Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

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Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part VII: Generalized matrix inverses)

Stand: 19. Dez. 2017

Teaching & learning contents according to the new modular description of LV 200 691.01

- Systems of linear equations, solutions set, solving a linear system,
- Matrix and vector algebra, determinants, inverses,
- Economic examples

As more and more introductory business mathematics textbooks present Moore-Penrose generalized matrix inverses as elementary part of the foundations of mathematical economics, generalized matrix inverses will be discussed from a Geometric Algebra viewpoint in the following.

Repetition: Basics of Geometric Algebra

$1 + 3 + 3 + 1 = 2^3 = 8$ different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: $\sigma_x, \sigma_y, \sigma_z$

Three base bivectors: $\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

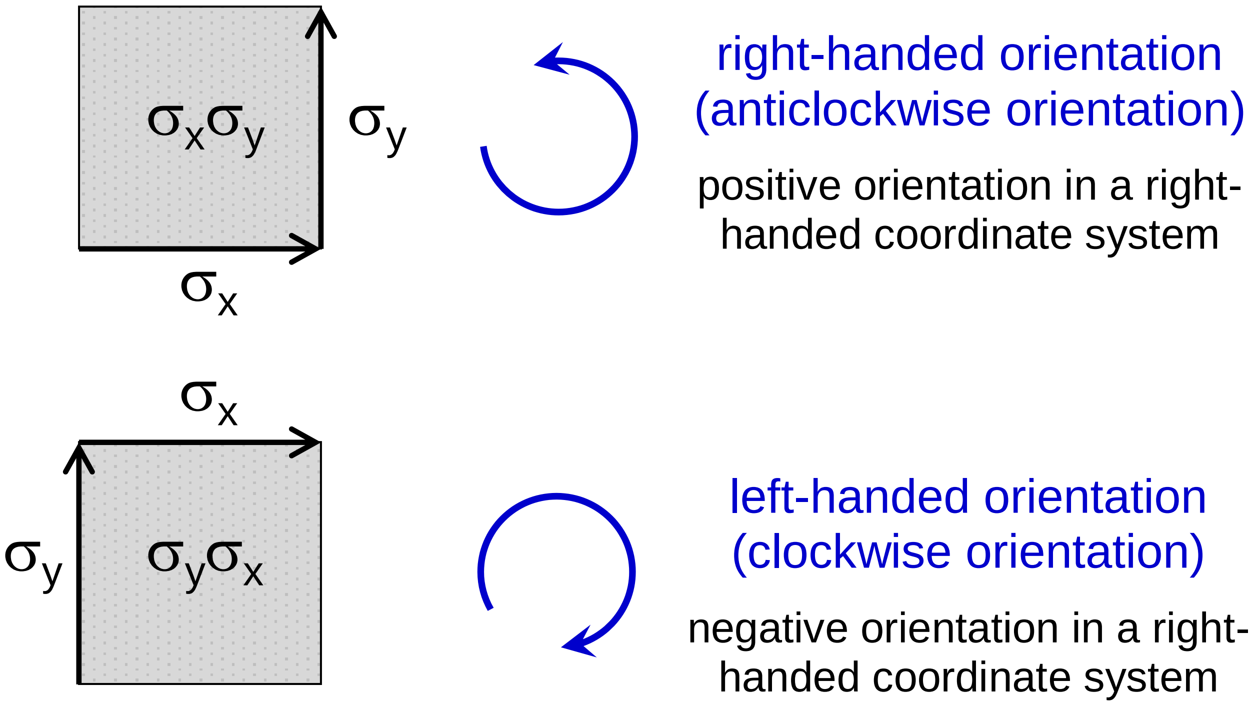
$$1^2 = \sigma_x{}^2 = \sigma_y{}^2 = \sigma_z{}^2 = 1$$

Base bivectors and base trivector square to minus one:

$$(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1$$

Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the resulting area elements.



Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

$$\sigma_x \sigma_y = -\,\sigma_y \sigma_x$$

$$\sigma_y \sigma_z = -\,\sigma_z \sigma_y$$

$$\sigma_z \sigma_x = -\,\sigma_x \sigma_z$$

Scalars

Scalars are geometric entities without direction. They can be expressed as multiples of the base scalar:

$$k = k\ 1$$

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

$$r = x\ \sigma_x + y\ \sigma_y + z\ \sigma_z$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

$$A = A_{xy}\ \sigma_x\sigma_y + A_{yz}\ \sigma_y\sigma_z + A_{zx}\ \sigma_z\sigma_x$$

Trivectors

Trivectors are oriented volume elements. They can be expressed as multiples of the base trivector:

$$V = V_{xyz}\ \sigma_x\sigma_y\sigma_z$$

Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$a\,b = a \bullet b + a \wedge b$$

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

$$a \bullet b = b \bullet a = \frac{1}{2}(a\,b + b\,a)$$

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

$$a \wedge b = -\,b \wedge a = \frac{1}{2}(a\,b - b\,a)$$

Geometric Multiplication of Vectors and Bivectors

The product of a bivector B and a vector a consists of a vector term and a trivector term. As the dimension of bivector B is reduced, the vector term is called inner product (dot product). And as the dimension of bivector B is increased, the trivector term is called outer product (exterior product or wedge product).

$$B\,a = B \bullet a + B \wedge a$$

In contrast to what was said on the last slide, the inner product of a bivector and a vector is an anti-commutative product as a reversion of the order of bivector and vector changes the sign of the inner product:

$$B \bullet a = -\,a \bullet B = \frac{1}{2}(B\,a - a\,B)$$

The outer product of a bivector and a vector is a commutative product as a reversion of the order of bivector and vector does not change it:

$$B \wedge a = a \wedge B = \frac{1}{2}(B\,a + a\,B)$$

Systems of Two Linear Equations

$$\begin{aligned} a_1\, x + b_1\, y &= r_1 \\ a_2\, x + b_2\, y &= r_2 \end{aligned} \qquad \Rightarrow \qquad a\, x + b\, y = r$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1\, \sigma_x + a_2\, \sigma_y)\, x + (b_1\, \sigma_x + b_2\, \sigma_y)\, y = r_1\, \sigma_x + r_2\, \sigma_y$$

Solutions:

$$x = \frac{1}{a \wedge b}(r \wedge b) = (a \wedge b)^{-1}\,(r \wedge b)$$

$$y = \frac{1}{a \wedge b}\,(a \wedge r) = (a \wedge b)^{-1}\,(a \wedge r)$$

Systems of Three Linear Equations

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= r_1 \\ a_2 x + b_2 y + c_2 z &= r_2 \qquad \Rightarrow \qquad a\,x + b\,y + c\,z = r \\ a_3 x + b_3 y + c_3 z &= r_3 \end{aligned}$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$\begin{aligned} (a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z)\, x + (b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z)\, y \\ + (c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z)\, z = r_1 \sigma_x + r_2 \sigma_y + r_3 \sigma_z \end{aligned}$$

Solutions:

$$x = (a \wedge b \wedge c)^{-1} (r \wedge b \wedge c)$$

$$y = (a \wedge b \wedge c)^{-1} (a \wedge r \wedge c)$$

$$z = (a \wedge b \wedge c)^{-1} (a \wedge b \wedge r)$$

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of former lessons and in Geometric Algebra textbooks.

The System of Two Linear Equations

of repetition page 7

$$\begin{matrix} a_1\, x + b_1\, y = r_1 \\ a_2\, x + b_2\, y = r_2 \end{matrix} \quad \Rightarrow \quad \mathbf{a}\, x + \mathbf{b}\, y = \mathbf{r}$$

can be expressed as matrix multiplication

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \qquad \text{(or in short: } \mathbf{A}\,\mathbf{q} = \mathbf{r}\text{)}$$

by the scheme of Falk

$$\begin{array}{r|l} & \left.\begin{matrix} x \\ y \end{matrix}\right\} \mathbf{q} \\ \hline \mathbf{A} \left\{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}\right. & \left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} \mathbf{r} \end{array}$$

The well-known solution values are:

$$x = \frac{1}{\mathbf{a} \wedge \mathbf{b}} (\mathbf{r} \wedge \mathbf{b}) = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r} \wedge \mathbf{b})$$

$$y = \frac{1}{\mathbf{a} \wedge \mathbf{b}} (\mathbf{a} \wedge \mathbf{r}) = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r})$$

