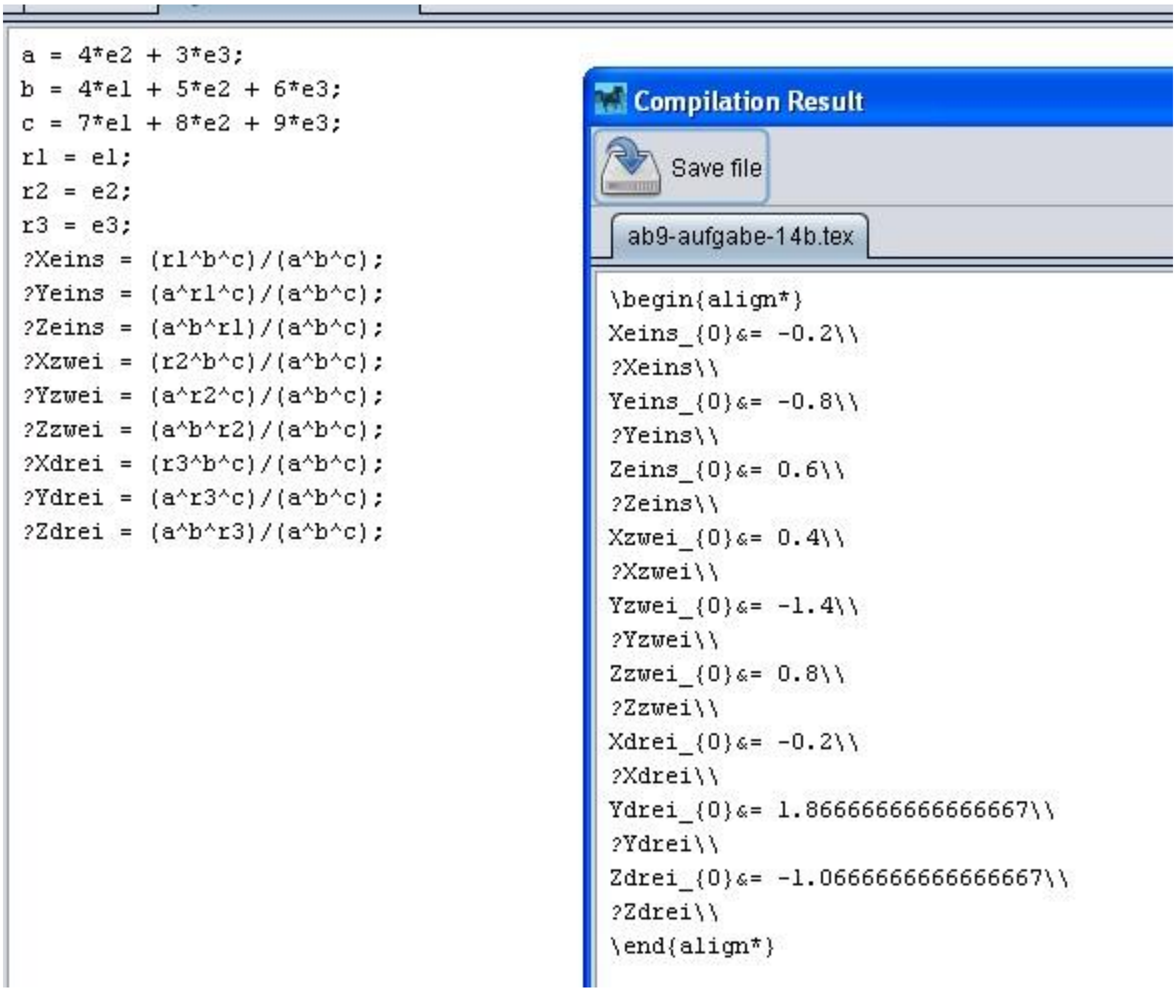
$$\begin{aligned} 4y_1 + 7z_1 &= 1 \\ 4x_1 + 5y_1 + 8z_1 &= 0 \\ 3x_1 + 6y_1 + 9z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 4y_2 + 7z_2 &= 0 \\ 4x_2 + 5y_2 + 8z_2 &= 1 \\ 3x_2 + 6y_2 + 9z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 4y_3 + 7z_3 &= 0 \\ 4x_3 + 5y_3 + 8z_3 &= 0 \\ 3x_3 + 6y_3 + 9z_3 &= 1 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 4\sigma_y + 3\sigma_z$ $\quad \mathbf{r_1} = \sigma_x$

$\mathbf{b} = 4\sigma_x + 5\sigma_y + 6\sigma_z$ $\quad \mathbf{r_2} = \sigma_y$

$\mathbf{c} = 7\sigma_x + 8\sigma_y + 9\sigma_z$ $\quad \mathbf{r_3} = \sigma_z$

GAALOP program and compilation result:




```
a = 4*e2 + 3*e3;
b = 4*e1 + 5*e2 + 6*e3;
c = 7*e1 + 8*e2 + 9*e3;
r1 = e1;
r2 = e2;
r3 = e3;
?Xeins = (r1^b^c)/(a^b^c);
?Yeins = (a^r1^c)/(a^b^c);
?Zeins = (a^b^r1)/(a^b^c);
?Xzwei = (r2^b^c)/(a^b^c);
?Yzwei = (a^r2^c)/(a^b^c);
?Zzwei = (a^b^r2)/(a^b^c);
?Xdrei = (r3^b^c)/(a^b^c);
?Ydrei = (a^r3^c)/(a^b^c);
?Zdrei = (a^b^r3)/(a^b^c);
```

```
\begin{align*}
Xeins_{0}&= -0.2\\
?Xeins\\
Yeins_{0}&= -0.8\\
?Yeins\\
Zeins_{0}&= 0.6\\
?Zeins\\
Xzwei_{0}&= 0.4\\
?Xzwei\\
Yzwei_{0}&= -1.4\\
?Yzwei\\
Zzwei_{0}&= 0.8\\
?Zzwei\\
Xdrei_{0}&= -0.2\\
?Xdrei\\
Ydrei_{0}&= 1.8666666666666667\\
?Ydrei\\
Zdrei_{0}&= -1.0666666666666667\\
?Zdrei\\
\end{align*}
```

