Pauli Algebras in Economics: Economathematics from Geometry to Didactics and back – The Geometry of Moore-Penrose Generalized Matrix Inverses –

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Abstract

According to Hestenes, geometry links the algebra to the physical world. Therefore the journey of economathematics told in this paper will present a closer analysis of the geometry of our world by questioning the Dirac belt trick: Obviously 4π periodicities are an elementary part of our world, and to describe this world, the mathematics of 4π periodicities – and thus Pauli algebras – are required.

Geometry also links the algebra to the physics of socio-economical systems. Consequently the mathematics of 4π periodicities – and thus Pauli algebras – can be applied to describe economic systems. A simple product engineering example will show how matrix inverses can be found by applying Pauli algebras.

But more and more introductory business mathematics textbooks present Generalized matrix inverses and Moore-Penrose matrix inverses as elementary part of the foundations of mathematical economics. A didactical approach to model these non-square matrix inverses with Pauli algebras will be presented. This didactical path will enable learners to understand that Moore-Penrose inverses only are scalar parts of more natural geometric matrix inverses which usually possess higher-dimensional terms, too.

At the turning point of this journey an interesting economathematical picture emerges: Problems which might be solved by using linear algebra can equally effective and sometimes even in a much simpler way be solved with Pauli algebra or generalized Pauli algebras.

1. A tremendous misunderstanding

Having sent my OHP slides about generalized matrix inverses based on Pauli algebra [1] to several colleagues, I received typical answers like: "The smell of relativistic quantum perfume is quite obvious. I liked it, but…" [3].

This is a tremendous misunderstanding! Pauli matrices and Pauli algebras do not have the nice smell of quantum mechanics. The mathematics of Pauli matrices and Pauli algebras will never smell out the riddles of the quantum world.

Pauli algebras show the spell of our classical world, and we will be able to spell out the riddles of classical physics by applying Pauli algebras.

Standard Pauli algebra not even is a relativistic mathematical language. It is the language of our classical, three-dimensional world with three purely spacelike directions.

Only higher-dimensional generalizations of Pauli algebra – like Dirac algebra – deliver us the mathematics of relativistic spacetimes. And again we will not be able to taste the smell of the quantum world. Pauli matrices and Dirac matrices are strictly classical objects.

2. A journey to Geometric Algebra

To understand the economathematical relevance of Pauli algebra and generalized Pauli algebras we will go on a journey along the following paths shown in figure 1.

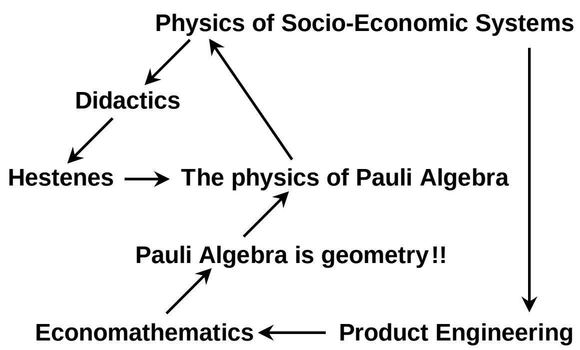

Fig.1: The journey to Geometric Algebra of this paper.

The sentence in the middle of figure 1 highlights the conceptual core of this journey: "Algebra equals geometry" – algebra will be geometry if the ideas of Grassmann and Pauli are blended into one strong mathematical tool. Therefore this mathematical tool is called Geometric Algebra.

To show this fascinating relation between algebra and geometry a simple product engineering example will be discussed in the following section.

3. A simple product engineering example

In this first example the number of raw materials is identical to the number of final products, thus resulting in a square demand matrix.

Non-square demand matrices with more raw material than final products will be discussed later (see sections 8 & 9) in a second example problem.

First Example: Problem

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

3 units of R_1 and 5 units of R_2 to produce 1 unit of P_1

2 units of R_1 and 4 units of R_2 to produce 1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed in the production process.

Fig.2: First product engineering problem of part VII of the BSEL Geometric Algebra Crash Course [1, p. 16].

The unknown quantities x of the first final product P_1 and y of the second final product P_2 can be found by solving the following system of two linear equations:

$$3\,x + 2\,y = 120$$
$$5\,x + 4\,y = 220 \qquad \{1\}$$

To implement to conceptual core of our economathematical journey, these algebraic equations {1} can now be geometricized by relating them to directions in space.

$3\,x + 2\,y = 120$ will point into a first direction

$5\,x + 4\,y = 220$ will point into a second direction

The base vectors which represent these directions will be named – historically appropriate – σ_x and σ_y (and later σ_z for a third direction). The wanted geometrization can then be achieved by simply multiplying the two linear equations {1} by these base vectors:

$$3\,x\,\sigma_x + 2\,y\,\sigma_x = 120\,\sigma_x$$
$$5\,x\,\sigma_y + 4\,y\,\sigma_y = 220\,\sigma_y \qquad \{2\}$$

But what are base vectors? This is a controversial question, as there is a struggle – or even a war – going on about the meaning of the mathematical instruments we use to describe the world.

Gian-Carlo Rota sadly exclaimed: "The neglect of exterior algebra is the mathematical tragedy of this century. … (Admired mathematicians) made sure that … no hint of mystifying concoctions of that crackpot Grassmann would be given out. … (Other admired mathematicians) did not believe in geometry and failed to provide the translation of geometry into algebra that exterior algebra makes possible. … Meanwhile we have to bear with mathematicians who are exterior algebra-blind" [4, p. 232/233].