Inverses of (2 x 2)-Matrices

These solution values

$$
\begin{aligned}
x &= (a \wedge b)^{-1}\,(r \wedge b)\\
&= (a \wedge b)^{-1}\,((r_1\,\sigma_x + r_2\,\sigma_y) \wedge b)\\
&= (a \wedge b)^{-1}\,(r_1\,\sigma_x \wedge b) + (a \wedge b)^{-1}\,(r_2\,\sigma_y \wedge b)\\
&= \underbrace{(a \wedge b)^{-1}\,(\sigma_x \wedge b)}_{\text{first element of first row of lead matrix}}\ \underset{\text{first element of column vector (lag matrix)}}{r_1} + \underbrace{(a \wedge b)^{-1}\,(\sigma_y \wedge b)}_{\text{second element of first row of lead matrix}}\ \underset{\text{second element of column vector (lag matrix)}}{r_2}
\end{aligned}
$$

$$
\begin{aligned}
y &= (a \wedge b)^{-1}\,(a \wedge r)\\
&= (a \wedge b)^{-1}\,(a \wedge (r_1\,\sigma_x + r_2\,\sigma_y))\\
&= (a \wedge b)^{-1}\,(a \wedge r_1\sigma_x) + (a \wedge b)^{-1}\,(a \wedge r_2\sigma_y)\\
&= \underbrace{(a \wedge b)^{-1}\,(a \wedge \sigma_x)}_{\text{first element of second row of lead matrix}}\ \underset{\text{first element of column vector (lag matrix)}}{r_1} + \underbrace{(a \wedge b)^{-1}\,(a \wedge \sigma_y)}_{\text{second element of second row of lead matrix}}\ \underset{\text{second element of column vector (lag matrix)}}{r_2}
\end{aligned}
$$

can be expressed as a matrix multiplication by the scheme of Falk, too.

Inverses of (2 x 2)-Matrices

Scheme of Falk of solution values:

$$\begin{array}{cc|c} & & r_1 \\ & & r_2 \\ \hline (a\wedge b)^{-1}(\sigma_x\wedge b) & (a\wedge b)^{-1}(\sigma_y\wedge b) & x \\ (a\wedge b)^{-1}(a\wedge \sigma_x) & (a\wedge b)^{-1}(a\wedge \sigma_y) & y \end{array}$$

Of course, the lead matrix is the inverse A^{-1} of the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

Thus the matrix equation

$$A\,q = r$$

can be solved by pre-multiplying the matrix inverse A^{-1} from the left (as already discussed earlier at the lessons):

$$A\,q = r$$

$$A^{-1} A\,q = A^{-1} r$$

$$\Rightarrow \qquad q = A^{-1} r$$

Never ever believe in the correctness of a result.
Please always check it!

$\Rightarrow$ First check of the inverse matrix:

$$A^{-1} A = \mathrm{I}$$

$$
\begin{array}{cc|cc}
 & & a_1 & b_1 \\
 & & a_2 & b_2 \\
\hline
(a \wedge b)^{-1} (\sigma_x \wedge b) & (a \wedge b)^{-1} (\sigma_y \wedge b) & \color{blue}{1} & 0 \\
(a \wedge b)^{-1} (a \wedge \sigma_x) & (a \wedge b)^{-1} (a \wedge \sigma_y) & 0 & 1
\end{array}
$$

$$
\begin{aligned}
&(a \wedge b)^{-1} (\sigma_x \wedge b)\, a_1 + (a \wedge b)^{-1} (\sigma_y \wedge b)\, a_2 \\
&\quad = (a \wedge b)^{-1} \big((a_1 \sigma_x \wedge b) + (a_2 \sigma_y \wedge b)\big) \\
&\quad = (a \wedge b)^{-1} \big((a_1 \sigma_x + a_2 \sigma_y) \wedge b\big) \\
&\quad = (a \wedge b)^{-1} (a \wedge b) \\
&\quad = 1
\end{aligned}
$$

$$
\begin{aligned}
&(a \wedge b)^{-1} (\sigma_x \wedge b)\, b_1 + (a \wedge b)^{-1} (\sigma_y \wedge b)\, b_2 \\
&\quad = (a \wedge b)^{-1} \big((b_1 \sigma_x \wedge b) + (b_2 \sigma_y \wedge b)\big) \\
&\quad = (a \wedge b)^{-1} \big((b_1 \sigma_x + b_2 \sigma_y) \wedge b\big) \\
&\quad = (a \wedge b)^{-1} (b \wedge b) \\
&\quad = 0
\end{aligned}
$$

Never ever believe in the correctness of a result.
Please always check it!