Solution of the three systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.2 = -\frac{1}{5}$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.8 = -\frac{4}{5}$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0.6 = \frac{3}{5}$

$x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0.4 = \frac{2}{5}$

$y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -1.4 = -\frac{7}{5}$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0.8 = \frac{4}{5}$

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.2 = -\frac{1}{5}$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 1.8\overline{6}.. = \frac{28}{15}$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -1.0\overline{6}.. = -\frac{16}{15}$

Check:

			– 0.2	0.4	– 0.2
			– 0.8	– 1.4	$1.8\overline{6}..$
			0.6	0.8	$-1.0\overline{6}..$
0	4	7	1	0	0
4	5	8	0	1	0
3	6	9	0	0	1

Alternative check:

			– 3	6	– 3
			– 12	– 21	28
			9	12	– 16
0	4	7	15	0	0
4	5	8	0	15	0
3	6	9	0	0	15

⇒ The resulting matrix $\mathbf{B}^{-1} = \begin{bmatrix} -0.2 & 0.4 & -0.2 \\ -0.8 & -1.4 & 1.8\overline{6}.. \\ 0.6 & 0.8 & -1.0\overline{6}.. \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 & 6 & -3 \\ -12 & -21 & 28 \\ 9 & 12 & -16 \end{bmatrix}$ is

the inverse of matrix $\mathbf{B} = \begin{bmatrix} 0 & 4 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

c) $$\underbrace{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{C}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}}$$ $\mathbf{C}\,\mathbf{C}^{-1} = \mathbf{I}$

I .... identity matrix

⇒ Three systems of linear equations:

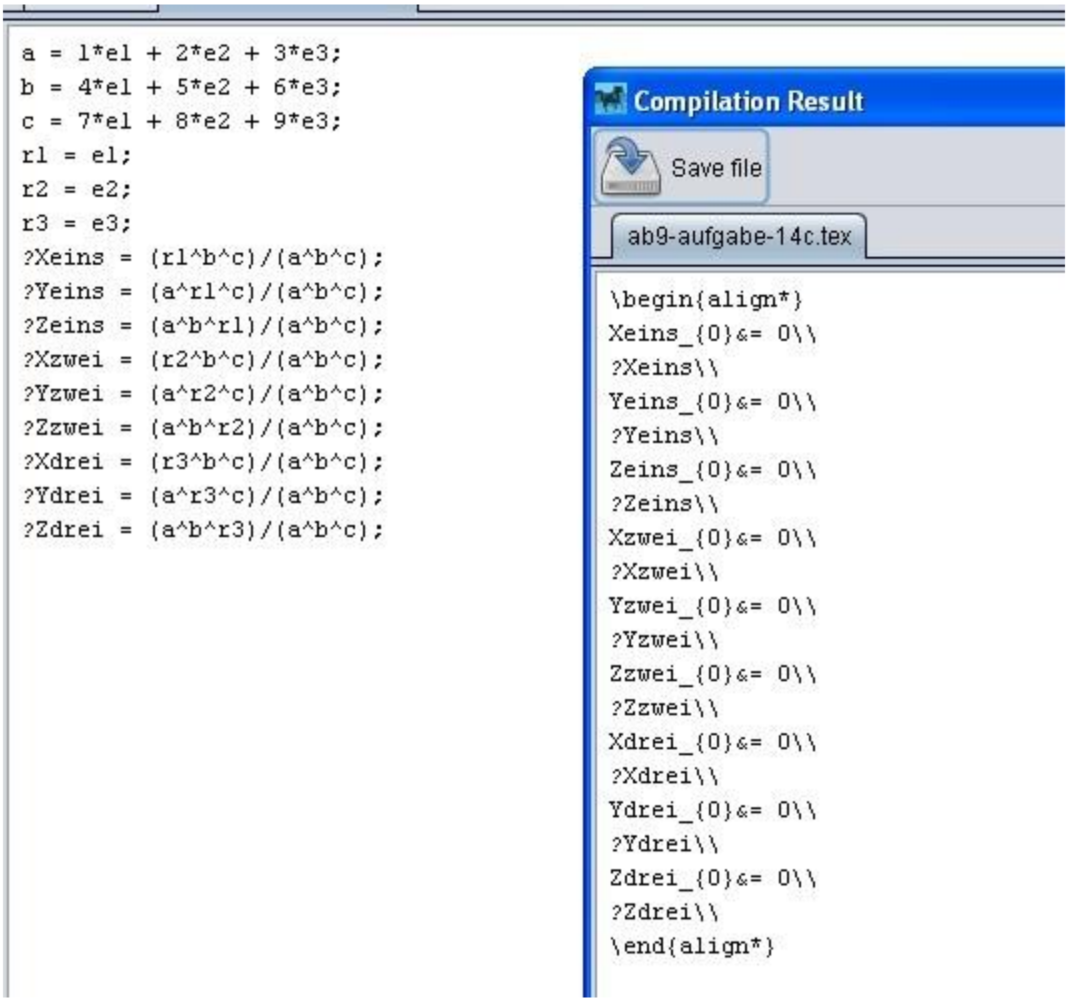
$$\begin{aligned} x_1 + 4\,y_1 + 7\,z_1 &= 1 \\ 2\,x_1 + 5\,y_1 + 8\,z_1 &= 0 \\ 3\,x_1 + 6\,y_1 + 9\,z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_2 + 4\,y_2 + 7\,z_2 &= 0 \\ 2\,x_2 + 5\,y_2 + 8\,z_2 &= 1 \\ 3\,x_2 + 6\,y_2 + 9\,z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_3 + 4\,y_3 + 7\,z_3 &= 0 \\ 2\,x_3 + 5\,y_3 + 8\,z_3 &= 0 \\ 3\,x_3 + 6\,y_3 + 9\,z_3 &= 1 \end{aligned}$$