Thus the first step to ensure that mathematicians, physicists and other scientists do not only learn castrated, exterior algebra-blind mathematics but encounter the full wealth of a complete and rich mathematical world, should be to present these mystifying concoctions of Grassmann and his theory of extensions.

What are now base vectors according to Grassmann? They surely have unit lengths

$[e_r \mid e_r] = 1$ [5, p. 378, *first equation*],

identical base vectors are parallel to each other

$[e_r\;\; e_r] = 0$ [5, p. 378, *third equation*],

and different base vectors are perpendicular to each other

$[e_r \mid e_s] = 0$ [5, p. 378, *second equation*],

$[e_r\;\; e_s] = -[e_s\;\; e_r]$ [5, p. 378, *fourth equation*].

In modern form these inner and outer products are written according to the Pauli notation as

$$\sigma_i \bullet \sigma_i = 1 \qquad \{3a\}$$
$$\sigma_i \wedge \sigma_i = 0 \qquad \{3b\}$$
$$\sigma_i \bullet \sigma_j = 0 \qquad i \neq j \qquad \{3c\}$$
$$\sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i \qquad i \neq j \qquad \{3d\}$$

And there can be no doubt that with these equations Grassmann already had written down the basic foundations of Pauli algebra:

$$\sigma_i^2 = (\sigma_i \bullet \sigma_i + \sigma_i \wedge \sigma_i) = 1 \qquad \{4a\}$$
$$\sigma_i \sigma_j = (\sigma_i \bullet \sigma_j + \sigma_i \wedge \sigma_j) = -\sigma_j \sigma_i \qquad i \neq j \qquad \{4b\}$$

It was this crackpot Grassmann who already had identified Pauli matrices (which had not been written as matrices in these days) with base vectors.

Of course it is allowed to say as a physicist or as another non-mathematician scientist: Pauli matrices **are** base vectors of three-dimensional, Euclidean space. Only admired mathematicians who do not believe in geometry must be a little bit more reluctant and have to say: Pauli matrices **represent** base vectors or three-dimensional, Euclidean space.

But it does not matter, how you speak about Grassmann's findings, the central fact is clear: This is not the nice and bewitching smell of quantum mechanics, this is the spell of our classical world.

These findings of Grassmann surely are disturbing for scientists who had been educated in an abstract quantum-mechanical way and who were told to neglect the mathematics of the classical world. Therefore Cambridge physicists Gull, Lasenby and Doran conclude that "our present thinking about quantum mechanics is infested with the deepest misconceptions" [6, p. 1185]. And the fact that "the familiar Pauli matrix relation … is now nothing more than an expression of the geometric product of orthonormal vectors" will then "cause the greatest intellectual shock" [6, p. 1184].

4. Solving the product engineering problem

What is now Grassmann's geometry-based way to find the solution of the given product engineering problem?

To understand his strategy we simply have to look into the first edition of his important theory of extensions [7]. It surely takes some time to get used to his peculiar style of writing and to grasp how "… the applicability of outer multiplication emerges with such a striking determination and firmness, … that algebra will gain a substantial different shape" [1] [7, p. 71].

But after having overcome all linguistic and syntax problems, a magnificent new mathematical world opens: Grassmann first constructed the two coefficient vectors

$$\mathbf{a} = 3\,\sigma_x + 5\,\sigma_y$$
$$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y \qquad \{5\}$$

and the resulting vector

$$\mathbf{r} = 120\,\sigma_x + 220\,\sigma_y \qquad \{6\}$$

then computed the oriented areas of the three parallelograms which can be formed by these vectors:

$$\mathbf{a} \wedge \mathbf{b} = 2\,\sigma_x\sigma_y$$
$$\mathbf{r} \wedge \mathbf{b} = 40\,\sigma_x\sigma_y \qquad \{7\}$$
$$\mathbf{a} \wedge \mathbf{r} = 60\,\sigma_x\sigma_y$$

and finally divided these oriented areas to get the two solution values x, y by the Geometric Algebra equivalent of Cramer's rule:

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = 20$$
$$y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = 30 \qquad \{8\}$$

As all oriented areas are parallel to each other, the post-divisions {9} will of course give the same results compared to the pre-divisions {8}

$$x = (\mathbf{r} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = 20$$
$$y = (\mathbf{a} \wedge \mathbf{r})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = 30 \qquad \{9\}$$

The expected answer to the product engineering problem of figure 2 is shown in figure 3.

In Geometric Algebra based on Grassmann's ideas the outer products of eqs. {7} have a clear and precise geometric meaning – they are oriented areas.

[1] Grassmann's words in German: "Aber desto interessanter ist es, zu bemerken, wie in der Algebra (…) auch die Anwendbarkeit der äusseren Multiplikation mit einer so schlagenden Entschiedenheit heraustritt, dass ich wohl behaupten darf, es werde durch diese Anwendung auch die Algebra eine wesentlich veränderte Gestalt gewinnen" [7, pp. 70/71].

First Example: Answer

If exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Fig.3: Answer of the first product engineering problem.

In contract to that standard textbooks of matrix algebra usually do not discuss this geometric background. Instead they present non-dimensional, purely algebraic scalar values, called determinants.

