The Geometry of Moore-Penrose Generalized Matrix Inverses

Martin Erik Horn

BSEL/HWR Berlin, FB 1 – Department of Business and Economics, FE Quantitative Methods

Hochschule für Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Email: *e_hornm@doz.hwr-berlin.de*

Economathematical Starting Point

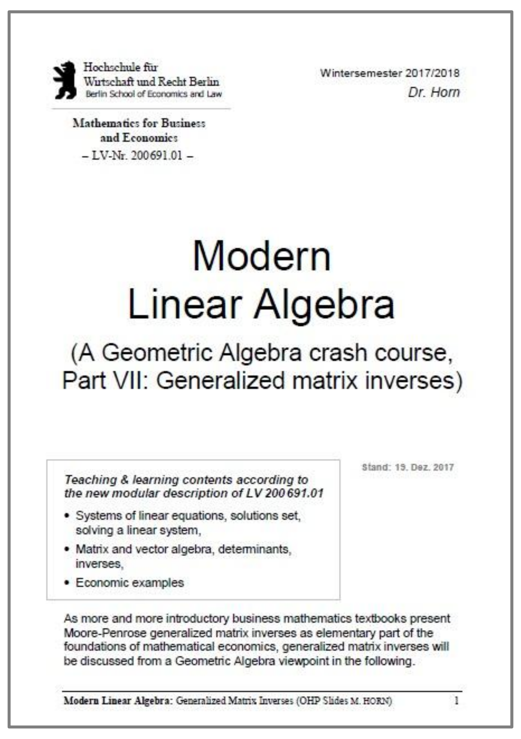
More and more introductory business mathematics textbooks present Moore-Penrose generalized matrix inverses as elementary part of the foundations of mathematical economics. Generalized matrix inverses are regularly discussed in introductory courses e.g. at FH Schmalkalden & TU Dortmund.

Didactical Problem

Most textbooks introduce generalized matrix inverses by purely algebraic reasoning and the discussion of Moore-Penrose generalized matrix inverses is based on the four Moore-Penrose conditions:

$$\mathbf{A}\,\mathbf{A}^+\mathbf{A} = \mathbf{A} \qquad (\mathbf{A}^+\mathbf{A})^T = \mathbf{A}^+\mathbf{A}$$
$$\mathbf{A}^+\mathbf{A}\,\mathbf{A}^+ = \mathbf{A}^+ \qquad (\mathbf{A}\,\mathbf{A}^+)^T = \mathbf{A}\,\mathbf{A}^+$$

But to give a complete picture of these mathematical structures it is helpful to introduce and to describe Moore-Penrose generalized matrix inverses also by using geometric representations based on the ideas of Grassmann's theory of extensions.



Hochschule für Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Mathematics for Business and Economics
– LV-Nr. 200691.01 –

Wintersemester 2017/2018
Dr. Horn

Modern
Linear Algebra

(A Geometric Algebra crash course, Part VII: Generalized matrix inverses)

Stand: 19. Dez. 2017

Teaching & learning contents according to the new modular description of LV 200691.01

- Systems of linear equations, solutions set, solving a linear system,
- Matrix and vector algebra, determinants, inverses,
- Economic examples

As more and more introductory business mathematics textbooks present Moore-Penrose generalized matrix inverses as elementary part of the foundations of mathematical economics, generalized matrix inverses will be discussed from a Geometric Algebra viewpoint in the following.

Modern Linear Algebra: Generalized Matrix Inverses (OHP Slides M. HORN) 1

After having discussed the basics of Geometric Algebra and the Geometric Algebra solution scheme of systems of linear equations with the students at previous lessons, a two hour lecture (2 x 45 min.) was required to introduce Pauli algebra generalized matrix inverses and Moore-Penrose generalized matrix inverses.

Overview of BSEL Geometric Algebra Crash Course:

Part 1: Basic Foundations of Geometric Algebra
Part 2: Solving Systems of Linear Equations with two or three Unknown Variables
Part 3: Direct Product & Solving higher-dimensional Systems of Linear Equations
— **winter semester 2014/2015** → DPG Wuppertal 2015

Part 4: Transformation of Coordinates & Gaussian Method of Solving Systems of Linear Equations
— **winter semester 2015/2016** → DPG Hannover 2016

Part 5: Eigenvalues and Eigenvectors
Part 6: Solving Systems of Linear Equations with Sandwich Products
— **winter semester 2016/2017** → DPG Dresden 2017