⇒ $\mathbf{a} = \sigma_x + 2\,\sigma_y + 3\,\sigma_z$ $\quad \mathbf{r_1} = \sigma_x$

$\mathbf{b} = 4\,\sigma_x + 5\,\sigma_y + 6\,\sigma_z$ $\quad \mathbf{r_2} = \sigma_y$

$\mathbf{c} = 7\,\sigma_x + 8\,\sigma_y + 9\,\sigma_z$ $\quad \mathbf{r_3} = \sigma_z$

GAALOP program and compilation result:



Supposed solution of the three systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

Check:					
			0	0	0
			0	0	0
			0	0	0
1	4	7	0	0	0
2	5	8	0	0	0
3	6	9	0	0	0

This zero matrix is not equal to the identity matrix **I** !

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ The check shows that the solutions found with GAALOP are wrong.

The reason for that can be found at the end of page one of the problem sheet (i.e. at the end of page 10 of this collection of worksheets), because …

> **… you will now find problems about systems of three linear equations. To solve these problems the mathematics of vectors, which point into three directions and which are situated in three-dimensional space, is required. Thus vectors will now have three components, representing x, y, and z directions.**

The three coefficient vectors do not form a basis of three-dimensional space, as they are linearly dependent. They are all situated in a plane, and every coefficient vector can be written as linear combination of the other two coefficient vectors:

$$\mathbf{a} = 2\,\mathbf{b} - \mathbf{c}$$

$$\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{a})$$

$$\mathbf{c} = 2\,\mathbf{b} - \mathbf{a}$$

Therefore every possible resulting vector $\mathbf{r}$ must be situated in the plane, which is formed by the coefficient vectors, to make sure that meaningful solution of the unknown variables x, y, and z can be found. Unfortunately, the three resulting vectors of this problem $\mathbf{r_1} = \sigma_x$, $\mathbf{r_2} = \sigma_y$, and $\mathbf{r_3} = \sigma_z$ do not point into the direction of this plane. Therefore (real number) solution values for x, y, and z do not exist.

This geometric explanation can also be interpreted algebraically, as the outer product of the three coefficient vectors $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is equal to zero:

$$\mathbf{a}\,\mathbf{b} = (\sigma_x + 2\,\sigma_y + 3\,\sigma_z)\,(4\,\sigma_x + 5\,\sigma_y + 6\,\sigma_z) = 32 - 3\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z + 6\,\sigma_z\sigma_x$$

$$\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} = -\,3\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z + 6\,\sigma_z\sigma_x$$

$$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (32 - 3\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z + 6\,\sigma_z\sigma_x)\,(7\,\sigma_x + 8\,\sigma_y + 9\,\sigma_z)$$
$$= 146\,\sigma_x + 250\,\sigma_y + 354\,\sigma_z + 0\,\sigma_x\sigma_y\sigma_z$$

$$\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0$$

Check: $(1^2 + 2^2 + 3^2)\,(4^2 + 5^2 + 6^2)\,(7^2 + 8^2 + 9^2) = (146^2 + 250^2 + 354^2) = 209\,132$

A short look at the solution formulas

$$x_i = (\mathbf{r_i} \wedge \mathbf{b} \wedge \mathbf{c}) \,/\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$
$$y_i = (\mathbf{a} \wedge \mathbf{r_i} \wedge \mathbf{c}) \,/\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$
$$z_i = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_i}) \,/\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$

shows that the numerators must be divided by the outer product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ (or in other words: the numerators must be divided by the determinant of matrix $\mathbf{C}$: $\det \mathbf{C} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}\ \sigma_z\sigma_y\sigma_x$).

As this outer product of coefficient vectors is zero, a division is not possible and solution values do not exist. Therefore the compilation result of GAALOP must be wrong, and the conclusion is:

$\Rightarrow$ Problem 14 c) is insoluble.

$\Rightarrow$ The inverse $\mathbf{C}^{-1}$ is not defined.

$\Rightarrow$ An inverse of matrix $\mathbf{C} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ does not exist.

d) $\underbrace{\begin{bmatrix} 3 & 4 & 8 \\ 10 & 5 & 10 \\ 10 & 20 & 15 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{D}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}}$ $\qquad \mathbf{D}\,\mathbf{D}^{-1} = \mathbf{I}$

I .... identity matrix

$\Rightarrow$ Three systems of linear equations:

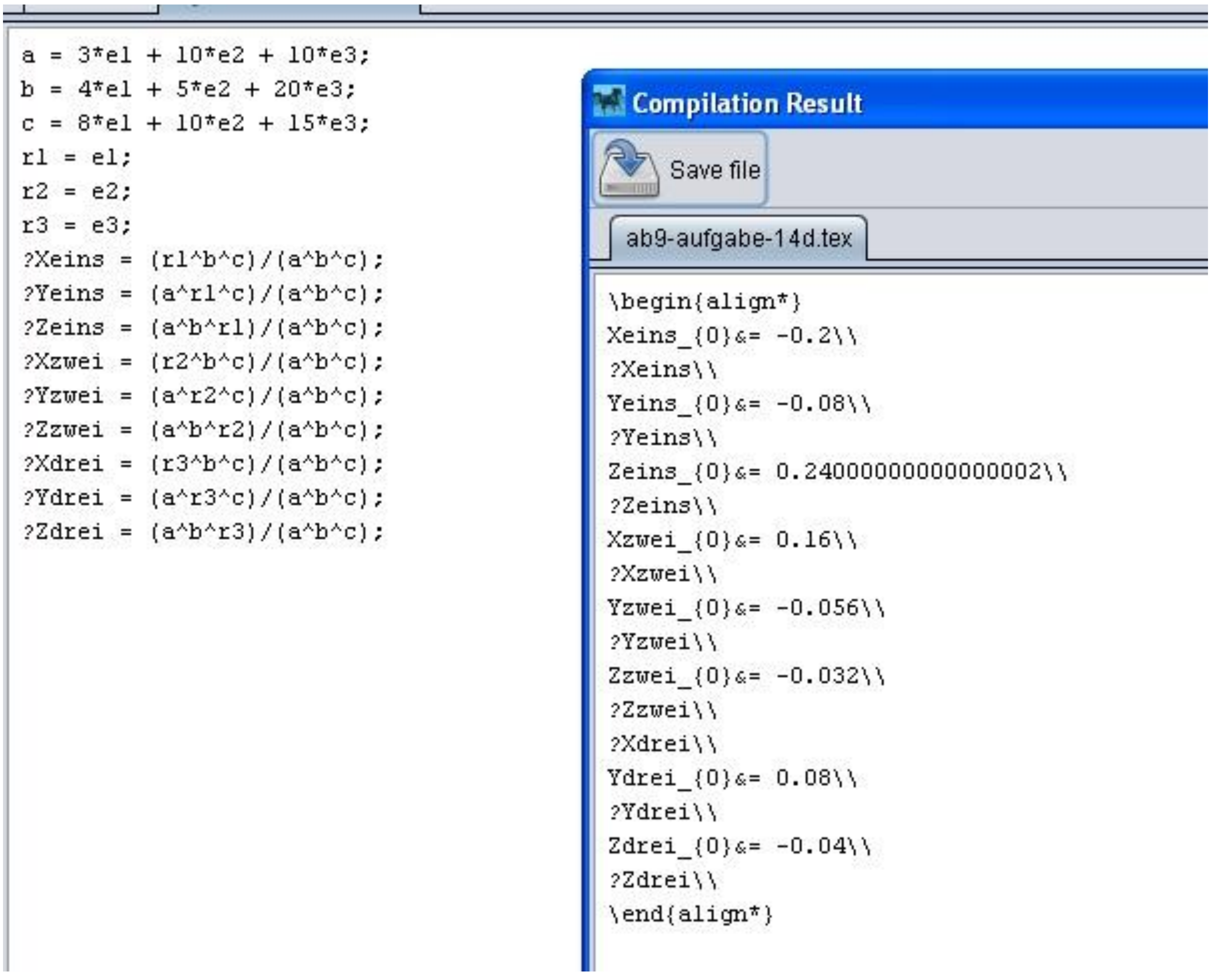
$$\begin{aligned} 3\,x_1 + 4\,y_1 + 8\,z_1 &= 1 \\ 10\,x_1 + 5\,y_1 + 10\,z_1 &= 0 \\ 10\,x_1 + 20\,y_1 + 15\,z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 3\,x_2 + 4\,y_2 + 8\,z_2 &= 0 \\ 10\,x_2 + 5\,y_2 + 10\,z_2 &= 1 \\ 10\,x_2 + 20\,y_2 + 15\,z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 3\,x_3 + 4\,y_3 + 8\,z_3 &= 0 \\ 10\,x_3 + 5\,y_3 + 10\,z_3 &= 0 \\ 10\,x_3 + 20\,y_3 + 15\,z_3 &= 1 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 3\,\sigma_x + 10\,\sigma_y + 10\,\sigma_z$ $\qquad \mathbf{r_1} = \sigma_x$

$\mathbf{b} = 4\,\sigma_x + 5\,\sigma_y + 20\,\sigma_z$ $\qquad \mathbf{r_2} = \sigma_y$

$\mathbf{c} = 8\,\sigma_x + 10\,\sigma_y + 15\,\sigma_z$ $\qquad \mathbf{r_3} = \sigma_z$

GAALOP program and compilation result:



As no scalar component Xdrei_{0} is shown in the compiler field, this component must be zero: Xdrei_{0} &= 0

Solution of the three systems of linear equations:

$x_1 = (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.20 = -\frac{1}{5}$

$y_1 = (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.08 = -\frac{2}{25}$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0.24 = \frac{6}{25}$

$x_2 = (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0.160 = \frac{4}{25}$

$y_2 = (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.056 = -\frac{7}{125}$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.032 = -\frac{4}{125}$

$x_3 = (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0$

$y_3 = (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = 0.08 = \frac{2}{25}$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = -0.04 = -\frac{1}{25}$

Check:					
			– 0.20	0.160	0
			– 0.08	– 0.056	0.08
			0.24	– 0.032	– 0.04
3	4	8	1	0	0
10	5	10	0	1	0
10	20	15	0	0	1

Alternative check:					
			– 25	20	0
			– 10	– 7	10
			30	– 4	– 5
3	4	8	125	0	0
10	5	10	0	125	0
10	20	15	0	0	125

$\Rightarrow$ The resulting matrix $\mathbf{D}^{-1} = \begin{bmatrix} -0.20 & 0.160 & 0 \\ -0.08 & -0.056 & 0.08 \\ 0.24 & -0.032 & -0.04 \end{bmatrix} = \frac{1}{125}\begin{bmatrix} -25 & 20 & 0 \\ -10 & -7 & 10 \\ 30 & -4 & -5 \end{bmatrix}$ is

the inverse of matrix $\mathbf{D} = \begin{bmatrix} 3 & 4 & 8 \\ 10 & 5 & 10 \\ 10 & 20 & 15 \end{bmatrix}$.