Therefore Arnold desperately says: "The determinant of a matrix is an (oriented) volume of the parallelepiped whose edges are its columns. If the students are told this secret – which is carefully hidden in the purified algebraic education, – then the whole theory of determinants becomes a clear chapter of the theory of poly-linear forms. If determinants are defined otherwise, then any sensible person will forever hate all the determinants …" [8].

Standard textbooks of matrix algebra thus create students who will forever hate determinants.

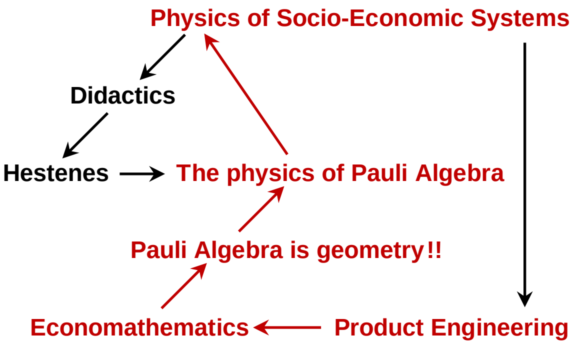

Fig.4: The first, red marked part of our Geometric Algebra journey.

At the end of this first part of our journey (see red path of fig. 4) the generalization of this solution strategy, already presented by Grassmann in [7, § 45, pp. 70-72], will be discussed.

If a system of n linear equations with n unknown variables x_i, $i \in \{1, 2, \dots, n\}$ is given, n coefficient vectors $\mathbf{a_i}$

$$\mathbf{a_i} = a_{1i}\,\sigma_1 + a_{2i}\,\sigma_2 + \dots + a_{ni}\,\sigma_n \qquad \{10\}$$

and the resulting vector

$$\mathbf{r} = r_1\,\sigma_1 + r_2\,\sigma_2 + \dots + r_n\,\sigma_n \qquad \{11\}$$

can be constructed.

Then the oriented hyper-volumes of the n-dimensional hyper-parallelepipeds formed of all coefficient vectors and the resulting vector

$$V_{det} = \mathbf{a_1} \wedge \mathbf{a_2} \wedge \dots \wedge \mathbf{a_i} \wedge \dots \wedge \mathbf{a_n} \qquad \{12\}$$
$$V_i = \mathbf{a_1} \wedge \mathbf{a_2} \wedge \dots \wedge \mathbf{r} \wedge \dots \wedge \mathbf{a_n} \qquad \{13\}$$

can be found. The oriented hyper-volume of eq. {12} represents the determinant of the coefficient matrix **A**

$$\det \mathbf{A} = \mathbf{V}_{\det}\, \sigma_n \ldots \sigma_3 \sigma_2 \sigma_1 \qquad \{14\}$$

while the n different oriented hyper-volumes $\mathbf{V}_i$ of eqs. {13} (in which the coefficient vectors $\mathbf{a_i}$ are replaced by the resulting vector **r)** represent the numerator determinants of Cramer's matrices $\mathbf{A_i}$

$$\det \mathbf{A_i} = \mathbf{V}_i\, \sigma_n \ldots \sigma_3 \sigma_2 \sigma_1 \qquad \{15\}$$

Finally the n solution values x_i of the system of linear equations can be computed by simple divisions of eqs. {12} and {13} or of eqs. {14} and {15}:

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}} = \mathbf{V}_{\det}^{-1}\, \mathbf{V}_i = \mathbf{V}_i\, \mathbf{V}_{\det}^{-1} \qquad \{16\}$$

And as all oriented hyper-volumes are parallel to each other, pre-division and post-division {16, *right side*} will get identical results again.

5. Didactical intermezzo: The Dirac belt trick

According to Hestenes, geometry links the algebra to the physical world, so a mathematical language which reflects this linkage is required to describe the physical world [9].

Because of that the journey of economathematics told in this paper will now take a turn to the byroad of didactics (see blue path of figure 5) before the route on the main road (see green path of figure 5) will be continued.

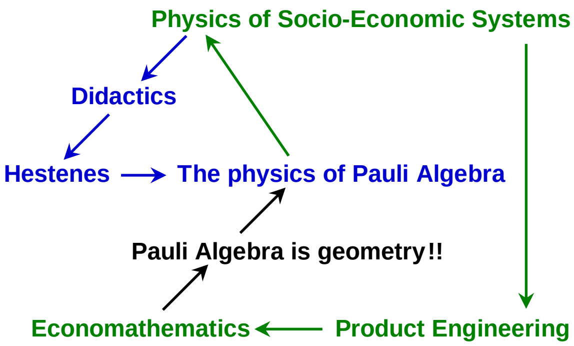

Fig.5: Second, blue marked and third, green marked parts of our Geometric Algebra journey.

It is obvious that human eyes are blind to spin – and we all are blind to spin, because the retina of a human eye is a two-dimensional curved plane. Therefore only information about a two-dimensional, distorted picture reaches our brain. The visual and thus the mathematical models our brain constructs are sometimes incomplete, they are partly unsuited for some situations or they are even faulty.