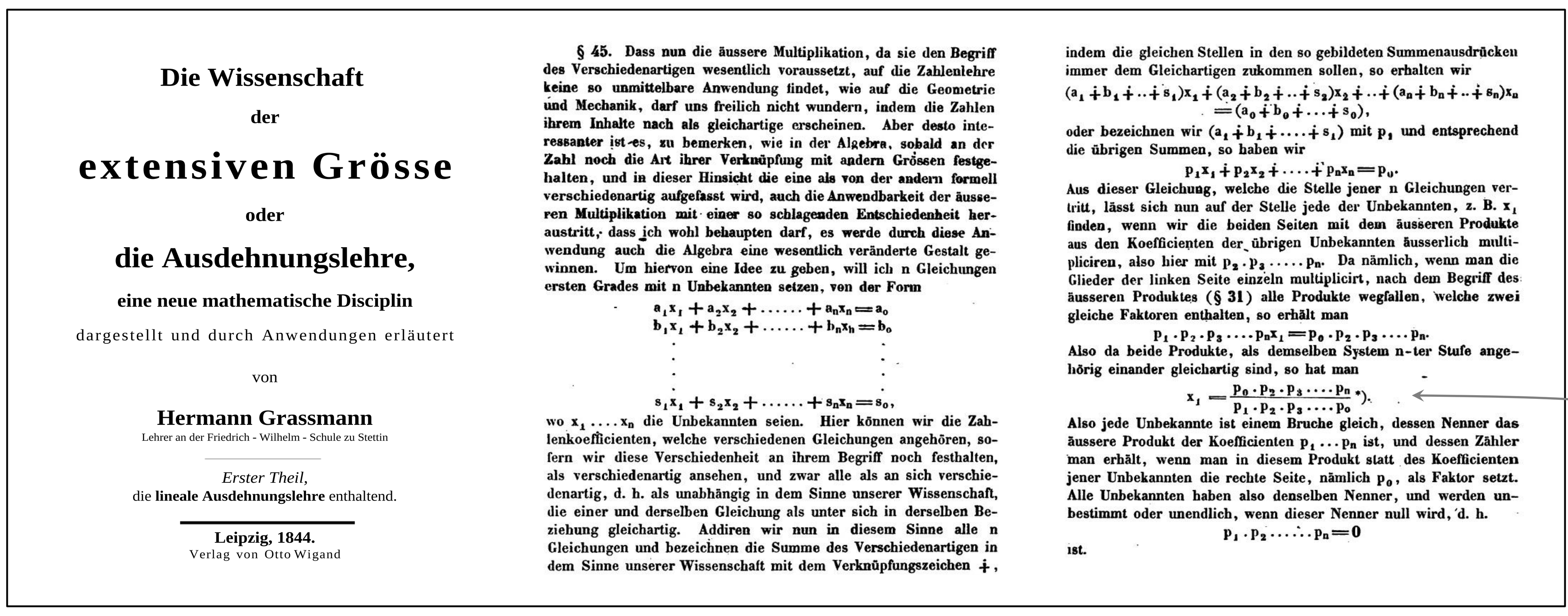
Part 7: Generalized Matrix Inverses
Addendum: Solving Systems of Linear Equations with the Geometric Algebra Algorithms Optimizer (GAALOP)
— **winter semester 2017/2018** → DPG Würzburg & Berlin 2018

Second Starting Point from the Perspective of Physics: Pauli Algebra and Generalized Pauli Algebra (Geometric Algebra)

Pauli Matrices represent base vectors of three-dimensional space.
Generalized Pauli Matrices represent base vectors of higher-dimensional spaces.

(And Dirac Matrices represent base vectors of four- or five-dimensional spacetimes.)

→ More about the foundation of this perspective will be discussed at the short talk SOE 9.2 "**Pauli Algebras in Economics: Economathematics from Geometry to Didactics and back**" at Tuesday, March 13, 2018, 10:45 – 11:00 h in room MA 001.



Die Wissenschaft
der
extensiven Grösse
oder
die Ausdehnungslehre,
eine neue mathematische Disciplin
dargestellt und durch Anwendungen erläutert
von
Hermann Grassmann
Lehrer an der Friedrich - Wilhelm - Schule zu Stettin

Erster Theil,
die **lineale Ausdehnungslehre** enthaltend.

