Solutions of two Problems about Systems of Simultaneous Linear Equations from Old Babylonia and from the Han Period with GAALOP as a Pocket Calculator Substitute

– Extended version of the paper "Lösung einer Aufgabe zu Linearen Gleichungssystemen aus der Han-Dynastie mit GAALOP als Taschenrechner-Ersatz" [1]
(Solution of a Problem about Systems of Simultaneous Linear Equations from the Han Dynasty with GAALOP as a Pocket Calculator Substitute), written in German for the annual meeting of the Society of Mathematics Education (GDM) 2017 in Potsdam –

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English Abstract

Present-day students very often are characterized by massive and excessive use of pocket calculators even for the simplest arithmetic calculations. Therefore it is a didactical problem to discuss mathematical topics with students if pocket calculators obtainable on the market are not able to perform required computations of the mathematical area discussed.
To perform these computations with students of such limited mathematical abilities when discussing Geometric Algebra, it might be helpful to use the Geometric Algebra program tool GAALOP (Geometric Algebra Algorithms Optimizer) as a pocket calculator substitute.

German Abstract

Wenn mit Lernenden mathematische Bereiche behandelt werden, in denen geforderte mathematischen Operationen nicht mit derzeit auf dem Markt befindlichen Taschenrechnern durchführbar sind, ergeben sich Probleme.
Um anfallende Rechnungen im Bereich der Geometrischen Algebra mit in ihren Rechenfähigkeiten reduzierten Studierenden, die durch einen exzessiven Taschenrechnereinsatz geprägt sind, behandeln zu können, bietet sich die Nutzung des Programm-Tools GAALOP (Geometric Algebra Algorithms Optimizer) im Sinne eines Taschenrechner-Ersatzes an.

1. Pocket calculators at school

Some decades ago a dramatic and radical change has reached mathematics lessons at schools and high schools in Germany. Looking back at my own school days[1] this change just started at these times: The use of pocket calculators at school was allowed only at the upper level of secondary school (Sekundarstufe II) after having been introduced at the very end of the lower level of secondary school (Sekundarstufe I).

And I still remember rather vividly how we tried to find values of logarithmic functions using slide rules and therefore having to discuss logarithmic properties in a very handy way.

Only a few years before these times, the use of pocket calculators was not allowed at schools.

And the years after my school time the implementation of pocket calculators shifted more and more to earlier grades.

The consequences of this process are depressing: Even for most simple arithmetic calculations present-day students immediately reach for their electronic devices. It is an automatically and well-trained reflex action: Do not bother your brain to find out the result of 3 times 10, the pocket calculator or your smart phone will tell you.

Even university students – at least these first-year students taking part at my business math lessons at BSEL, a leading University of Applied Sciences in Germany [2] – are for the most part arithmetic illiterates.

This fact is regrettable, but obviously arithmetic illiteracy will be inevitable if the massive and excessive use of pocket calculators will continue at our schools.

[1] The school leaving exams (Abitur examination) took place at Baden-Wuerttemberg in 1983.

Therefore it is a didactical problem to discuss mathematical topics with students if pocket calculators obtainable on the market are not able to perform required computations of the mathematical area discussed.

As multiplications of non-commutative numbers or non-commutative entities cannot be performed by present-day pocket calculators, Geometric Algebra (the didactical version of Clifford Algebra) is such a mathematical area.

The didactical consequences are straight-forward: Possible discussions of Geometric Algebra at school or university lessons have to be restricted to very simple examples and easy problems, for not expecting too much of our students and for not exceeding time resources.

If arithmetic questions cover the discussion of central mathematical concepts, the main aim of Geometric Algebra lessons – to understand the mathematical concept of non-commutativity in an algebraic as well as in a geometric based way – might not be reached.

Therefore a Geometric Algebra teaching unit of only restricted arithmetic complexity [3], [4], [5], [6] has been developed some time ago: A modern version of Linear Algebra based on Geometric Algebra is discussed from geometric and algebraic perspectives on the basis of systems of only two simultaneous linear equations with two unknown variables only.

Of course it is worthwhile to enable students to solve systems of simultaneous linear equations of more than two equations and more than two unknown variables. But as appropriate pocket calculators are not available and as most students seem to be not capable to find the required arithmetic results within the given restricted time of the lessons, it is necessary to find an alternative way of computing simple calculations.