Dirac's belt trick [10, p. 1149, fig. 41.6], [11, p. 12/13] (or scissor problem [12, p. 43, fig. 1-13]) shows this incompleteness and faultiness as our human brains usually do not expect to have a topological situation which is identical to the original situation if an object attached with strings to the surroundings is rotated by 4 π. But they are identical and the attached object of figure 6 indeed equals a spin ½ particle. The necessary conclusions we should draw are straightforward:

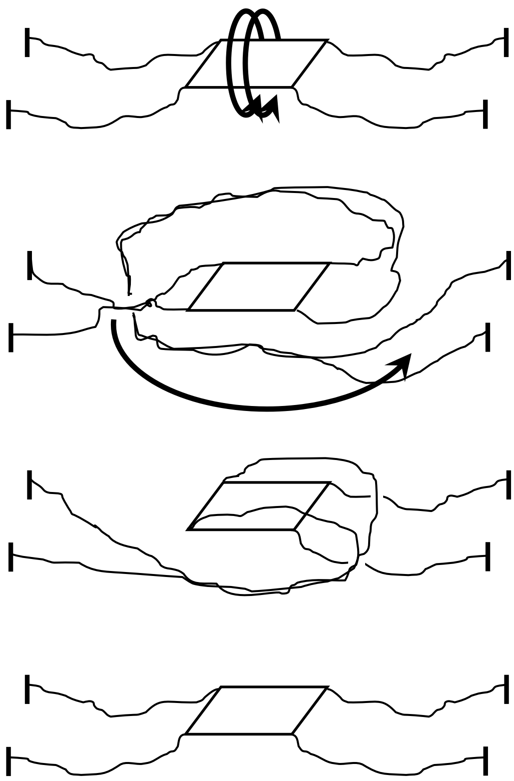

Fig.6: Visualization of Dirac's belt trick.

- 4 π symmetries are nothing exclusively quantum mechanical,
- 4 π symmetries are an essential part of our everyday world (our space) we live in.
- We need an appropriate mathematical language to describe 4 π symmetries.

This language is Geometric Algebra. The basic objects of three-dimensional Euclidean space are then conclusively encoded by Pauli Algebra: Pauli matrices represent base vectors, and products of two different base vectors are base bivectors, which represent the oriented unit area elements of eqs. {4b}, shown in figure 7.

Hestenes then not only concludes that "mathematics is too important to be left to the mathematicians", but also states that "the most impressive benefits of Geometric Algebra arise from surprising new insights into the structure of physics" [9, p. 107].

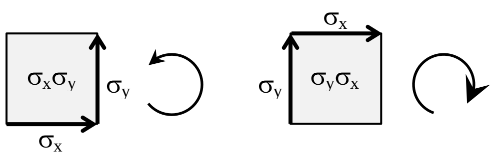

Fig.7: Bivector anti-commutativity $\sigma_x\sigma_y = -\,\sigma_y\sigma_x$ describes algebraically the geometric orientation of area elements.

6. Inverse matrices

New insights into the structure of the given product engineering problem of figure 2 can be gained by solving this problem with the help of the inverse (or inverses) of the demand matrix **A**

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} \qquad \{17\}$$

To find this inverse (or more precise: these two identical inverses $\mathbf{A}^{-1}$ as pre-inverse and $\underline{\mathbf{A}}^{-1}$ as post-inverse),

$$\mathbf{A}^{-1} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \underline{\mathbf{A}}^{-1} = \begin{pmatrix} \underline{x}_1 & \underline{x}_2 \\ \underline{y}_1 & \underline{y}_2 \end{pmatrix} \qquad \{18\}$$

the two systems of linear equations

$$\mathbf{a}\, x_1 + \mathbf{b}\, y_1 = \sigma_x$$
$$\mathbf{a}\, x_2 + \mathbf{b}\, y_2 = \sigma_y \qquad \{19\}$$

have to be solved. According to eqs. {8} and {9}, the solutions will be

$$\begin{aligned} x_1 &= (\mathbf{a} \wedge \mathbf{b})^{-1} (\sigma_x \wedge \mathbf{b}) \\ &= \underline{x}_1 = (\sigma_x \wedge \mathbf{b})(\mathbf{a} \wedge \mathbf{b})^{-1} = \; 2 \\ y_1 &= (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \sigma_x) \\ &= \underline{y}_1 = (\mathbf{a} \wedge \sigma_x)(\mathbf{a} \wedge \mathbf{b})^{-1} = -2.5 \\ x_2 &= (\mathbf{a} \wedge \mathbf{b})^{-1} (\sigma_y \wedge \mathbf{b}) \\ &= \underline{x}_2 = (\sigma_y \wedge \mathbf{b})(\mathbf{a} \wedge \mathbf{b})^{-1} = -1 \\ y_2 &= (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \sigma_y) \\ &= \underline{y}_2 = (\mathbf{a} \wedge \sigma_y)(\mathbf{a} \wedge \mathbf{b})^{-1} = \; 1.5 \end{aligned} \qquad \{20\}$$

Thus the two identical inverse matrices {18} are

$$\mathbf{A}^{-1} = \underline{\mathbf{A}}^{-1} = \begin{pmatrix} 2 & -1 \\ -5/2 & 3/2 \end{pmatrix} \qquad \{21\}$$

and the production vector of the already known solution according to figure 3 can be computed to

$$\vec{p} = \mathbf{A}^{-1} \begin{pmatrix} 120 \\ 220 \end{pmatrix} = \underline{\mathbf{A}}^{-1} \begin{pmatrix} 120 \\ 220 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix} \qquad \{22\}$$