Leipzig, 1844.
Verlag von Otto Wigand

§ 45. Dass nun die äussere Multiplikation, da sie den Begriff des Verschiedenartigen wesentlich voraussetzt, auf die Zahlenlehre keine so unmittelbare Anwendung findet, wie auf die Geometrie und Mechanik, darf uns freilich nicht wundern, indem die Zahlen ihrem Inhalte nach als gleichartige erscheinen. Aber desto interessanter ist es, zu bemerken, wie in der Algebra, sobald an der Zahl noch die Art ihrer Verknüpfung mit andern Grössen festgehalten, und in dieser Hinsicht die eine als von der andern formell verschiedenartig aufgefasst wird, auch die Anwendbarkeit der äusseren Multiplikation mit einer so schlagenden Entschiedenheit heraustritt, dass ich wohl behaupten darf, es werde durch diese Anwendung auch die Algebra eine wesentlich veränderte Gestalt gewinnen. Um hiervon eine Idee zu geben, will ich n Gleichungen ersten Grades mit n Unbekannten setzen, von der Form

$$\begin{aligned} a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n &= a_0 \\ b_1x_1 + b_2x_2 + \ldots\ldots + b_nx_n &= b_0 \\ &\vdots \\ s_1x_1 + s_2x_2 + \ldots\ldots + s_nx_n &= s_0, \end{aligned}$$

wo $x_1 \ldots . x_n$ die Unbekannten seien. Hier können wir die Zahlenkoefficienten, welche verschiedenen Gleichungen angehören, sofern wir diese Verschiedenheit an ihrem Begriff noch festhalten, als verschiedenartig ansehen, und zwar alle als an sich verschiedenartig, d. h. als unabhängig in dem Sinne unserer Wissenschaft, die einer und derselben Gleichung als unter sich in derselben Beziehung gleichartig. Addiren wir nun in diesem Sinne alle n Gleichungen und bezeichnen die Summe des Verschiedenartigen in dem Sinne unserer Wissenschaft mit dem Verknüpfungszeichen $\dotplus$, indem die gleichen Stellen in den so gebildeten Summenausdrücken immer dem Gleichartigen zukommen sollen, so erhalten wir

$$(a_1 \dotplus b_1 \dotplus .. \dotplus s_1)x_1 \dotplus (a_2 \dotplus b_2 \dotplus .. \dotplus s_2)x_2 \dotplus .. \dotplus (a_n \dotplus b_n \dotplus .. \dotplus s_n)x_n = (a_0 \dotplus b_0 \dotplus \ldots \dotplus s_0),$$

oder bezeichnen wir $(a_1 \dotplus b_1 \dotplus \ldots . \dotplus s_1)$ mit p_1 und entsprechend die übrigen Summen, so haben wir

$$p_1x_1 \dotplus p_2x_2 \dotplus \ldots . \dotplus p_nx_n = p_0.$$

Aus dieser Gleichung, welche die Stelle jener n Gleichungen vertritt, lässt sich nun auf der Stelle jede der Unbekannten, z. B. x_1 finden, wenn wir die beiden Seiten mit dem äusseren Produkte aus den Koefficienten der übrigen Unbekannten äusserlich multipliciren, also hier mit $p_2 . p_3 \ldots . p_n$. Da nämlich, wenn man die Glieder der linken Seite einzeln multiplicirt, nach dem Begriff des äusseren Produktes (§ 31) alle Produkte wegfallen, welche zwei gleiche Faktoren enthalten, so erhält man

$$p_1 . p_2 . p_3 \ldots . p_nx_1 = p_0 . p_2 . p_3 \ldots . p_n.$$

Also da beide Produkte, als demselben System n-ter Stufe angehörig einander gleichartig sind, so hat man

$$x_1 = \frac{p_0 . p_2 . p_3 \ldots . p_n}{p_1 . p_2 . p_3 \ldots . p_0} \, ^{*}).$$

Also jede Unbekannte ist einem Bruche gleich, dessen Nenner das äussere Produkt der Koefficienten $p_1 \ldots p_n$ ist, und dessen Zähler man erhält, wenn man in diesem Produkt statt des Koefficienten jener Unbekannten die rechte Seite, nämlich p_0, als Faktor setzt. Alle Unbekannten haben also denselben Nenner, und werden unbestimmt oder unendlich, wenn dieser Nenner null wird, d. h.

$$p_1 . p_2 \ldots . p_n = 0$$

ist.

With his theory of extensions Herman Günther Grassmann (1809 – 1877) already invented generalized Pauli Algebra and generalized Dirac Algebra. The solution of a system of linear equations can be found by applying his solution equations of the first edition of his ***Ausdehnungslehre*** of 1844.