Using the program GAALOP (Geometric Algebra Algorithms Optimizer) can be a suitable alternative to prevent lengthy calculations by hand.

2. GAALOP

GAALOP (Geometric algebra algorithms optimizer) is a software to optimize geometric algebra files [7], [8], [9], [10], [11]. The optimized code has no more geometric algebra operations and can be run very efficiently on various platforms. There are currently two versions of GAALOP: A GUI based standalone version, that allows for quick and easy experiments and a more development-focused variant named Gaalop Precompiler [7]. Thus GAALOP is a very effective and powerful program tool which makes Geometric Algebra programming easier.

The standalone version can be used as a substitute for non-existing Geometric Algebra pocket calculators. This version is based on simple CluCalc and LaTex program codes, which are accessible rather intuitively and can be used right from the start without longer preparations.

The download of GAALOP is free and can be done within seconds from the GAALOP download page (www.gaalop.de/download [7]). After having extracted the zip file GAALOP can be started directly by activating the start icon.

Fig.1: Download of the Geometric Algebra Algorithms Optimizer GAALOP [7].

In the following two examples of historically relevant linear algebra problems will be discussed and it will be shown how these problems can be solved with the help of GAALOP.

3. A cuneiform problem from Old Babylonia

One of the earliest cultural techniques developed by mankind in the area of mathematics has been to find solutions of simple systems of simultaneous linear equations. Already over 4000 years ago Babylonian mathematicians had been able to solve systems of linear equations with two unknown variables.

“Problems that can be interpreted as simultaneous linear equations are present, but not prevalent, in the earliest written mathematics” [12, p. 782]. Solution schemes, which had been formulated as early preversions of the scheme of Gauss in these Old Babylonian cuneiform texts [12], [13], were routinely taught to generations of students at Old Babylonian schools.

The first problem of cuneiform tablet VAT 8389, which is part of the collection of the "Vorderasiatisches Museum (VAT)" in Berlin, presents the following problem [12, p. 782], [13, p. 167]:

> *There are two fields whose total area is 1800 sar.*
>
> *The rent for one field is 2 silà of grain per 3 sar, the rent for the other is 1 silà per 2 sar, and the total rent on the first exceeds that on the other by 500 silà.*
>
> *What is the size of each field?*

Tablets VAT 8389 and another tablet "from the Old Babylonian period, 2000 – 1600 BC, contain what are believed to be among the earliest problems that can be interpreted as systems of linear equations" [13, p. 166].

In modern form – after the introduction of variables by Diophantus and his colleagues [14, chap. 2] – this cuneiform problem can be written as the following system of two linear equations:

$$\begin{aligned} x + \quad y &= 1800 \\ \frac{2}{3}\,x - \frac{1}{2}\,y &= \ \ 500 \end{aligned} \qquad \{1\}$$

Following the discussion of the Old Babylonian scribes, mathematicians and school teachers in [13, p. 167, fig. 2, eq. 3], which "for brevity … expresses the solution symbolically", the Old Babylonian mathematicians replaced the variables x, y by splitting the resulting area of 1800 sar into two different terms:

$$\begin{aligned} x + \quad y &= 1800 = (900 + s)\,(900 - s) \\ \frac{2}{3}\,x - \frac{1}{2}\,y &= \ \ 500 \end{aligned} \qquad \{2\}$$

Therefore the system of linear equations {1} is then expressed with the help of only one variable s, thus identifying the variables x, y as

$$\begin{aligned} x &= 900 + s \\ y &= 900 - s \end{aligned} \qquad \{3\}$$

Replacing the variables of the second linear equation

$$\begin{aligned} x + \quad y &= 1800 = (900 + s)\,(900 - s) \\ \frac{2}{3}\,x - \frac{1}{2}\,y &= \ \ 500 = \frac{2}{3}\,(900 + s) - \frac{1}{2}\,(900 - s) \end{aligned} \qquad \{4\}$$

will result in

$$\frac{2}{3}\,(900 + s) - \frac{1}{2}\,(900 - s) = 600 + \frac{2}{3}\,s - 450 + \frac{1}{2}\,s$$

$$500 = 150 + \frac{7}{6}\,s \qquad \{5\}$$

The solutions can then be found as

$$\begin{aligned} & s = 300 \\ \Rightarrow\quad & x = 900 + 300 = 1200 \\ & y = 900 - 300 = \ \ 600 \end{aligned} \qquad \{6\}$$

> *The size of the first field is 1200 sar.*
> *And the size of the second field is 600 sar.*

In a similar way Old Babylonian mathematicians solved quadratic equations, which they expressed as a system of a linear and a non-linear equation for didactical reasons [15].