Again this can be generalized, and the equations to find the elements of the inverse(s) $\mathbf{A}^{-1} = \underline{\mathbf{A}}^{-1} = (x_{ij})$ of an (n x n) square matrix **A** can be constructed analogous to eqs. {12} – {16}:

$$\mathbf{V}_{det} = \mathbf{a_1} \wedge \mathbf{a_2} \wedge \ldots \wedge \mathbf{a_i} \wedge \ldots \wedge \mathbf{a_n} \qquad \{23\}$$

$$\mathbf{V}_{ij} = \mathbf{a_1} \wedge \mathbf{a_2} \wedge \ldots \wedge \sigma_j \wedge \ldots \wedge \mathbf{a_n} \qquad \{24\}$$

$$x_{ij} = \mathbf{V}_{det}^{-1}\, \mathbf{V}_{ij} = \underline{x}_{ij} = \mathbf{V}_{ij}\, \mathbf{V}_{det}^{-1} \qquad \{25\}$$

And as all oriented hyper-volumes are parallel to each other, pre-divisions and post-divisions of eq. {25} will of course get identical results again.

7. Economathematical intermezzo: Generalized matrix inverses

More and more introductory business mathematics textbooks present Moore-Penrose generalized matrix inverses as elementary part of the foundations of mathematical economics, see for example [13, chap. 7] or [14, chap. 6].

And more and more often Generalized matrix inverses are regularly discussed in introductory courses e.g. at University of Applied Sciences Schmalkalden, at Technical University Dortmund, or at Leibniz University Hannover [14].

Unfortunately in most of these textbooks and business mathematics courses Generalized matrix inverses are introduced by purely algebraic reasoning. Usually the discussion of Moore-Penrose generalized matrix inverses is based on the four – algebraic – Moore-Penrose conditions:

$$\mathbf{A}\,\mathbf{A}^{+}\mathbf{A} = \mathbf{A} \qquad \mathbf{A}^{+}\mathbf{A}\,\mathbf{A}^{+} = \mathbf{A}^{+}$$
$$(\mathbf{A}^{+}\mathbf{A})^{T} = \mathbf{A}^{+}\mathbf{A} \qquad (\mathbf{A}\,\mathbf{A}^{+})^{T} = \mathbf{A}\,\mathbf{A}^{+} \qquad \{26\}$$

This restriction to a purely algebraic reasoning is a didactical disadvantage, as geometrical insights are then neglected and the geometry of our physical and of our economathematical world is ignored.

To open this situation for a wider and more geometrical view and for changing the tragedy described by Rota into a happy ending, Generalized matrix inverses will be discussed by applying exterior algebra in the following.

And to show how it is possible to model non-square matrix inverses with Pauli algebras, the product engineering example of section 3 will now be extended.

8. A second product engineering example

In this second example problem there are more raw materials than final products, thus resulting in a non-square demand matrix.

Second Example: Problem

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

3 units of R_1, 5 units of R_2, and 4 units of R_3 to produce 1 unit of P_1

2 units of R_1, 4 units of R_2, and 8 units of R_3 to produce 1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 120 units of the first raw material R_1, 220 units of the second raw material R_2, and 320 units of the third raw material R_3 are consumed in the production process.

Fig.8: Second product engineering problem of part VII of the BSEL Geometric Algebra Crash Course [1, p. 33].

The unknown quantities x of the first final product P_1 and y of the second final product P_2 can be found by solving the following overconstrained system of three linear equations:

$$\begin{aligned} 3x + 2y &= 120 \\ 5x + 4y &= 220 \\ 4x + 8y &= 320 \end{aligned} \qquad \{27\}$$

To implement the conceptual core of our economathematical journey again, these algebraic equations {27} can be geometricized by relating them to directions in space, which were represented by now three base vectors or Pauli matrices:

$$3\,x\,\sigma_x + 2\,y\,\sigma_x = 120\,\sigma_x$$
$$5\,x\,\sigma_y + 4\,y\,\sigma_y = 220\,\sigma_y \qquad \{28\}$$
$$4\,x\,\sigma_z + 8\,y\,\sigma_z = 320\,\sigma_z$$

The two coefficient vectors **a**, **b** and the resulting vector **r** then are:

$$\mathbf{a} = 3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z$$
$$\mathbf{b} = 2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z \qquad \{29\}$$
$$\mathbf{r} = 120\,\sigma_x + 220\,\sigma_y + 320\,\sigma_z$$

If the two coefficient vectors are not linearly dependent and if the resulting vector and the two coefficient vectors are linearly dependent, the system of linear equations will be consistent and a solution will exist.

This algebraic condition can also be stated geometrically: If the two coefficient vectors do not point into the same direction and if the two coefficient vectors and the resulting vector are lying in the same plane, the system of linear equations will be consistent and a solution will exist.

Obviously, the two coefficient vectors are not parallel as their outer product does not equal zero. And all vectors are lying in the same plane, as the bivector directions of their outer products {30} are identical:

$$\mathbf{a} \wedge \mathbf{b} = 2\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z - 16\,\sigma_z\sigma_x = 2\,(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)$$
$$\mathbf{r} \wedge \mathbf{b} = 40\,\sigma_x\sigma_y + 480\,\sigma_y\sigma_z - 320\,\sigma_z\sigma_x = 40\,(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x) \qquad \{30\}$$
$$\mathbf{a} \wedge \mathbf{r} = 60\,\sigma_x\sigma_y + 720\,\sigma_y\sigma_z - 480\,\sigma_z\sigma_x = 60\,(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)$$

Pre- or post-divisions {31} of the outer products will then give the quantities produced (see fig. 9).

$$x = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{r}\wedge\mathbf{b}) = (\mathbf{r}\wedge\mathbf{b})\,(\mathbf{a}\wedge\mathbf{b})^{-1} = 20$$
$$y = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{a}\wedge\mathbf{r}) = (\mathbf{a}\wedge\mathbf{r})\,(\mathbf{a}\wedge\mathbf{b})^{-1} = 30 \qquad \{31\}$$

Second Example: Answer

If exactly 120 units of the first raw material R_1, 220 units of the second raw material R_2, and 320 units of the third raw material R_3 and are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Fig.9: Answer of the second product engineering problem.