Written in modern form, the solution of consistent systems of linear equations with two variables x, y

$$\mathbf{a}\,x + \mathbf{b}\,y = \mathbf{r}$$

can be found with the following solution equations:

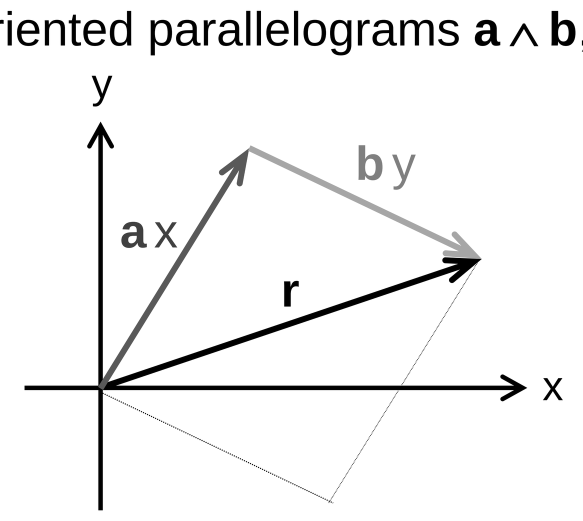
$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{r} \wedge \mathbf{b}) \quad \text{or} \quad x = (\mathbf{r} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1}$$
$$y = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{a} \wedge \mathbf{r}) \quad \text{or} \quad y = (\mathbf{a} \wedge \mathbf{r})\,(\mathbf{a} \wedge \mathbf{b})^{-1}$$

Geometric Interpretation of the Solution Equations

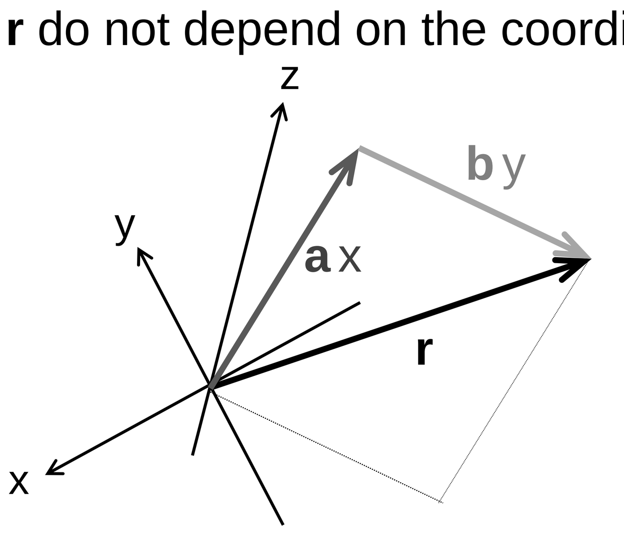
Outer products of two vectors can be interpreted as oriented area elements. And as a change of the coordinate system does not change the geometric situation, all ratios of the areas of the oriented parallelograms $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{r} \wedge \mathbf{b}$, and $\mathbf{a} \wedge \mathbf{r}$ do not depend on the coordinate system.

coefficient vectors:
$\mathbf{a} = a_1\,\sigma_x + a_2\,\sigma_y$
$\mathbf{b} = b_1\,\sigma_x + b_2\,\sigma_y$
resulting vector:
$\mathbf{r} = r_1\,\sigma_x + r_2\,\sigma_y$

$$a_1\,x + b_1\,y = r_1$$
$$a_2\,x + b_2\,y = r_2$$



⟹



$$a_1\,x + b_1\,y = r_1$$
$$a_2\,x + b_2\,y = r_2$$
$$a_3\,x + b_3\,y = r_3$$

coefficient vectors:
$\mathbf{a} = a_1\,\sigma_x + a_2\,\sigma_y + a_3\,\sigma_z$
$\mathbf{b} = b_1\,\sigma_x + b_2\,\sigma_y + b_3\,\sigma_z$
resulting vector:
$\mathbf{r} = r_1\,\sigma_x + r_2\,\sigma_y + r_3\,\sigma_z$

But a change of the coordinate system will change the algebraic description of the vectors, which now have three components instead of only two. Therefore the matrix inverse will have a different algebraic description, too. This generalized Pauli Algebra matrix inverse can be used to solve **consistent** systems of linear equations, even if theses systems have more equations than variables.