4. Geometric Algebra

With his theory of extensions Hermann Grassmann presented an alternative way of solving systems of simultaneous linear equations which started a dramatic algebraic revolution. Expressed in his own words: „the applicability of outer multiplication emerges with such a striking determination and firmness, that (…) algebra will gain a substantial different shape.[2]"

But this algebraic revolution indeed is at the same time a geometric revolution, as Grassmann deeply connected algebra and geometry by attaching geometric directions to the algebraic equations. Thus the Old Babylonian system {1} of two algebraic linear equations is now considered as a geometric system of linear equations:

$$\begin{aligned} x + \quad y &= 1800 \quad \rightarrow \text{ x-direction} \\ \frac{2}{3}\,x - \frac{1}{2}\,y &= \ \ 500 \quad \rightarrow \text{ y-direction} \end{aligned} \qquad \{7\}$$

The first equation thus will point into the direction of the x-axis, and the second equation will point into the direction of the y-axis.

The geometric-algebraic revolution of Grassmann reached its climax by identifying the base vectors of three-dimensional space with Pauli matrices.

In his later papers Grassmann presented his "Bedingungsgleichungen" – "conditional equations" [17, p. 376, reprint p. 269] describing the base vectors of three-dimensional space as

$$\begin{aligned} & e_1 e_2 = -\,e_2 e_1 \qquad e_2 e_3 = -\,e_3 e_2 \qquad e_3 e_1 = -\,e_1 e_3 \\ & e_1^2 = e_2^2 = e_3^2 = 1 \end{aligned} \qquad \{8\}$$

These equations are identical to the basic equations of Pauli Algebra:

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[2] Grassmann's statement in German: "Aber desto interessanter ist es, zu bemerken, wie in der Algebra, sobald an der Zahl noch die Art ihrer Verknüpfung mit anderen Größen festgehalten, und in dieser Hinsicht die eine als von der anderen formell verschiedenartig aufgefasst wird, auch die Anwendbarkeit der äusseren Multiplikation mit einer so schlagenden Entschiedenheit heraustritt, dass ich wohl behaupten darf, es werde durch diese Anwendung auch die Algebra eine wesentlich veränderte Gestalt gewinnen." [16, p. 71]

$$\sigma_x \sigma_y = -\sigma_y \sigma_x \quad \sigma_y \sigma_z = -\sigma_z \sigma_y \quad \sigma_z \sigma_x = -\sigma_x \sigma_z$$
$$e_x^2 = e_y^2 = e_z^2 = 1 \qquad \{9\}$$

Hermann Grassmann had already invented Pauli Algebra![3] Now multiplying the system of linear equations {7} by theses base vectors will result in:

$$\begin{aligned} x\,\sigma_x + \quad y\,\sigma_x &= 1800\,\sigma_x \\ \frac{2}{3}\,x\,\sigma_y - \frac{1}{2}\,y\,\sigma_y &= \;\;500\,\sigma_y \end{aligned} \qquad \{10\}$$

Grassmann then constructed the following coefficient vectors

$$\begin{aligned} \mathbf{a} &= 1\,\sigma_x + \frac{2}{3}\,\sigma_y \\ \mathbf{b} &= 1\,\sigma_x - \frac{1}{2}\,\sigma_y \end{aligned} \qquad \{11\}$$

and the resulting vector of constant terms

$$\mathbf{r} = 1800\,\sigma_x + 500\,\sigma_y \qquad \{12\}$$

to re-formulate the system of simultaneous linear equations {10} as

$$\mathbf{a}\,x + \mathbf{b}\,y = \mathbf{r} \qquad \{13\}$$

Outer multiplications of {13} by the coefficient vectors **a** and **b** [20], [21]

$$\begin{aligned} (\mathbf{a}\,x + \mathbf{b}\,y) \wedge \mathbf{b} &= (\mathbf{a} \wedge \mathbf{b})\,x + 0\,y = \mathbf{r} \wedge \mathbf{b} \\ \mathbf{a} \wedge (\mathbf{a}\,x + \mathbf{b}\,y) &= 0\,x + (\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r} \end{aligned} \qquad \{14\}$$

directly results in Grassmann's solution formulas

$$\begin{aligned} x &= (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) \\ y &= (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) \end{aligned} \qquad \{15\}$$