$\Rightarrow$ Second check of the inverse matrix:

$$A\,A^{-1} = I$$

		$(a\wedge b)^{-1}(\sigma_x\wedge b)$ $(a\wedge b)^{-1}(a\wedge\sigma_x)$	$(a\wedge b)^{-1}(\sigma_y\wedge b)$ $(a\wedge b)^{-1}(a\wedge\sigma_y)$
a_1	b_1	1	0
a_2	b_2	0	1

$$\begin{aligned}
&a_1\,(a\wedge b)^{-1}(\sigma_x\wedge b) + b_1\,(a\wedge b)^{-1}(a\wedge\sigma_x)\\
&\quad= (a\wedge b)^{-1}\big(a_1\,\sigma_x\wedge b + a\wedge(b_1\,\sigma_x)\big)\\
&\quad= (a\wedge b)^{-1}\big(a_1\,\sigma_x\wedge(b_1\,\sigma_x + b_2\,\sigma_y) + (a_1\,\sigma_x + a_2\,\sigma_y)\wedge(b_1\,\sigma_x)\big)\\
&\quad= (a\wedge b)^{-1}\big(a_1\,b_2\,\sigma_x\sigma_y + a_2\,b_1\,\sigma_y\sigma_x\big)\\
&\quad= (a\wedge b)^{-1}\big(a_1\,b_2\,\sigma_x\sigma_y - a_2\,b_1\,\sigma_x\sigma_y\big)\\
&\quad= (a\wedge b)^{-1}(a\wedge b)\\
&\quad= 1
\end{aligned}$$

$$
\begin{aligned}
a_1\,(a\wedge b)^{-1}(\sigma_y\wedge b) + b_1\,(a\wedge b)^{-1}(a\wedge\sigma_y)
&= (a\wedge b)^{-1}\big((a_1\,\sigma_y\wedge b) + ((a\wedge b_1\,\sigma_y)\big) \\
&= (a\wedge b)^{-1}\big(a_1\,\sigma_y\wedge(b_1\,\sigma_x + b_2\,\sigma_y) \\
&\qquad + (a_1\,\sigma_x + a_2\,\sigma_y)\wedge(b_1\,\sigma_y\big) \\
&= (a\wedge b)^{-1}\big(a_1\,b_1\,\sigma_y\sigma_x + a_1\,b_1\,\sigma_x\sigma_y\big) \\
&= (a\wedge b)^{-1}\big(-a_1\,b_1\,\sigma_x\sigma_y + a_1\,b_1\,\sigma_x\sigma_y\big) \\
&= 0
\end{aligned}
$$

Please note, that the first check is based on simple reconstructions of the coefficient vectors a and b as linear combinations of the base vectors σ_x and σ_y, …

… while the second check is based on decomposing the coefficient vectors into its terms.

Summary

If

$$a \wedge b \neq 0 \qquad \text{(or det } A \neq 0\text{)},$$

the inverse of the (2 x 2)-matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

with coefficient vectors

$$a = a_1\,\sigma_x + a_2\,\sigma_y$$

$$b = b_1\,\sigma_x + b_2\,\sigma_y$$

will be

$$A^{-1} = \begin{bmatrix} (a \wedge b)^{-1}\,(\sigma_x \wedge b) & (a \wedge b)^{-1}\,(\sigma_y \wedge b) \\ (a \wedge b)^{-1}\,(a \wedge \sigma_x) & (a \wedge b)^{-1}\,(a \wedge \sigma_y) \end{bmatrix}$$

$$= \frac{1}{a \wedge b} \begin{bmatrix} (\sigma_x \wedge b) & (\sigma_y \wedge b) \\ (a \wedge \sigma_x) & (a \wedge \sigma_y) \end{bmatrix}$$

First Example: Problem

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

> 3 units of R_1 and 5 units of R_2
> to produce 1 unit of P_1
>
> 2 units of R_1 and 4 units of R_2
> to produce 1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed in the production process.