Of course the outer products {30} represent determinants. If these determinants are defined according to eq. {14}, they are no longer scalars, but vectors.

$$\det\begin{pmatrix}3 & 2\\ 5 & 4\\ 4 & 8\end{pmatrix} = (\mathbf{a}\wedge\mathbf{r})\,\sigma_z\sigma_y\sigma_x \qquad \{32\}$$
$$= 24\,\sigma_x - 16\,\sigma_y + 2\,\sigma_z$$

Therefore Grassmann's solution strategy shown in eqs. {10} to {16} still is valid for non-square matrices and overconstrained systems of linear equations.

9. Finding non-square matrix inverses

In a similar way, eqs. {17} to {25} still are valid with the only difference, that the pre-inverse $\mathbf{A}^{-1}$ and the post-inverse $\underline{\mathbf{A}}^{-1}$ are no longer identical. We now have to carefully distinguish between pre-multiplication and post-multiplication and between pre-division and post-division.

To find these two Generalized matrix inverses of the non-square matrix **A**

$$\mathbf{A} = \begin{pmatrix}3 & 2\\ 5 & 4\\ 4 & 8\end{pmatrix} \qquad \{33\}$$

the three (double) systems of linear equations

$$\mathbf{a}\,x_1 + \mathbf{b}\,y_1 = \mathbf{a}\,\underline{x}_1 + \mathbf{b}\,\underline{y}_1 = \sigma_x$$
$$\mathbf{a}\,x_2 + \mathbf{b}\,y_2 = \mathbf{a}\,\underline{x}_2 + \mathbf{b}\,\underline{y}_2 = \sigma_y \qquad \{34\}$$
$$\mathbf{a}\,x_3 + \mathbf{b}\,y_3 = \mathbf{a}\,\underline{x}_3 + \mathbf{b}\,\underline{y}_3 = \sigma_z$$

now have to be solved to get the 12 elements of

$$\mathbf{A}^{-1} = \begin{pmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix} \neq \underline{\mathbf{A}}^{-1} = \begin{pmatrix}\underline{x}_1 & \underline{x}_2 & \underline{x}_3\\ \underline{y}_1 & \underline{y}_2 & \underline{y}_3\end{pmatrix} \qquad \{35\}$$

With the help of the duals **N**, **M** of the two coefficient vectors **a**, **b** (which can be found by multiplying by the unit trivector or pseudoscalar $I = \sigma_x\sigma_y\sigma_z$)

$$\mathbf{N} = I\,\mathbf{a} = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$
$$\mathbf{M} = I\,\mathbf{b} = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x \qquad \{36\}$$

the elements of the Generalized matrix inverses can be written as

$$x_1 = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\sigma_x\wedge\mathbf{b}) = \frac{1}{418}\,(\;68 - 12\,\mathbf{M})$$
$$y_1 = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{a}\wedge\sigma_x) = \frac{1}{418}\,(-37 + 12\,\mathbf{N})$$
$$x_2 = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\sigma_y\wedge\mathbf{b}) = \frac{1}{418}\,(\;94 + 8\,\mathbf{M})$$
$$y_2 = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{a}\wedge\sigma_y) = \frac{1}{418}\,(-45 - 8\,\mathbf{N})$$
$$x_3 = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\sigma_z\wedge\mathbf{b}) = \frac{1}{418}\,(-64 - \mathbf{M})$$
$$y_3 = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{a}\wedge\sigma_z) = \frac{1}{418}\,(84 + \mathbf{N}) \qquad \{37\}$$

or with reversed bivector terms as

$$\underline{x}_1 = (\sigma_x\wedge\mathbf{b})\,(\mathbf{a}\wedge\mathbf{b})^{-1} = \frac{1}{418}\,(\;68 + 12\,\mathbf{M})$$
$$\underline{y}_1 = (\mathbf{a}\wedge\sigma_x)\,(\mathbf{a}\wedge\mathbf{b})^{-1} = \frac{1}{418}\,(-37 - 12\,\mathbf{N})$$
$$\underline{x}_2 = (\sigma_y\wedge\mathbf{b})\,(\mathbf{a}\wedge\mathbf{b})^{-1} = \frac{1}{418}\,(\;94 - 8\,\mathbf{M})$$
$$\underline{y}_2 = (\mathbf{a}\wedge\sigma_y)\,(\mathbf{a}\wedge\mathbf{b})^{-1} = \frac{1}{418}\,(-45 + 8\,\mathbf{N})$$

$$\underline{x}_3 = (\sigma_z \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{418}\,(-64 + \mathbf{M})$$

$$\underline{y}_3 = (\mathbf{a} \wedge \sigma_z)\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{418}\,(84 - \mathbf{N}) \qquad \{38\}$$