Thus the Old Babylonian cuneiform problem can be solved after finding the coefficient vector outer products

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= -\frac{1}{2}\,\sigma_x\sigma_y - \frac{2}{3}\,\sigma_x\sigma_y = -\frac{7}{6}\,\sigma_x\sigma_y \\ \Rightarrow\quad (\mathbf{a} \wedge \mathbf{b})^{-1} &= -\frac{6}{7}\,\sigma_y\sigma_x = \frac{6}{7}\,\sigma_x\sigma_y \end{aligned} \qquad \{16\}$$

and

$$\begin{aligned} \mathbf{r} \wedge \mathbf{b} &= -900\,\sigma_x\sigma_y - 500\,\sigma_x\sigma_y = -1400\,\sigma_x\sigma_y \\ \mathbf{a} \wedge \mathbf{r} &= 500\,\sigma_x\sigma_y - 1200\,\sigma_x\sigma_y = -700\,\sigma_x\sigma_y \end{aligned}$$

even by students with only limited mathematical skills and without lengthy arithmetic calculations.

$$\begin{aligned} x &= \left(-\frac{6}{7}\,\sigma_y\sigma_x\right)(-1400\,\sigma_x\sigma_y) = 1200 \\ y &= \left(-\frac{6}{7}\,\sigma_y\sigma_x\right)(-700\,\sigma_x\sigma_y) \;=\; 600 \end{aligned} \qquad \{17\}$$

[3] And he had already invented Dirac Algebra. But this is another, more relativistic story of spacetime [18], [19].

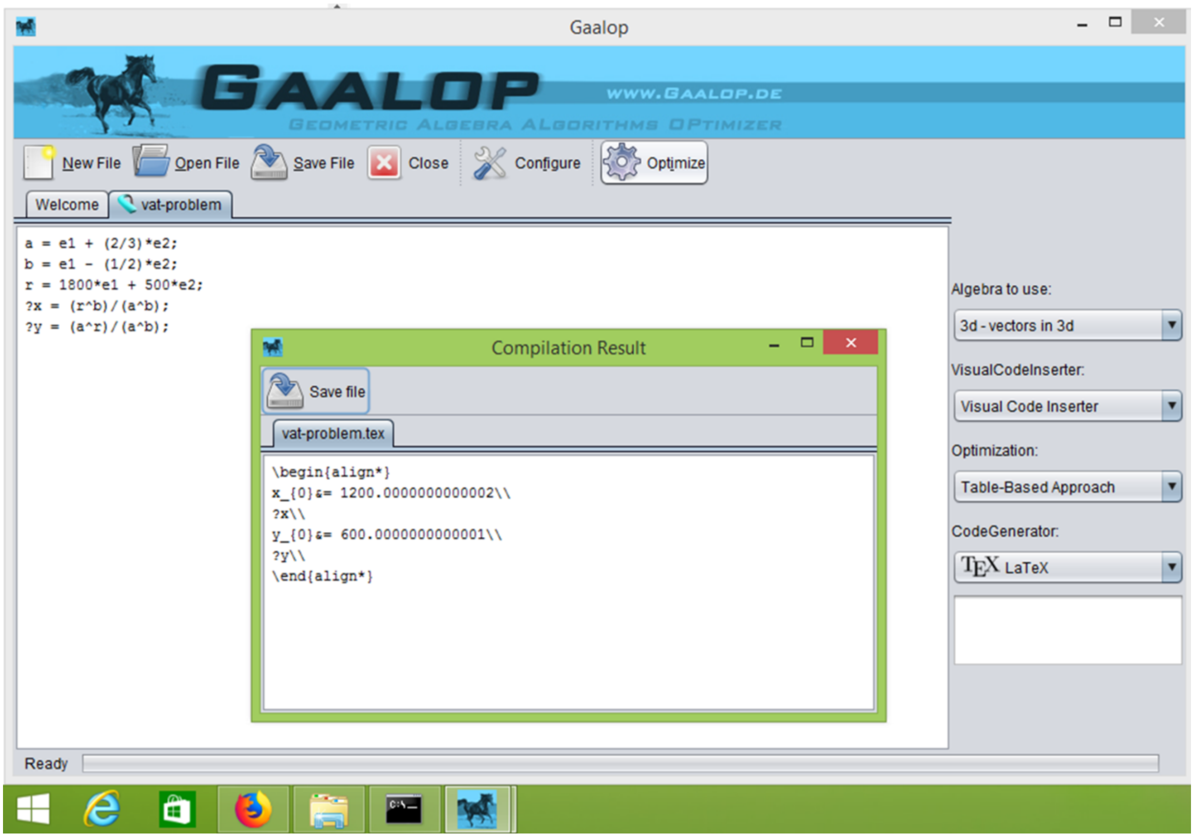

Fig.2: GAALOP program to solve the first Old Babylonian cuneiform problem of VAT 8389.