Business Mathematics Example

As you already know this product engineering problem

$$3\,x + 2\,y = 120$$

$$5\,x + 4\,y = 220$$

can be modelled by a simple matrix multiplication

$$D\,q = r$$

with the following matrices:

Demand of raw materials to produce one unit of each final product (matrix of total demand)

$$D = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

Quantities of final products which are produced (production vector)

$$q = \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

Total demand of raw materials which are consumed (vector of consumption of raw materials)

$$r = \begin{bmatrix} 120 \\ 220 \end{bmatrix}$$

Scheme of Falk:

$$\begin{array}{rcc|cl} & & & x & \multirow{2}{*}{$\left.\vphantom{\begin{matrix}x\\y\end{matrix}}\right\}q$} \\ & & & y & \\ \hline D\left\{\vphantom{\begin{matrix}3\\5\end{matrix}}\right. & 3 & 2 & 120 & \left.\vphantom{\begin{matrix}1\\2\end{matrix}}\right\}r \\ & 5 & 4 & 220 & \end{array}$$

Business Mathematics Example

To find the solution values x and y the inverse of the matrix of total demand D^{-1} will be pre-multiplied from the left:

$$q = D^{-1}\, r$$

To construct the matrix inverse D^{-1} the outer products of the coefficient vectors

$$a = 3\,\sigma_x + 5\,\sigma_y$$
$$b = 2\,\sigma_x + 4\,\sigma_y$$

and the base vectors

$$r_1 = \sigma_x$$
$$r_2 = \sigma_y$$

are required: $a \wedge b = (3\,\sigma_x + 5\,\sigma_y) \wedge (2\,\sigma_x + 4\,\sigma_y)$

$$= 12\,\sigma_x\sigma_y + 10\,\sigma_y\sigma_x$$
$$= 2\,\sigma_x\sigma_y$$

$$\Rightarrow \quad (a \wedge b)^{-1} = \frac{1}{2\,\sigma_x\sigma_y} = -\frac{1}{2}\,\sigma_x\sigma_y$$

$$\sigma_x \wedge b = \sigma_x \wedge (2\,\sigma_x + 4\,\sigma_y) = 4\,\sigma_x\sigma_y$$
$$a \wedge \sigma_x = (3\,\sigma_x + 5\,\sigma_y) \wedge \sigma_x = -5\,\sigma_x\sigma_y$$

$$\sigma_y \wedge b = \sigma_y \wedge (2\,\sigma_x + 4\,\sigma_y) = -2\,\sigma_x\sigma_y$$
$$a \wedge \sigma_y = (3\,\sigma_x + 5\,\sigma_y) \wedge \sigma_y = 3\,\sigma_x\sigma_y$$

Business Mathematics Example

The matrix inverse D^{-1} then is:

$$D^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2.5 & 1.5 \end{bmatrix}$$

First check of matrix inverse D^{-1}

$$\begin{array}{cc|cc} & & 3 & 2 \\ & & 5 & 4 \\ \hline 2 & -1 & 1 & 0 \\ -2.5 & 1.5 & 0 & 1 \end{array}$$

Second check of matrix inverse D^{-1}

$$\begin{array}{cc|cc} & & 2 & -1 \\ & & -2.5 & 1.5 \\ \hline 3 & 2 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}$$

Result of First Example

$q = D^{-1}\, r$

$$
\begin{array}{cc|c}
 & & 120 \\
 & & 220 \\
\hline
2 & -1 & 20 \\
-2.5 & 1.5 & 30
\end{array}
$$

$\Rightarrow$ If exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Check of result:

$$
\begin{array}{cc|c}
 & & 20 \\
 & & 30 \\
\hline
3 & 2 & 120 \\
5 & 4 & 220
\end{array}
$$

Non-Square Matrices

Some books claim that matrix inverses only exist if the inverted matrix is a square matrix.

> ***Conditions for the existence of the matrix inverse***
> … Since the number of unknown variables must equal the number of equations the matrix of co-efficients **A** must be *square*, i.e. the number of rows equals the number of columns.
>
> Mike Rosser: Basic Mathematics for Economists. 2nd edition, Routledge / Taylor & Francis Group, London, New York 2003, chap. 15.4.

This statement is not perfectly true!

Of course the usual definition of determinants is only applicable to square matrices. If now matrix inverses are tried to be found according to Cramer's rule (which is based a division of determinants), it will not be possible to determine them.

But there are other and more general mathematical strategies to define matrix inverses. Therefore more and more math books discuss inverses of non-square matrices.