Conventional Matrix Inverses and Generalized Matrix Inverses

As Grassmann's solution equations can always be written as a matrix multiplication, the lead matrix of this matrix multiplication will be the matrix inverse:

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{r} \wedge \mathbf{b}) = (\mathbf{a} \wedge \mathbf{b})^{-1}\big((\sigma_x \wedge \mathbf{b})\,r_1 + (\sigma_y \wedge \mathbf{b})\,r_2\big)$$
$$y = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{a} \wedge \mathbf{r}) = (\mathbf{a} \wedge \mathbf{b})^{-1}\big((\mathbf{a} \wedge \sigma_x)\,r_1 + (\mathbf{a} \wedge \sigma_y)\,r_2\big)$$

Inverse of a 2 x 2 square matrix: $\mathbf{A}^{-1} = \dfrac{1}{\mathbf{a} \wedge \mathbf{b}} \begin{pmatrix} \sigma_x \wedge \mathbf{b} & \sigma_y \wedge \mathbf{b} \\ \mathbf{a} \wedge \sigma_x & \mathbf{a} \wedge \sigma_y \end{pmatrix}$

If $(\mathbf{a} \wedge \mathbf{b}) \neq 0$, every element of $\mathbf{A}^{-1}$ will be a scalar.

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{r} \wedge \mathbf{b}) = (\mathbf{a} \wedge \mathbf{b})^{-1}\big((\sigma_x \wedge \mathbf{b})\,r_1 + (\sigma_y \wedge \mathbf{b})\,r_2 + (\sigma_z \wedge \mathbf{b})\,r_3\big)$$
$$y = (\mathbf{a} \wedge \mathbf{b})^{-1}(\mathbf{a} \wedge \mathbf{r}) = (\mathbf{a} \wedge \mathbf{b})^{-1}\big((\mathbf{a} \wedge \sigma_x)\,r_1 + (\mathbf{a} \wedge \sigma_y)\,r_2 + (\mathbf{a} \wedge \sigma_z)\,r_3\big)$$

Inverse of a 3 x 2 matrix: $\mathbf{A}^{-1} = \dfrac{1}{\mathbf{a} \wedge \mathbf{b}} \begin{pmatrix} \sigma_x \wedge \mathbf{b} & \sigma_y \wedge \mathbf{b} & \sigma_z \wedge \mathbf{b} \\ \mathbf{a} \wedge \sigma_x & \mathbf{a} \wedge \sigma_y & \mathbf{a} \wedge \sigma_z \end{pmatrix}$
(if the bivector $\mathbf{a} \wedge \mathbf{b}$ is pre-multiplied from the left)

Or alternatively: $\underline{\mathbf{A}}^{-1} = \begin{pmatrix} \sigma_x \wedge \mathbf{b} & \sigma_y \wedge \mathbf{b} & \sigma_z \wedge \mathbf{b} \\ \mathbf{a} \wedge \sigma_x & \mathbf{a} \wedge \sigma_y & \mathbf{a} \wedge \sigma_z \end{pmatrix} \dfrac{1}{\mathbf{a} \wedge \mathbf{b}}$
(if the bivector $\mathbf{a} \wedge \mathbf{b}$ is post-multiplied from the right)

As all elements of $\mathbf{A}^{-1}$ are products of two different bivectors, every element will be a linear combination of a scalar and a bivector.

⇒ Pauli Algebra generalized matrix inverses are left-sided matrix inverses:

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathrm{I} \neq \mathbf{A}\,\mathbf{A}^{-1}$$

$$\mathbf{A}\,\mathbf{A}^{-1}\mathbf{A} = \mathbf{A} \qquad (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{A}^{-1}\mathbf{A}$$
$$\mathbf{A}^{-1}\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1} \qquad (\mathbf{A}\,\mathbf{A}^{-1})^T \neq \mathbf{A}\,\mathbf{A}^{-1} \quad \text{but} \quad (\mathbf{A}\,\langle \mathbf{A}^{-1}\rangle_0)^T = \mathbf{A}\,\langle \mathbf{A}^{-1}\rangle_0$$

The fourth Moore-Penrose condition is only met by the scalar terms of $\mathbf{A}^{-1}$.

Conclusion:

Moore-Penrose generalized matrix inverses $\mathbf{A}^+$ consist of the scalar terms of Pauli algebra generalized matrix inverses $\mathbf{A}^{-1}$, which usually possess higher-dimensional terms, too.

$$\mathbf{A}^+ = \begin{pmatrix} \langle(\mathbf{a}\wedge\mathbf{b})^{-1}(\sigma_x\wedge\mathbf{b})\rangle_0 & \langle(\mathbf{a}\wedge\mathbf{b})^{-1}(\sigma_y\wedge\mathbf{b})\rangle_0 & \langle(\mathbf{a}\wedge\mathbf{b})^{-1}(\sigma_z\wedge\mathbf{b})\rangle_0 \\ \langle(\mathbf{a}\wedge\mathbf{b})^{-1}(\mathbf{a}\wedge\sigma_x)\rangle_0 & \langle(\mathbf{a}\wedge\mathbf{b})^{-1}(\mathbf{a}\wedge\sigma_y)\rangle_0 & \langle(\mathbf{a}\wedge\mathbf{b})^{-1}(\mathbf{a}\wedge\sigma_z)\rangle_0 \end{pmatrix}$$