5. Old Babylonian GAALOP example

To prepare students for more complicated GAALOP programming, the simple system of simultaneous linear equations of the Old Babylonian cuneiform problem can be used to implement a first and easy GAALOP example.

As the syntax of GAALOP is quite simple, it is only necessary to give the three Pauli vectors {11} & {12} as input (see first three lines of fig. 2) and then to give the two formulas {15} (see fourth and fifth line of fig. 2).

After activating the optimize button, a compilation result field will show the results (which sometimes are not rounded perfectly). The result named as x_{0}&= will be the scalar value x = 1200 and the result named as y_{0}&= will be the scalar value y = 600 (see fig. 2). The other lines of the compilation result field can be ignored as they are only of importance if the results should be used as part of a LaTex program.

6. A problem from the Early Han Dynasty

In China mathematicians solved systems of simultaneous linear equations at very early times as well. “The Early Han dynasty (200 BC – 9 AD) was a period of mathematical creativity in China” [14, p. 163].

One of the oldest Chinese mathematical books is the book “Nine Chapters on the Art of Calculation” from this Early Han period, very probably a compilation of lost earlier mathematical texts. This is ”by far the most impressive treatment of simultaneous linear equations known from antiquity” [13, p. 166] and it “played a part in the subsequent mathematical culture of China comparable to that played by Euclid’s Elements in Europe” [14, p.163].

In chapter 8 of this book the following rice paddy problem can be found [12, p. 783], [13, p. 167], [14, p. 161], [22, p.57/58]:

> *There are three types of rice.*
>
> *Three baskets of the first, two of the second, and one of the third weigh 39 measures.*
>
> *Two baskets of the first, three of the second, and one of the third weigh 34 measures.*
>
> *And one basket of the first, two of the second, and three of the third weigh 26 measures.*
>
> *How many measures of rice are contained in one basket of each type?*

In modern form this rice paddy problem can be written as the following system of three linear equations:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned} \qquad \{18\}$$

Mathematicians in ancient China used rectangular patterns to solve systems of simultaneous linear equations. These patterns, wooden boards used as counting tables [22, p. 52], resembled the augmented matrices of Gauss, simply rotated by 90°. It is sort of a matrix calculation[4] “in any number of unknowns – a method that is still taught to beginning students of matrix algebra today. And all this took place over 2,000 years ago! [14, p. 163]”

	×3	×2
1	2	3
2	3	2
3	1	1
26	34	39

		–
1	6	6
2	9	4
3	3	2
26	102	78

×3		×1
1	0	3
2	5	2
3	1	1
26	24	39

		–
3	0	3
6	5	2
9	1	1
78	24	39

×5	×4	
0	0	3
4	5	2
8	1	1
39	24	39

	–	
0	0	3
20	20	2
40	4	1
185	96	39

0	0	3
0	5	2
36	1	1
99	24	39

$\Downarrow$

$$z = \frac{99}{36} = \frac{11}{4} = 2.75$$

$$y = \frac{1}{5}\left(24 - \frac{11}{4}\right) = \frac{17}{4} = 4.25$$

$$x = \frac{1}{3}\left(39 - \frac{11}{4} - 2\cdot\frac{17}{4}\right) = \frac{37}{4} = 9.25$$

Fig.3: Chinese solution of the rice paddy problem according to Grcar [12, p. 783], [13, p. 167 (5)].

And all this really looks modern.

[4] Wußing writes: “Es handelt sich um eine Art Matrizenrechnung zur Lösung von Systemen linearer Gleichungen nach einem – auch in diesem Falle – allgemeingültigen Verfahren“ [22, p. 57].

7. Second Geometric Algebra interlude

All this really looks modern, but it isn't. Grassmann would have solved the rice paddy problem in a completely different way.