Non-Square Matrices

Generalized matrix inverses may exist if the following condition is met:

> ***Generalized Inverse of a Matrix***
>
> … For the nonsquare matrix A with dimensions $n \times k$ where $k < n$, the generalized inverse is defined. … The Moore-Penrose generalized inverse is defined only for matrices with full column rank.
>
> Kamran Dadkhah: Foundations of Mathematical and Computational Economics. 2nd edition, Springer, Heidelberg, Dordrecht 2011, chap. 7.2

$\Rightarrow$ The system of linear equations must not be underconstrained.
The number of equations must not be smaller than the number of variables.

$\Rightarrow$ Matrices should have full column rank.
The number of rows must be equal to or greater than the number of columns and the column vectors must not be linearly dependent.

A System of Three Linear Equations With Two Variables

$$\begin{aligned} a_1\, x + b_1\, y &= r_1 \\ a_2\, x + b_2\, y &= r_2 \qquad \Rightarrow \qquad a\, x + b\, y = r \\ a_3\, x + b_3\, y &= r_3 \end{aligned}$$

can be expressed as matrix multiplication

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \qquad \text{(or in short: } A\, q = r\text{)}$$

by the scheme of Falk

$$\begin{array}{r|l} & \left.\begin{matrix} x \\ y \end{matrix}\right\} q \\ \hline A \left\{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right. & \left.\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}\right\} r \end{array}$$

This system of linear equations is over-constrained as it has fewer variables than equations.

A System of Three Linear Equations With Two Variables

This overconstrained system of three linear equations now possesses two coefficient vectors

$$a = a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z$$

$$b = b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z$$

and a resulting vector

$$r = r_1 \sigma_x + r_2 \sigma_y + r_3 \sigma_z$$

with three components pointing into the three different directions x, y, and z.

Consistency

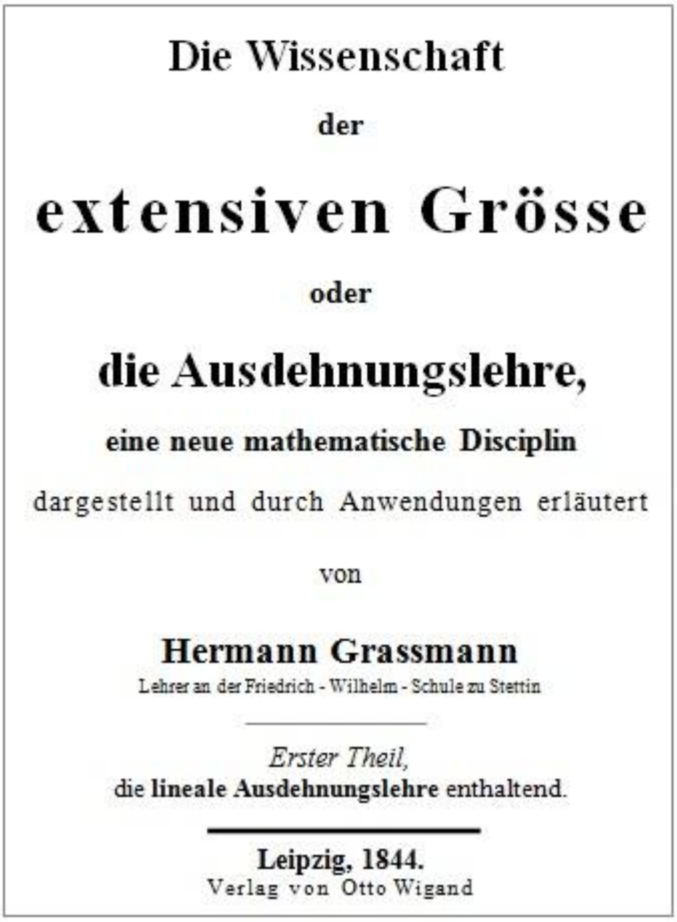
If the two coefficient vectors are not linearly dependent and if the resulting vector and the two coefficient vectors are linearly dependent (the two coefficient vectors do not point into the same direction, but the two coefficient vectors and the resulting vector are in the same plane), the system of linear equations will be consistent and a solution will exist.