DPG-Frühjahrstagung 2018 – TU Berlin, SOE 7.2

The Geometry of Moore-Penrose Generalized Matrix Inverses (Example Problem)

Martin Erik Horn

BSEL/HWR Berlin, FB 1 – Department of Business and Economics, FE Quantitative Methods

Hochschule für Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Email: *e_hornm@doz.hwr-berlin.de*

Hochschule für Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Wintersemester 2017/2018
Dr. Horn

Mathematics for Business and Economics
– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part VII: Generalized matrix inverses)

Stand: 19. Dez. 2017

Teaching & learning contents according to the new modular description of LV 200 691.01

- Systems of linear equations, solutions set, solving a linear system,
- Matrix and vector algebra, determinants, inverses,
- Economic examples

As more and more introductory business mathematics textbooks present Moore-Penrose generalized matrix inverses as elementary part of the foundations of mathematical economics, generalized matrix inverses will be discussed from a Geometric Algebra viewpoint in the following.

Summary

If a (3 x 2)-matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

with non-parallel

$$a \wedge b \neq 0$$

coefficient vectors

$$a = a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z$$
$$b = b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z$$

is given, the (2 x 3)-matrix

$$A^{-1} = \frac{1}{a \wedge b} \begin{bmatrix} (\sigma_x \wedge b) & (\sigma_y \wedge b) & (\sigma_z \wedge b) \\ (a \wedge \sigma_x) & (a \wedge \sigma_y) & (a \wedge \sigma_z) \end{bmatrix}$$

is called Pauli algebra generalized matrix inverse of A.

A^{-1} is a left-sided non-square matrix inverse.

Second Example: Problem

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

3 units of R_1, 5 units of R_2, and 4 units of R_3 to produce 1 unit of P_1

2 units of R_1, 4 units of R_2, and 8 units of R_3 to produce 1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 120 units of the first raw material R_1, 220 units of the second raw material R_2, and 320 units of the third raw material R_3 are consumed in the production process.

⇒ In real life it is quite common that a larger number of raw materials is required to produce a small number of final products.

Business Mathematics Example

This product engineering problem

$$3x + 2y = 120$$
$$5x + 4y = 220$$
$$4x + 8y = 320$$

can be modelled by a simple matrix multiplication

$$D\,q = r$$

with the following matrices:

Demand of raw materials to produce one unit of each final product (matrix of total demand) $D = \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 4 & 8 \end{bmatrix}$

Quantities of final products which are produced (production vector) $q = \begin{bmatrix} x \\ y \end{bmatrix} = ?$

Total demand of raw materials which are consumed (vector of consumption of raw materials) $r = \begin{bmatrix} 120 \\ 220 \\ 320 \end{bmatrix}$

Business Mathematics Example

Scheme of Falk:

			x
			y } q
D	3	2	120
	5	4	220 } r
	4	8	320

To find the solution values x and y the generalized inverse of the matrix of total demand D^{-1} will be pre-multiplied from the left:

$$q = D^{-1} r$$

To construct the generalized matrix inverse D^{-1} the outer products of the coefficient vectors

$$a = 3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z$$
$$b = 2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z$$

and the base vectors

$$r_1 = \sigma_x \quad r_2 = \sigma_y \quad r_3 = \sigma_z$$

are required.

Business Mathematics Example

Outer products:

$$a \wedge b = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z)$$
$$= 12\,\sigma_x\sigma_y + 24\,\sigma_x\sigma_z + 10\,\sigma_y\sigma_x + 40\,\sigma_y\sigma_z + 8\,\sigma_z\sigma_x + 16\,\sigma_z\sigma_y$$
$$= (12-10)\,\sigma_x\sigma_y + (40-16)\,\sigma_y\sigma_z + (8-24)\,\sigma_z\sigma_x$$
$$= 2\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z - 16\,\sigma_z\sigma_x$$

$$(a \wedge b)^2 = (2\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z - 16\,\sigma_z\sigma_x)^2$$
$$= (2\,\sigma_x\sigma_y)^2 + (24\,\sigma_y\sigma_z)^2 + (-16\,\sigma_z\sigma_x)^2$$
$$= -4 - 576 - 256$$
$$= -836 \qquad = -2 \cdot 2 \cdot 11 \cdot 19$$

$$(a \wedge b)^{-1} = \frac{a \wedge b}{(a \wedge b)^2}$$
$$= -\frac{1}{836}(2\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z - 16\,\sigma_z\sigma_x)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)$$