Now the three linear equations {18} are multiplied by three different base vectors:

$$\begin{aligned} 3\,x\,\sigma_x + 2\,y\,\sigma_x + \,z\,\sigma_x &= 39\,\sigma_x \\ 2\,x\,\sigma_y + 3\,y\,\sigma_y + \,z\,\sigma_y &= 34\,\sigma_y \\ x\,\sigma_z + 2\,y\,\sigma_z + 3\,z\,\sigma_z &= 26\,\sigma_z \end{aligned} \qquad \{19\}$$

The following three coefficient vectors

$$\begin{aligned} \mathbf{a} &= 3\,\sigma_x + 2\,\sigma_y + \sigma_x \\ \mathbf{b} &= 2\,\sigma_x + 3\,\sigma_y + 2\,\sigma_z \\ \mathbf{c} &= \sigma_x + \sigma_y + 3\,\sigma_z \end{aligned} \qquad \{20\}$$

and the following resulting vector of constant terms

$$\mathbf{r} = 39\,\sigma_x + 34\,\sigma_y + 26\,\sigma_z \qquad \{21\}$$

can be constructed to re-formulate the system of simultaneous linear equations {19} as

$$\mathbf{a}\,x + \mathbf{b}\,y + \mathbf{c}\,z = \mathbf{r} \qquad \{22\}$$

Outer multiplications of {22} by two of the three coefficient vectors **a**, **b**, and **c** [20], [21]

$$\begin{aligned} (\mathbf{a}\,x + \mathbf{b}\,y + \mathbf{c}\,z) \wedge \mathbf{b} \wedge \mathbf{c} &= \mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} \\ \mathbf{a} \wedge (\mathbf{a}\,x + \mathbf{b}\,y + \mathbf{c}\,z) \wedge \mathbf{c} &= \mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} \\ \mathbf{a} \wedge \mathbf{b} \wedge (\mathbf{a}\,x + \mathbf{b}\,y + \mathbf{c}\,z) &= \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} \end{aligned} \qquad \{23\}$$

will again directly result in Grassmann's solution formulas [16, p. 72, eq.1], [17, p. 385, $x = \frac{[dbc]}{[abc]}$]

$$\begin{aligned} x &= (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) \\ y &= (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) \\ z &= (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) \end{aligned} \qquad \{24\}$$

Again the given system of simultaneous linear equations can be solved after finding the coefficient vector outer products, which can now be identified with the oriented volume of the parallelepiped [5] of the three Pauli vectors {20}

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 12\,\sigma_x\sigma_y\sigma_z$$

$$\Rightarrow\ (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{12}\,\sigma_z\sigma_y\sigma_x = -\frac{1}{12}\,\sigma_x\sigma_y\sigma_z \qquad \{25\}$$

and

$$\begin{aligned} \mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} &= 111\,\sigma_x\sigma_y\sigma_z \\ \mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} &= 51\,\sigma_x\sigma_y\sigma_z \\ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} &= 33\,\sigma_x\sigma_y\sigma_z \end{aligned} \qquad \{26\}$$

[5] It is enlightening to read Grassmann's original papers. He wrote: "Die schönste Anwendung der Quaternionen ist die auf die sphärische Trigonometrie. Doch glaube ich, dass auch hier die Verknüpfung der Strecken der Rechnung mit Quaternionen überlegen ist. (…) Ich nenne [*abc*] das äussere Product der drei Strecken *a*, *b*, *c*. Es ergeben sich (…) begrifflich, dass [*abc*] gleich dem Parallelepipedon (Spat) ist, in welchem 3 sich aneinander schliessende Kanten gleich *a*, *b* und *c* sind" [17, p. 384].