HWR Berlin, Wintersemester 2017/2018

Mathematics for Business and Economics

Berlin School of Economics and Law

Worksheet 21 – Answers

Problem 8:

a) $\mathbf{a} = 20\ \sigma_x + 6\ \sigma_y$

$\mathbf{b} = -2\ \sigma_x + 10\ \sigma_y$

$\mathbf{a\ b} = (20\ \sigma_x + 6\ \sigma_y)\ (-2\ \sigma_x + 10\ \sigma_y)$

$= -40\ \sigma_x^2 + 200\ \sigma_x\sigma_y - 12\ \sigma_y\sigma_x + 60\ \sigma_y^2$

$= -40 + 200\ \sigma_x\sigma_y + 12\ \sigma_x\sigma_y + 60$

$= 20 + 212\ \sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{b} = 212\ \sigma_x\sigma_y \quad \Rightarrow \quad |\mathbf{A}| = 212\ \text{cm}^2$

b) $\mathbf{a} = 18\ \sigma_x + 4\ \sigma_y$

$\mathbf{d} = 10\ \sigma_x + 12\ \sigma_y$

$\mathbf{a\ d} = (18\ \sigma_x + 4\ \sigma_y)\ (10\ \sigma_x + 12\ \sigma_y)$

$= 180\ \sigma_x^2 + 216\ \sigma_x\sigma_y + 40\ \sigma_y\sigma_x + 48\ \sigma_y^2$

$= 180 + 216\ \sigma_x\sigma_y - 40\ \sigma_x\sigma_y + 48$

$= 228 + 176\ \sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{d} = 176\ \sigma_x\sigma_y \quad \Rightarrow \quad |\mathbf{A}| = 176\ \text{cm}^2$

Alternative solution:

$\mathbf{b} = \mathbf{d} - \mathbf{a} = (10\ \sigma_x + 12\ \sigma_y) - (18\ \sigma_x + 4\ \sigma_y)$

$= 10\ \sigma_x + 12\ \sigma_y - 18\ \sigma_x - 4\ \sigma_y$

$= -8\ \sigma_x + 8\ \sigma_y$

$\mathbf{a\ b} = (18\ \sigma_x + 4\ \sigma_y)\ (-8\ \sigma_x + 8\ \sigma_y)$

$= -144\ \sigma_x^2 + 144\ \sigma_x\sigma_y - 32\ \sigma_y\sigma_x + 32\ \sigma_y^2$

$= -144 + 144\ \sigma_x\sigma_y + 32\ \sigma_x\sigma_y + 32$

$= -112 + 176\ \sigma_x\sigma_y$

$\mathbf{a} \wedge \mathbf{b} = 176\ \sigma_x\sigma_y \quad \Rightarrow \quad |\mathbf{A}| = 176\ \text{cm}^2$

Problem 9:

$8\ x + 10\ y = 280 \quad \Rightarrow \quad \mathbf{a} = 8\ \sigma_x + 5\ \sigma_y$

$5\ x + 15\ y = 280 \qquad\qquad \mathbf{b} = 10\ \sigma_x + 15\ \sigma_y$

$\mathbf{r} = 280\ \sigma_x + 280\ \sigma_y$

$\mathbf{a\,b} = (8\,\sigma_x + 5\,\sigma_y)\,(10\,\sigma_x + 15\,\sigma_y) = 155 + 70\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 70\,\sigma_x\sigma_y$

$\mathbf{r\,b} = (280\,\sigma_x + 280\,\sigma_y)\,(10\,\sigma_x + 15\,\sigma_y) = 7000 + 1400\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{r} \wedge \mathbf{b} = 1400\,\sigma_x\sigma_y$

$\mathbf{a\,r} = (8\,\sigma_x + 5\,\sigma_y)\,(280\,\sigma_x + 280\,\sigma_y) = 3640 + 840\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{r} = 840\,\sigma_x\sigma_y$

$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \frac{1400}{70} = 20 \qquad y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \frac{840}{70} = 12$

Check of results:

		20
		12
8	10	280
5	15	280

20 units of the first final product P_1 and 12 units of the second final product P_2 will be produced.

Problem 10:

$7\,x_1 + 3\,y_1 = 94 \quad \Rightarrow \quad \mathbf{a} = 7\,\sigma_x + 8\,\sigma_y$

$8\,x_1 + 9\,y_1 = 152 \qquad \mathbf{b} = 3\,\sigma_x + 9\,\sigma_y$

$\mathbf{r_1} = 94\,\sigma_x + 152\,\sigma_y$

$\mathbf{a\,b} = (7\,\sigma_x + 8\,\sigma_y)\,(3\,\sigma_x + 9\,\sigma_y) = 93 + 39\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 39\,\sigma_x\sigma_y$

$\mathbf{r_1\,b} = (94\,\sigma_x + 152\,\sigma_y)\,(3\,\sigma_x + 9\,\sigma_y) = 1650 + 390\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{r_1} \wedge \mathbf{b} = 390\,\sigma_x\sigma_y$

$\mathbf{a\,r_1} = (7\,\sigma_x + 8\,\sigma_y)\,(94\,\sigma_x + 152\,\sigma_y) = 1874 + 312\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{r_1} = 312\,\sigma_x\sigma_y$

$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{390}{39} = 10 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{312}{39} = 8$

$7\,x_2 + 3\,y_2 = 80 \quad \Rightarrow \quad \mathbf{a} = 7\,\sigma_x + 8\,\sigma_y$

$8\,x_2 + 9\,y_2 = 175 \qquad \mathbf{b} = 3\,\sigma_x + 9\,\sigma_y$

$\mathbf{r_2} = 80\,\sigma_x + 175\,\sigma_y$

$\mathbf{r_2\,b} = (80\,\sigma_x + 175\,\sigma_y)\,(3\,\sigma_x + 9\,\sigma_y) = 1815 + 195\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{r_2} \wedge \mathbf{b} = 195\,\sigma_x\sigma_y$