Thus the two Generalized matrix inverses are

$$\mathbf{A}^{-1} = \frac{1}{418}\begin{pmatrix} 68-12\mathbf{M} & 94+8\mathbf{M} & -64-\mathbf{M} \\ -37+12\mathbf{N} & -45-8\mathbf{N} & 84+\mathbf{N} \end{pmatrix} \{39\}$$

and

$$\underline{\mathbf{A}}^{-1} = \frac{1}{418}\begin{pmatrix} 68+12\mathbf{M} & 94-8\mathbf{M} & -64+\mathbf{M} \\ -37-12\mathbf{N} & -45+8\mathbf{N} & 84-\mathbf{N} \end{pmatrix} \{40\}$$

Again the production vector of the already known solution according to figure 9 can be computed to

$$\vec{p} = \mathbf{A}^{-1}\begin{pmatrix}120\\220\\320\end{pmatrix} = \underline{\mathbf{A}}^{-1}\begin{pmatrix}120\\220\\320\end{pmatrix} = \begin{pmatrix}20\\30\end{pmatrix} \qquad \{41\}$$

It is interesting to see, that all bivector terms cancel because

$$-12 \cdot 120 + 8 \cdot 220 - 320 = 0 \qquad \{42\}$$

and the overconstrained consistent system of linear equations results as expected with pure scalar solution values.

This mathematical procedure can be generalized again, and the equations to find the elements of the inverses $\mathbf{A}^{-1} = (x_{ij}) \neq \underline{\mathbf{A}}^{-1} = (\underline{x}_{ij})$ of a non-square matrix $\mathbf{A}$ can be constructed analogous to eqs. {23} – {25}:

$$\mathbf{V}_{det} = \mathbf{a_1} \wedge \mathbf{a_2} \wedge \ldots \wedge \mathbf{a_i} \wedge \ldots \wedge \mathbf{a_n} \qquad \{43\}$$

$$\mathbf{V}_{ij} = \mathbf{a_1} \wedge \mathbf{a_2} \wedge \ldots \wedge \sigma_j \wedge \ldots \wedge \mathbf{a_n} \qquad \{44\}$$

$$x_{ij} = \mathbf{V}_{det}^{-1}\,\mathbf{V}_{ij} \neq \underline{x}_{ij} = \mathbf{V}_{ij}\,\mathbf{V}_{det}^{-1} \qquad \{45\}$$

But now the oriented hyper-volumes are not parallel to each other, and pre-divisions and post-divisions of eq. {45} will get different results.

10. Constructing Moore-Penrose matrix inverses

Obviously the Generalized matrix inverses $\mathbf{A}^{-1}$ and $\underline{\mathbf{A}}^{-1}$ are not Moore-Penrose matrix inverses $\mathbf{A}^{+}$. They can be called Pauli Algebra generalized matrix inverses.

It can be seen that only the scalar parts of Pauli Algebra generalized matrix inverses are identical to Moore-Penrose matrix inverses. Therefore Moore-Penrose matrix inverses $\mathbf{A}^{+}$ can be defined as

$$\mathbf{A}^{+} = \frac{1}{2}\,(\mathbf{A}^{-1} + \underline{\mathbf{A}}^{-1}) \qquad \{46\}$$

$$= \frac{1}{2}\begin{pmatrix} x_1 + \underline{x}_1 & x_2 + \underline{x}_2 & x_3 + \underline{x}_3 \\ y_1 + \underline{y}_1 & y_2 + \underline{y}_2 & y_3 + \underline{y}_3 \end{pmatrix}$$

and the second product engineering problem will have the following Moore-Penrose matrix inverse:

$$\mathbf{A}^{+} = \begin{pmatrix} 68 & 94 & -64 \\ -37 & -45 & 84 \end{pmatrix} \qquad \{47\}$$

Just for fun an anti-Moore-Penrose matrix inverse $\mathbf{A}^{-}$ can also be found

$$\mathbf{A}^{-} = \frac{1}{2}\,(\mathbf{A}^{-1} - \underline{\mathbf{A}}^{-1}) \qquad \{48\}$$

$$= \frac{1}{2}\begin{pmatrix} x_1 - \underline{x}_1 & x_2 - \underline{x}_2 & x_3 - \underline{x}_3 \\ y_1 - \underline{y}_1 & y_2 - \underline{y}_2 & y_3 - \underline{y}_3 \end{pmatrix}$$

and the second product engineering problem will have the following anti-Moore-Penrose matrix inverse:

$$\mathbf{A}^{-} = \begin{pmatrix} -12\mathbf{M} & 8\mathbf{M} & -\mathbf{M} \\ 12\mathbf{N} & -8\mathbf{N} & \mathbf{N} \end{pmatrix} \qquad \{49\}$$

But while the Moore-Penrose matrix inverse $\mathbf{A}^{+}$ can be used to get the solution of figure 9 again

$$\vec{p} = \mathbf{A}^{+}\begin{pmatrix}120\\220\\320\end{pmatrix} = \begin{pmatrix}20\\30\end{pmatrix} \qquad \{50\}$$

anti-Moore-Penrose matrix inverse $\mathbf{A}^{-}$ seem to be rather useless as all values {42} disappear to zero

$$\mathbf{A}^{-}\begin{pmatrix}120\\220\\320\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad \{51\}$$

11. Teaching Pauli Algebra generalized matrix inverses

Pauli Algebra generalized matrix inverses have been taught as seventh part [1], [2] of a Geometric algebra lecture series [15], [16], [17], [18] with English-speaking students at BSEL at winter semester 2017/ 2018. The slides of the different lecture series parts can be downloaded from *PhyDid*, see electronic resources at [19].