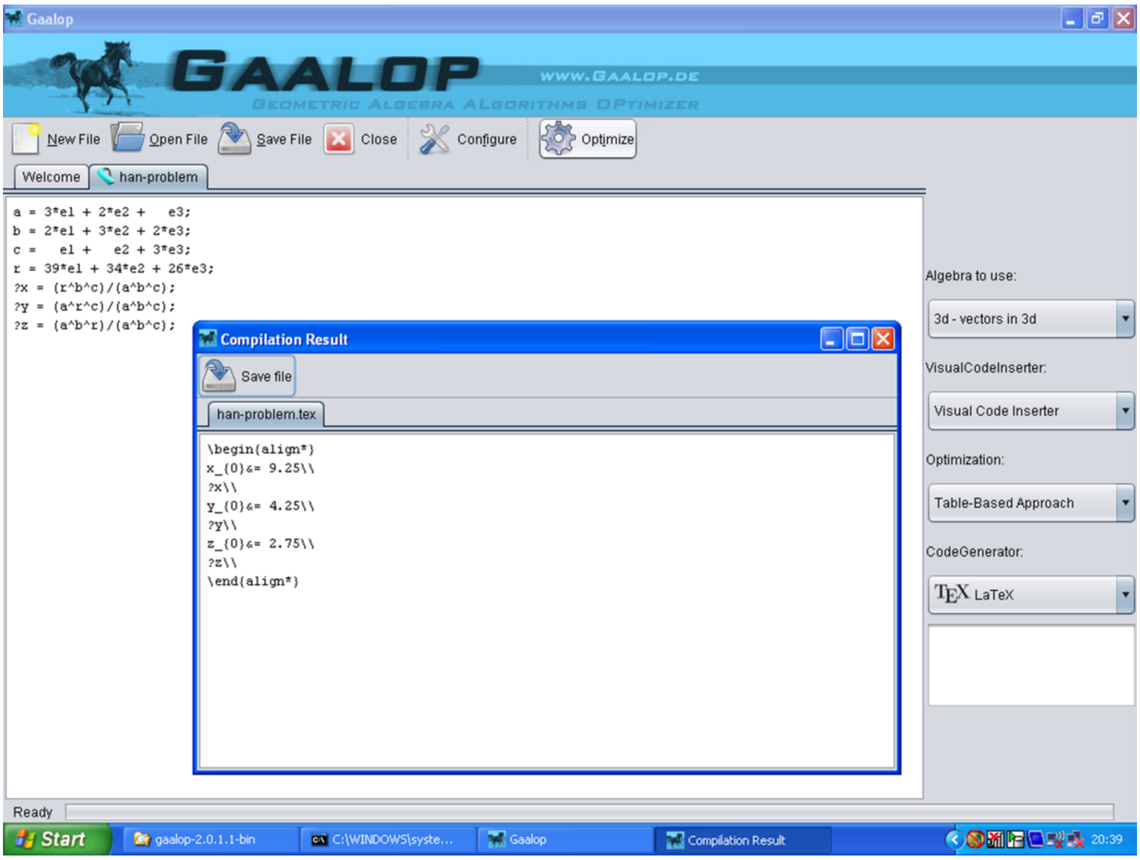

Fig.4: GAALOP program to solve the rice paddy problem of the Early Han period.

As all base units cancel, the three results will again be scalar values, identical to the results of fig 3.

$$x = \left(\frac{1}{12}\ \sigma_z\sigma_y\sigma_x\right)(111\ \sigma_x\sigma_y\sigma_z) = 9.25$$

$$y = \left(\frac{1}{12}\ \sigma_z\sigma_y\sigma_x\right)(\ 51\ \sigma_x\sigma_y\sigma_z) = 4.25 \quad \{27\}$$

$$z = \left(\frac{1}{12}\ \sigma_z\sigma_y\sigma_x\right)(\ 33\ \sigma_x\sigma_y\sigma_z) = 2.75$$

One basket of the first type of rice contains 9 ¼ measures of rice.

One basket of the second type of rice contains 4 ¼ measures of rice.

And one basket of the third type of rice contains 2 ¾ measures of rice.

8. Early Han period GAALOP example

The GAALOP program to solve the rice paddy problem of the Early Han period is shown in fig. 4. Again it is only necessary to give the four Pauli vectors {20} & {21} as input (see first four lines of fig. 4) and then to state the three formulas {24} (see last three lines of fig. 4).

After activating the optimize button, the compilation result field will show the expected results. The result named as x_{0}&= will be the scalar value x = 9.25, the result named as y_{0}&= will be the scalar value y = 4.25, and the third result named as z_{0}&= will be the scalar value z = 2.75 (see fig. 4). And again the other lines of the compilation result field can be ignored.

9. Conclusion

To perform Geometric Algebra calculations with students who have only limited mathematical and arithmetical skills, the Geometric Algebra program tool GAALOP (Geometric Algebra Algorithms Optimizer) can be used as a pocket calculator substitute.

GAALOP is a nice and helpful program tool with a wide range of possible didactical applications. It can be used to model purely Euclidean situation (as shown in this paper), relativistic effects of spacetime [23], [24] and projective or conformal Geometric Algebra models [11].

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