$\mathbf{a\,r_2} = (7\,\sigma_x + 8\,\sigma_y)\,(80\,\sigma_x + 175\,\sigma_y) = 1960 + 585\,\sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{r_2} = 585\,\sigma_x\sigma_y$

$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{195}{39} = 5 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{585}{39} = 15$

Check of results:

		10	5
		8	15
7	3	94	80
8	9	152	175

⇒ Demand matrix **B** of second production step: $\mathbf{B} = \begin{bmatrix} 10 & 5 \\ 8 & 15 \end{bmatrix}$

Problem 11:

a) Scheme of Falk:

			x
			y
			z
7	9	5	7 x + 9 y + 5 z = 359
6	8	4	6 x + 8 y + 4 z = 308
5	7	3	5 x + 7 y + 3 z = 257

} System of simultaneous linear equations

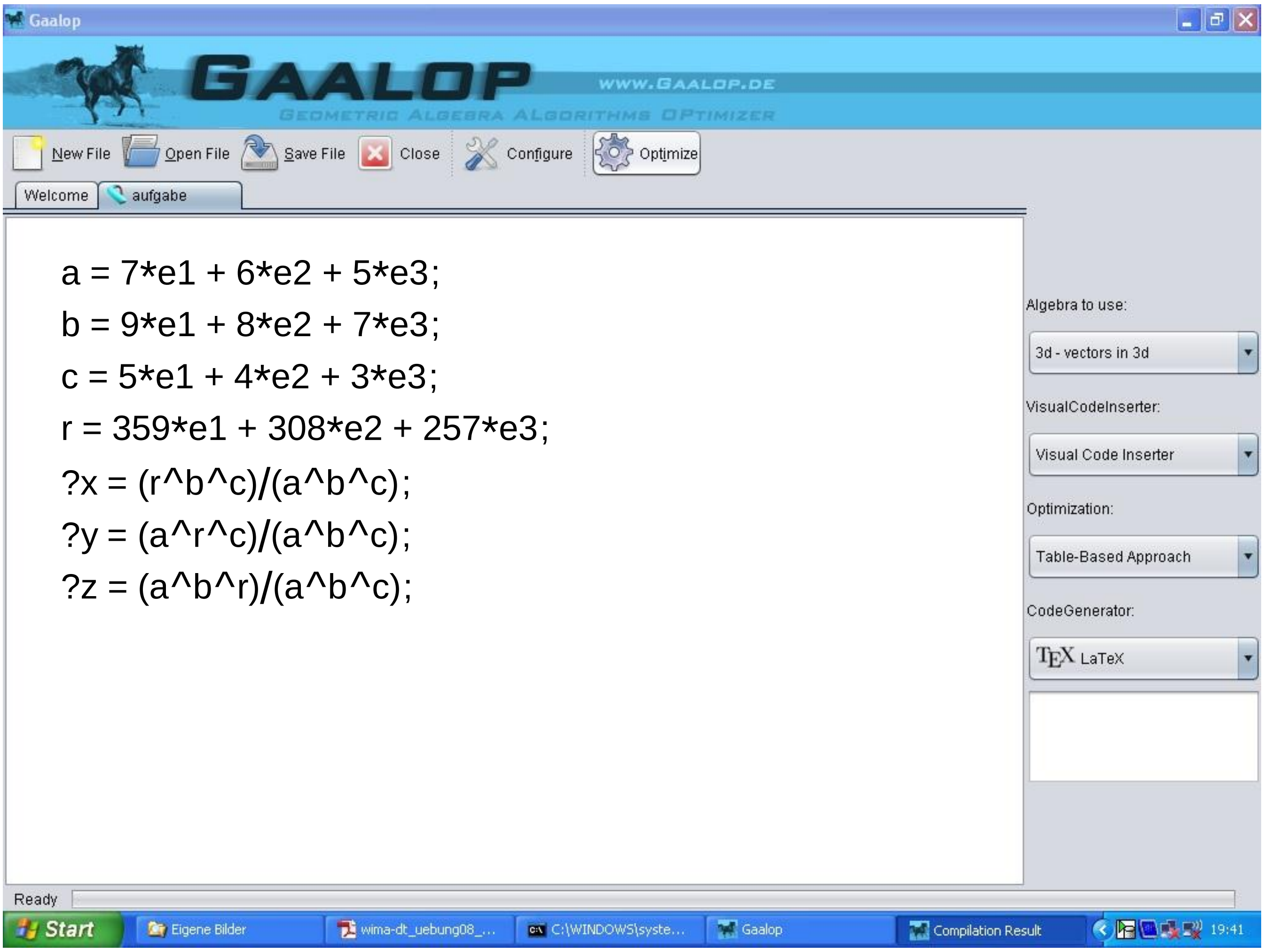
b) $\mathbf{a} = 7\,\sigma_x + 6\,\sigma_y + 5\,\sigma_z$ → `a = 7*e1 + 6*e2 + 5*e3;`

$\mathbf{b} = 9\,\sigma_x + 8\,\sigma_y + 7\,\sigma_z$ → `b = 9*e1 + 8*e2 + 7*e3;`

$\mathbf{c} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z$ → `c = 5*e1 + 4*e2 + 3*e3;`

$\mathbf{r} = 359\,\sigma_x + 308\,\sigma_y + 257\,\sigma_z$ → `r = 359*e1 + 308*e2 + 257*e3;`

GAALOP user interface:



c) Check of the expected solution values:

			18
			17
			16
7	9	5	359
6	8	4	308
5	7	3	257

$\Rightarrow$ The production vector $\mathbf{q} = \begin{bmatrix} 18 \\ 17 \\ 16 \end{bmatrix}$ is a solution of the given system of linear equations.

d) The solution given in part (c) is not the only solution of the system of linear equations, because there are an unlimited number of solutions, e.g. x = 20, y = 16, z = 15 or x = 16, y = 18, z = 17

Check of the two alternative solutions:

			20
			16
			15
7	9	5	359
6	8	4	308
5	7	3	257

			16
			18
			17
7	9	5	359
6	8	4	308
5	7	3	257

Mathematical reason why it is not possible to find solution values:

The solution values are undefined, because the outer product of the three coefficient values is zero ...