As in this special winter semester lecture time was too short to discuss all previous lecture slides in detail, a quick start into Geometric Algebra was made by using GAALOP as a Geometric Algebra pocket calculator substitute [20], [21].

After this quick start, the students had been able to solve systems of linear equations and to find inverses of square matrices by applying Geometric Algebra solution strategies (see sections 4 and 6 of this paper).

Thus a good basic knowledge of Geometric Algebra was already established before the discussion of non-square matrix inverses started which required the time of two lesson hours (2 x 45 min.).

To be able to understand and to comprehend all steps of finding Pauli Algebra generalized matrix inverse elements, only pre-division non-square matrix elements {37} have been discussed at this course [21, pp. 38 – 43].

Based on the vivid discussion and on comments the students made, it can be concluded that this seventh part of the lecture series was successful.

12. This is geometry

We are living in a geometrical world. Therefore we can use geometry to solve systems of linear equations like

$$x\,\mathbf{a} + y\,\mathbf{b} = \mathbf{r} \qquad \{52\}$$

of the first or second example problem (figure 2 & 8). For example, we only have to reflect eq. {52} at an axis into the direction of coefficient vector **a** by sandwiching [22], [23] eq.{52} by this vector **a**:

$$x\,a^2\,\mathbf{a} + y\,\mathbf{a}\,\mathbf{b}\,\mathbf{a} = \mathbf{a}\,\mathbf{r}\,\mathbf{a} \qquad \{53\}$$

Another geometrical operation, a dilation of eq. {52} can be constructed by simply multiplying by $\mathbf{a}^2 = a^2$ to get

$$x\,a^2\,\mathbf{a} + y\,a^2\,\mathbf{b} = a^2\,\mathbf{r} \qquad \{54\}$$

Now subtracting eq. {54} from eq. {53} results in

$$y\,(\mathbf{a}\,\mathbf{b}\,\mathbf{a} - a^2\,\mathbf{b}) = \mathbf{a}\,\mathbf{r}\,\mathbf{a} - a^2\,\mathbf{r} \qquad \{55\}$$

Thus the solution value y then can be found straightforward [18] as

$$y = (\mathbf{a}\,\mathbf{b}\,\mathbf{a} - a^2\,\mathbf{b})^{-1}\,(\mathbf{a}\,\mathbf{r}\,\mathbf{a} - a^2\,\mathbf{r}) \qquad \{56\}$$

In a similar, geometry-based way, the solution value x can be constructed as

$$x = (\mathbf{b}\,\mathbf{a}\,\mathbf{b} - b^2\,\mathbf{a})^{-1}\,(\mathbf{b}\,\mathbf{r}\,\mathbf{b} - b^2\,\mathbf{r}) \qquad \{57\}$$

All this is geometry. And it explains, why Grassmann was able to write down his equations, which are hidden behind eqs. {56} and {57}, e.g.

$$x = (\mathbf{b}\,(\mathbf{a} \wedge \mathbf{b}))^{-1}\,(\mathbf{b}\,(\mathbf{r} \wedge \mathbf{b}))$$
$$= (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) \qquad \{58\}$$

As a physicist my mathematical world view is based on geometry. Therefore I even consider the elements of standard matrix inverses {20} or Generalized matrix inverses {37}, {38} as elements which can be seen through the spectacles of geometry analogous to eqs. {52} to {58}.

For example, the first element x_{11} of a matrix inverse is constructed geometrically by a reflection of the system of linear equations

$$x_{11}\,\mathbf{a} + x_{12}\,\mathbf{b} = \sigma_x \qquad \{59\}$$

at an axis into the direction of coefficient vector **b**

$$x_{11}\,\mathbf{b}\,\mathbf{a}\,\mathbf{b} + x_{12}\,b^2\,\mathbf{b} = \mathbf{b}\,\sigma_x\,\mathbf{b} \qquad \{60\}$$

and by a dilation of eq. {59} by the factor $\mathbf{b}^2 = b^2$

$$x_{11}\,b^2\,\mathbf{a} + x_{12}\,b^2\,\mathbf{b} = b^2\,\sigma_x \qquad \{61\}$$

Now again subtracting eq. {61} from eq. {60} results in

$$x_{11}\,(\mathbf{b}\,\mathbf{a}\,\mathbf{b} - b^2\,\mathbf{a}) = \mathbf{b}\,\sigma_x\,\mathbf{b} - b^2\,\sigma_x \qquad \{62\}$$

Thus the value of the first matrix element can be computed as

$$x_{11} = (\mathbf{b}\,\mathbf{a}\,\mathbf{b} - b^2\,\mathbf{a})^{-1}\,(\mathbf{b}\,\sigma_x\,\mathbf{b} - b^2\,\sigma_x)$$
$$= (\mathbf{b}\,(\mathbf{a} \wedge \mathbf{b}))^{-1}\,(\mathbf{b}\,(\sigma_x \wedge \mathbf{b})) \qquad \{63\}$$
$$= (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_x \wedge \mathbf{b})$$

And this is not only algebra. This is based on geometry. This is geometry.

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