Business Mathematics Example

(Sub-matrix or minor) outer products:

$$\sigma_x \wedge b = \sigma_x \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z) = 4\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x$$

$$a \wedge \sigma_x = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge \sigma_x = -5\,\sigma_x\sigma_y + 4\,\sigma_z\sigma_x$$

$$\sigma_y \wedge b = \sigma_y \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z) = -2\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z$$

$$a \wedge \sigma_y = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge \sigma_y = 3\,\sigma_x\sigma_y - 4\,\sigma_y\sigma_z$$

$$\sigma_z \wedge b = \sigma_z \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z) = 2\,\sigma_z\sigma_x - 4\,\sigma_y\sigma_z$$

$$a \wedge \sigma_z = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge \sigma_z = -3\,\sigma_z\sigma_x + 5\,\sigma_y\sigma_z$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$x_1 = (a \wedge b)^{-1}(\sigma_x \wedge b)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(4\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x)$$
$$= -\frac{1}{418}(4\,\sigma_x\sigma_y\sigma_x\sigma_y - 8\,\sigma_x\sigma_y\sigma_z\sigma_x + 48\,\sigma_y\sigma_z\sigma_x\sigma_y - 96\,\sigma_y\sigma_z\sigma_z\sigma_x - 32\,\sigma_z\sigma_x\sigma_x\sigma_y + 64\,\sigma_z\sigma_x\sigma_z\sigma_x)$$
$$= -\frac{1}{418}(-4 - 8\,\sigma_y\sigma_z + 48\,\sigma_z\sigma_x + 96\,\sigma_x\sigma_y + 32\,\sigma_y\sigma_z - 64)$$
$$= -\frac{1}{418}(-68 + 96\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z + 48\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(68 - 96\,\sigma_x\sigma_y - 24\,\sigma_y\sigma_z - 48\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(68 - 12\,M)$$

$$M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$x_2 = (a \wedge b)^{-1}(\sigma_y \wedge b)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(-2\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(-2\,\sigma_x\sigma_y\sigma_x\sigma_y + 8\,\sigma_x\sigma_y\sigma_y\sigma_z - 24\,\sigma_y\sigma_z\sigma_x\sigma_y + 96\,\sigma_y\sigma_z\sigma_y\sigma_z + 16\,\sigma_z\sigma_x\sigma_x\sigma_y - 64\,\sigma_z\sigma_x\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(2 - 8\,\sigma_z\sigma_x - 24\,\sigma_z\sigma_x - 96 - 16\,\sigma_y\sigma_z - 64\,\sigma_x\sigma_y)$$
$$= -\frac{1}{418}(-94 - 64\,\sigma_x\sigma_y - 16\,\sigma_y\sigma_z - 32\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(94 + 64\,\sigma_x\sigma_y + 16\,\sigma_y\sigma_z + 32\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(94 + 8\,M)$$

$$M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$x_3 = (a \wedge b)^{-1}(\sigma_z \wedge b)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(2\,\sigma_z\sigma_x - 4\,\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(2\,\sigma_x\sigma_y\sigma_z\sigma_x - 4\,\sigma_x\sigma_y\sigma_y\sigma_z + 24\,\sigma_y\sigma_z\sigma_z\sigma_x - 48\,\sigma_y\sigma_z\sigma_y\sigma_z - 16\,\sigma_z\sigma_x\sigma_z\sigma_x + 32\,\sigma_z\sigma_x\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x - 24\,\sigma_x\sigma_y + 48 + 16 + 32\,\sigma_x\sigma_y)$$
$$= -\frac{1}{418}(64 + 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(-64 - 8\,\sigma_x\sigma_y - 2\,\sigma_y\sigma_z - 4\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(-64 - M)$$