$\Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0\ \sigma_x\sigma_y\sigma_z = 0$

... or in other words:

The solution values are undefined, because the determinant of the demand matrix **A** is zero ...

$$\Rightarrow \quad \det \mathbf{A} = \det \begin{bmatrix} 7 & 9 & 5 \\ 6 & 8 & 4 \\ 5 & 7 & 3 \end{bmatrix} = 168 + 180 + 210 - 196 - 162 - 200 = 0$$

To find the solution values

$$x = (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$
$$y = (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$
$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) / (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$

GAALOP has to divide by the outer product of the coefficient vectors (or by the determinant of demand matrix **A**), which is not possible. **It is impossible to divide by zero.**

Problem 12:

a) Scheme of Falk:

			x
			y
			z
7	9	5	7 x + 9 y + 5 z = 422
6	8	4	6 x + 8 y + 4 z = 362
5	7	2	5 x + 7 y + 2 z = 283

} System of simultaneous linear equations

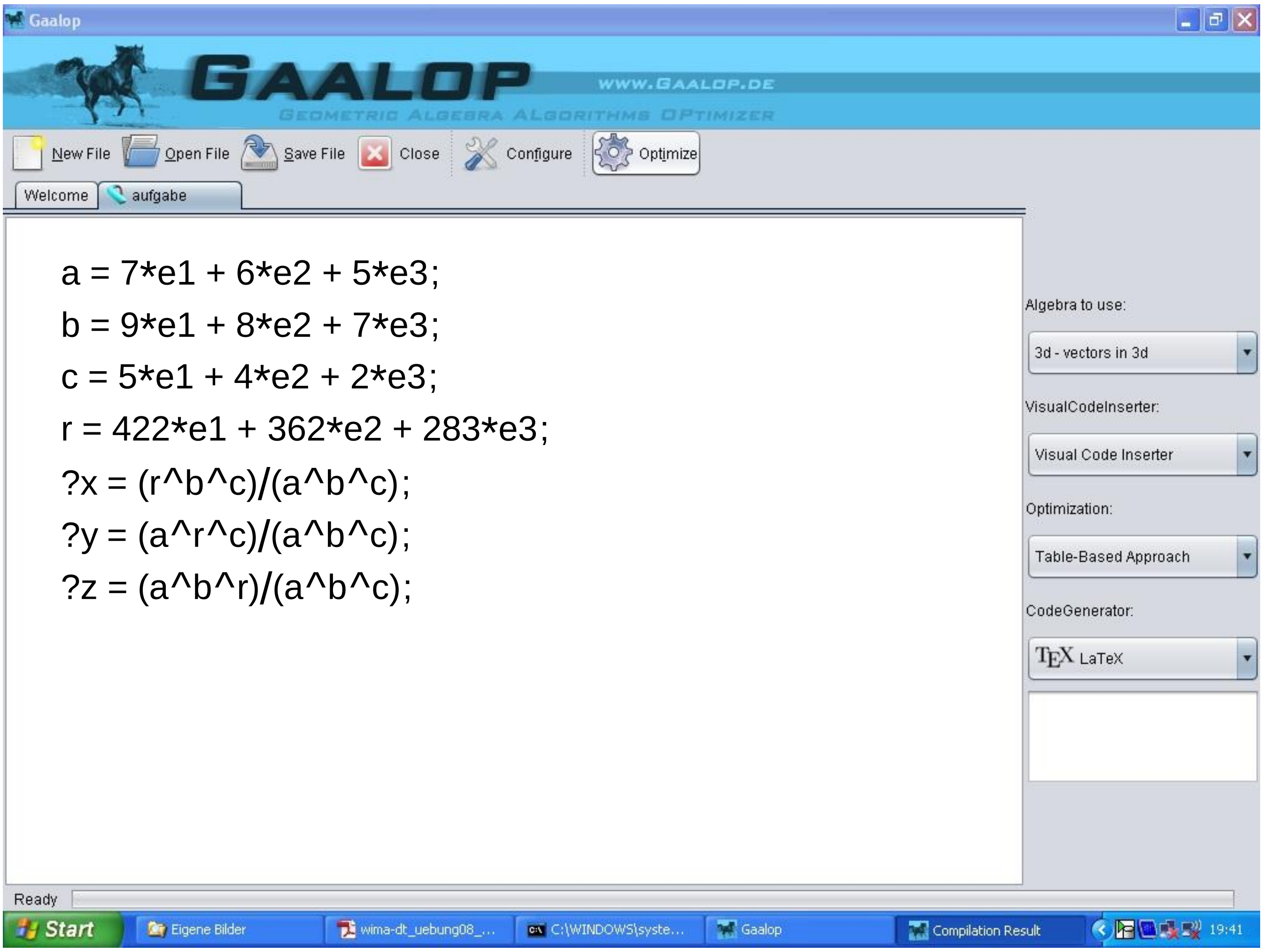
b) $\mathbf{a} = 7\,\sigma_x + 6\,\sigma_y + 5\,\sigma_z$ → a = 7*e1 + 6*e2 + 5*e3;

$\mathbf{b} = 9\,\sigma_x + 8\,\sigma_y + 7\,\sigma_z$ → b = 9*e1 + 8*e2 + 7*e3;

$\mathbf{c} = 5\,\sigma_x + 4\,\sigma_y + 2\,\sigma_z$ → c = 5*e1 + 4*e2 + 2*e3;

$\mathbf{r} = 422\,\sigma_x + 362\,\sigma_y + 283\,\sigma_z$ → r = 422*e1 + 362*e2 + 283*e3;

GAALOP user interface:

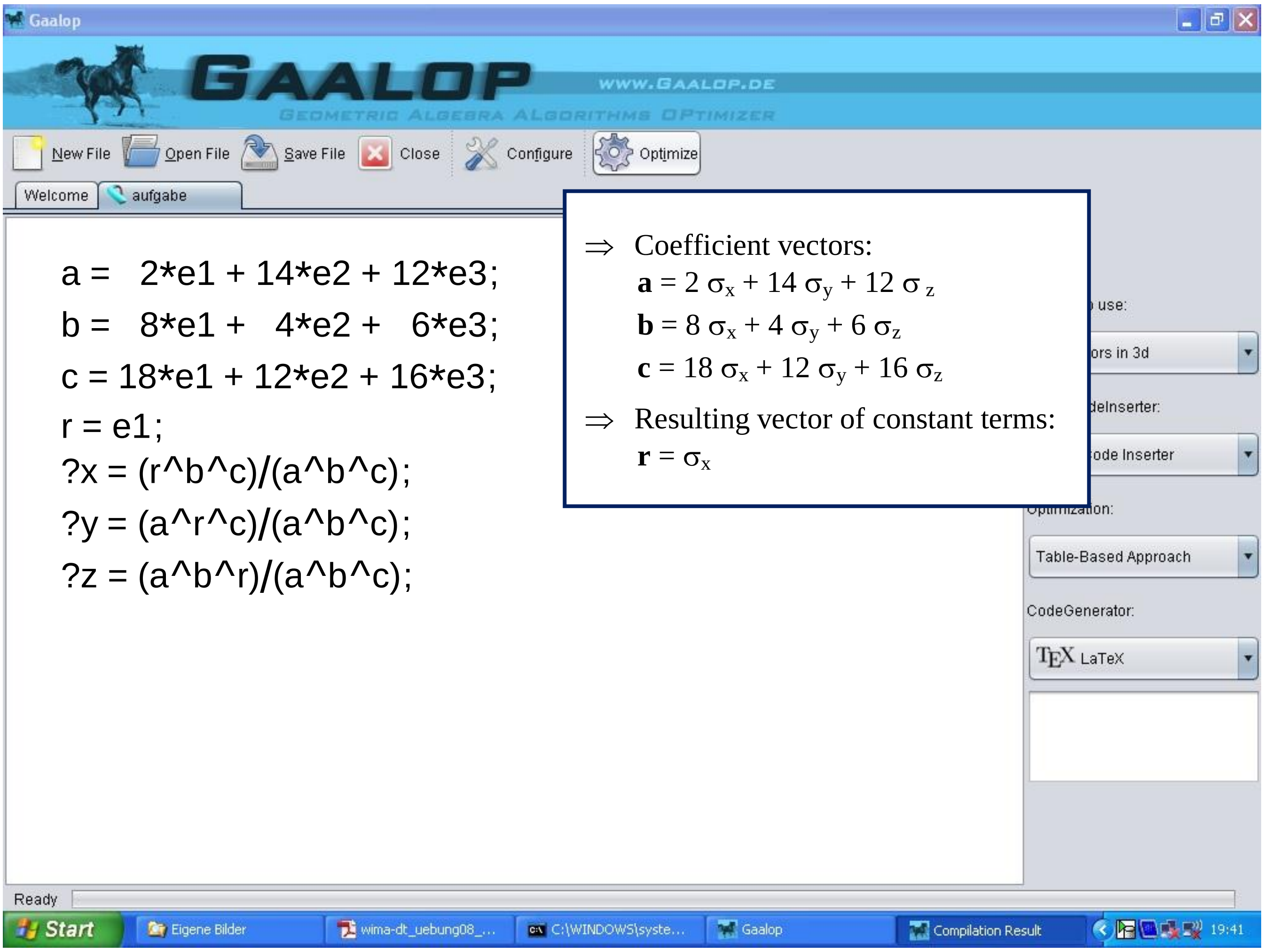


c) Check of the expected solution values:

			21
			20
			19
7	9	5	422
6	8	4	362
5	7	2	283

$\Rightarrow$ The production vector $\mathbf{q} = \begin{bmatrix} 21 \\ 20 \\ 19 \end{bmatrix}$ is a solution of the given system of linear equations.

Problem 13:



$\Rightarrow$ Coefficient vectors:

$\mathbf{a} = 2\,\sigma_x + 14\,\sigma_y + 12\,\sigma_z$

$\mathbf{b} = 8\,\sigma_x + 4\,\sigma_y + 6\,\sigma_z$

$\mathbf{c} = 18\,\sigma_x + 12\,\sigma_y + 16\,\sigma_z$

$\Rightarrow$ Resulting vector of constant terms:

$\mathbf{r} = \sigma_x$

a) Scheme of Falk:

			x
			y
			z
2	8	18	1
14	4	12	0
12	6	16	0

This is the first part of the following complete calculation:

			x_1	x_2	x_3
			y_1	y_2	y_3
			z_1	z_2	z_3
2	8	18	1	0	0
14	4	12	0	1	0
12	6	16	0	0	1

(x_i, y_i, z_i: inverse matrix $\mathbf{A}^{-1}$; left block: matrix $\mathbf{A}$; right block: identity matrix $\mathbf{I}$)

⇒ The given GAALOP program calculates the first column of the inverse of a matrix which consists of coefficient vectors **a**, **b**, and **c**.

b) Check of results:

			1
			10
			– 4.5
2	8	18	$2 \cdot 1 + 8 \cdot 10 + 18 \cdot (-4.5) = 1$
14	4	12	$14 \cdot 1 + 4 \cdot 10 + 12 \cdot (-4.5) = 0$
12	6	16	$12 \cdot 1 + 6 \cdot 10 + 16 \cdot (-4.5) = 0$

⇒ The given values are correct solutions of the GAALOP problem.