Inverses of (3 x 2)-Matrices

Again, the solution values of a consistent system of three linear equations with two variables x and y can be found with the formula Grassmann has published 1844 in his important theory of extensions:

$$x = \frac{1}{a \wedge b}(r \wedge b) = (a \wedge b)^{-1}\,(r \wedge b)$$

$$y = \frac{1}{a \wedge b}(a \wedge r) = (a \wedge b)^{-1}\,(a \wedge r)$$



Die Wissenschaft

der

extensiven Grösse

oder

die Ausdehnungslehre,

eine neue mathematische Disciplin

dargestellt und durch Anwendungen erläutert

von

Hermann Grassmann

Lehrer an der Friedrich - Wilhelm - Schule zu Stettin

Erster Theil,

die lineale Ausdehnungslehre enthaltend.

Leipzig, 1844.

Verlag von Otto Wigand

See: Geometric Algebra Crash Course of last winter semester 2016/ 2017 (Part VI: Solving systems of linear equations with sandwich products)

Inverses of (3 x 2)-Matrices

These solution values

$$x = (a \wedge b)^{-1}\,(r \wedge b)$$
$$= (a \wedge b)^{-1}\,((r_1\,\sigma_x + r_2\,\sigma_y + r_3\,\sigma_z) \wedge b)$$
$$= (a \wedge b)^{-1}\,(r_1\,\sigma_x \wedge b) + (a \wedge b)^{-1}\,(r_2\,\sigma_y \wedge b) + (a \wedge b)^{-1}\,(r_3\,\sigma_z \wedge b)$$

first element of column vector (lag matrix)

second element of column vector (lag matrix)

$$= (a \wedge b)^{-1}\,(\sigma_x \wedge b)\; r_1$$

third element of column vector (lag matrix)

$$+ (a \wedge b)^{-1}\,(\sigma_y \wedge b)\; r_2$$

first element of first row of lead matrix

$$+ (a \wedge b)^{-1}\,(\sigma_z \wedge b)\; r_3$$

second element of first row of lead matrix

third element of first row of lead matrix

and …

Inverses of (3 x 2)-Matrices

$$
\begin{aligned}
y &= (a \wedge b)^{-1} (a \wedge r) \\
&= (a \wedge b)^{-1} (a \wedge (r_1 \sigma_x + r_2 \sigma_y + r_3 \sigma_z)) \\
&= (a \wedge b)^{-1} (a \wedge r_1 \sigma_x) + (a \wedge b)^{-1} (a \wedge r_2 \sigma_y) \\
&\qquad + (a \wedge b)^{-1} (a \wedge r_3 \sigma_z)
\end{aligned}
$$

first element of column vector (lag matrix)

second element of column vector (lag matrix)

$$
\begin{aligned}
&= (a \wedge b)^{-1} (a \wedge \sigma_x)\, r_1 \\
&\qquad + (a \wedge b)^{-1} (a \wedge \sigma_y)\, r_2 \\
&\qquad\qquad + (a \wedge b)^{-1} (a \wedge \sigma_z)\, r_3
\end{aligned}
$$

third element of column vector (lag matrix)

first element of second row of lead matrix

second element of second row of lead matrix

third element of second row of lead matrix

can be expressed as a matrix multiplication by the scheme of Falk, too.

Inverses of (3 x 2)-Matrices

Scheme of Falk of solution values:

$$\begin{array}{ccc|c}
 & & & r_1 \\
 & & & r_2 \\
\hline
(a\wedge b)^{-1}(\sigma_x\wedge b) & (a\wedge b)^{-1}(\sigma_y\wedge b) & (a\wedge b)^{-1}(\sigma_z\wedge b) & x \\
(a\wedge b)^{-1}(a\wedge\sigma_x) & (a\wedge b)^{-1}(a\wedge\sigma_y) & (a\wedge b)^{-1}(a\wedge\sigma_z) & y
\end{array}$$

The lead matrix can now be called an inverse A^{-1} of the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$

It is a generalized matrix inverse.

Thus the matrix equation

$$A\,q = r$$

can be solved by pre-multiplying the gener-alized matrix inverse A^{-1} from the left:

$$A\,q = r$$

$$A^{-1}A\,q = A^{-1}r$$

$$\Rightarrow \qquad q = A^{-1}r$$

Never ever believe in the correctness of a result. Please always check it!