$$M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_1 = (a \wedge b)^{-1}(a \wedge \sigma_x)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(-5\,\sigma_x\sigma_y + 4\,\sigma_z\sigma_x)$$
$$= -\frac{1}{418}(-5\,\sigma_x\sigma_y\sigma_x\sigma_y + 4\,\sigma_x\sigma_y\sigma_z\sigma_x - 60\,\sigma_y\sigma_z\sigma_x\sigma_y + 48\,\sigma_y\sigma_z\sigma_z\sigma_x + 40\,\sigma_z\sigma_x\sigma_x\sigma_y - 32\,\sigma_z\sigma_x\sigma_z\sigma_x)$$
$$= -\frac{1}{418}(5 + 4\,\sigma_y\sigma_z - 60\,\sigma_z\sigma_x - 48\,\sigma_x\sigma_y - 40\,\sigma_y\sigma_z + 32)$$
$$= -\frac{1}{418}(37 - 48\,\sigma_x\sigma_y - 36\,\sigma_y\sigma_z - 60\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(-37 + 48\,\sigma_x\sigma_y + 36\,\sigma_y\sigma_z + 60\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(-37 + 12\,N)$$

$$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_2 = (a \wedge b)^{-1}(a \wedge \sigma_y)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(3\,\sigma_x\sigma_y - 4\,\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(3\,\sigma_x\sigma_y\sigma_x\sigma_y - 4\,\sigma_x\sigma_y\sigma_y\sigma_z + 36\,\sigma_y\sigma_z\sigma_x\sigma_y - 48\,\sigma_y\sigma_z\sigma_y\sigma_z - 24\,\sigma_z\sigma_x\sigma_x\sigma_y + 32\,\sigma_z\sigma_x\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(-3 + 4\,\sigma_z\sigma_x + 36\,\sigma_z\sigma_x + 48 + 24\,\sigma_y\sigma_z + 32\,\sigma_x\sigma_y)$$
$$= -\frac{1}{418}(45 + 32\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z + 40\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(-45 - 32\,\sigma_x\sigma_y - 24\,\sigma_y\sigma_z - 40\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(-45 - 8\,N)$$

$$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_3 = (a \wedge b)^{-1}(a \wedge \sigma_z)$$
$$= -\frac{1}{418}(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(-3\,\sigma_z\sigma_x + 5\,\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(-3\,\sigma_x\sigma_y\sigma_z\sigma_x + 5\,\sigma_x\sigma_y\sigma_y\sigma_z - 36\,\sigma_y\sigma_z\sigma_z\sigma_x + 60\,\sigma_y\sigma_z\sigma_y\sigma_z + 24\,\sigma_z\sigma_x\sigma_z\sigma_x - 40\,\sigma_z\sigma_x\sigma_y\sigma_z)$$
$$= -\frac{1}{418}(-3\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x + 36\,\sigma_x\sigma_y - 60 - 24 - 40\,\sigma_x\sigma_y)$$
$$= -\frac{1}{418}(-84 - 4\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(84 + 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x)$$
$$= \frac{1}{418}(84 + N)$$

$$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$

Business Mathematics Example

The generalized matrix inverse D^{-1} then is:

$$D^{-1} = \frac{1}{418}\begin{bmatrix} 68 - 12\,M & 94 + 8\,M & -64 - M \\ -37 + 12\,N & -45 - 8\,N & 84 + N \end{bmatrix}$$

with $M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$

$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$

Left-sided check of generalized matrix inverse D^{-1}

			3	2
			5	4
			4	8
68 − 12 M	94 + 8 M	−64 − M	418	0
−37 + 12 N	−45 − 8 N	84 + N	0	418

Result of Second Example

$q = D^{-1} r$

			120
			220
			320
68 − 12 M	94 + 8 M	−64 − M	8360
−37 + 12 N	−45 − 8 N	84 + N	12540

⇓

$$x = \frac{8360}{418} = 20 \qquad y = \frac{12540}{418} = 30$$

⇒ If exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Check of result:

		20
		30
3	2	120
5	4	220
4	8	320

Moore-Penrose Generalized Matrix Inverses of Second Example

The Pauli algebra generalized matrix inverse D^{-1} of this problem has been:

$$D^{-1} = \frac{1}{418}\begin{bmatrix} 68 - 12\,M & 94 + 8\,M & -64 - M \\ -37 + 12\,N & -45 - 8\,N & 84 + N \end{bmatrix}$$

with $M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$

$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$

Thus the Moore-Penrose generalized matrix inverse will be

$$D^{+} = \frac{1}{418}\begin{bmatrix} 68 & 94 & -64 \\ -37 & -45 & 84 \end{bmatrix}$$

because Moore and Penrose have decided to neglect all bivector terms: $M = 0$, $N = 0$

Result of Second Example Revisited

$q = D^{+} r$

			120
			220
			320
$\frac{68}{418}$	$\frac{94}{418}$	$-\frac{64}{418}$	20
$-\frac{37}{418}$	$-\frac{45}{418}$	$\frac{84}{418}$	30

⇒ If exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.