$\Rightarrow$ First check of the generalized matrix inverse:

$$A^{-1} A = \mathrm{I}$$

$$\begin{array}{ccc|cc}
 & & & a_1 & b_1 \\
 & & & a_2 & b_2 \\
 & & & a_3 & b_3 \\
\hline
(a\wedge b)^{-1}(\sigma_x\wedge b) & (a\wedge b)^{-1}(\sigma_y\wedge b) & (a\wedge b)^{-1}(\sigma_z\wedge b) & 1 & 0 \\
(a\wedge b)^{-1}(a\wedge \sigma_x) & (a\wedge b)^{-1}(a\wedge \sigma_y) & (a\wedge b)^{-1}(a\wedge \sigma_z) & 0 & 1
\end{array}$$

$$\begin{aligned}
&(a\wedge b)^{-1}(\sigma_x\wedge b)\, a_1 + (a\wedge b)^{-1}(\sigma_y\wedge b)\, a_2 + (a\wedge b)^{-1}(\sigma_z\wedge b)\, a_3 \\
&\quad = (a\wedge b)^{-1}\big((a_1\sigma_x\wedge b) + (a_2\sigma_y\wedge b) + (a_3\sigma_z\wedge b)\big) \\
&\quad = (a\wedge b)^{-1}\big((a_1\sigma_x + a_2\sigma_y + a_3\sigma_z)\wedge b\big) \\
&\quad = (a\wedge b)^{-1}(a\wedge b) \\
&\quad = 1
\end{aligned}$$

$$\begin{aligned}
&(a\wedge b)^{-1}(\sigma_x\wedge b)\, b_1 + (a\wedge b)^{-1}(\sigma_y\wedge b)\, b_2 + (a\wedge b)^{-1}(\sigma_z\wedge b)\, b_3 \\
&\quad = (a\wedge b)^{-1}\big((b_1\sigma_x\wedge b) + (b_2\sigma_y\wedge b) + (b_3\sigma_z\wedge b)\big) \\
&\quad = (a\wedge b)^{-1}\big((b_1\sigma_x + b_2\sigma_y + b_3\sigma_z)\wedge b\big) \\
&\quad = (a\wedge b)^{-1}(b\wedge b) \\
&\quad = 0
\end{aligned}$$

Never ever believe in the correctness of a result. Please always check it!

$\Rightarrow$ Second check of the generalized matrix in-verse:

$$A\,A^{-1} = \mathrm{I}$$

	$(a\wedge b)^{-1}(\sigma_x\wedge b)$ $(a\wedge b)^{-1}(a\wedge\sigma_x)$	$(a\wedge b)^{-1}(\sigma_y\wedge b)$ $(a\wedge b)^{-1}(a\wedge\sigma_y)$	$(a\wedge b)^{-1}(\sigma_z\wedge b)$ $(a\wedge b)^{-1}(a\wedge\sigma_z)$
$a_1\quad b_1$			
$a_2\quad b_2$			
$a_3\quad b_3$			

$$\begin{aligned}
a_1\,(a\wedge b)^{-1}(\sigma_x\wedge b) + b_1\,(a\wedge b)^{-1}(a\wedge\sigma_x) &\neq 1\\
&= (a\wedge b)^{-1}\big(a_1\,\sigma_x\wedge b + a\wedge(b_1\,\sigma_x)\big)\\
&= (a\wedge b)^{-1}\big(a_1\,\sigma_x\wedge(b_1\,\sigma_x+b_2\,\sigma_y+b_3\,\sigma_z)\\
&\qquad + (a_1\,\sigma_x+a_2\,\sigma_y+a_3\,\sigma_z)\wedge(b_1\,\sigma_x)\big)\\
&= (a\wedge b)^{-1}\big(a_1\,b_2\,\sigma_x\sigma_y + a_1\,b_3\,\sigma_x\sigma_z + a_2\,b_1\,\sigma_y\sigma_x + a_3\,b_1\,\sigma_x\sigma_z\big)\\
&= (a\wedge b)^{-1}\underbrace{\big((a_1\,b_2 - b_1\,a_2)\,\sigma_x\sigma_y + (a_3\,b_1 - b_1\,a_3)\,\sigma_z\sigma_x\big)}_{\text{This bracket does not equal}} \neq 1
\end{aligned}$$

$$a\wedge b = (a_1b_2-b_1a_2)\sigma_x\sigma_y + (a_2b_3-b_2a_3)\sigma_y\sigma_z + (a_3b_1-b_1a_3)\sigma_z\sigma_x$$

Inverses of (3 x 2)-Matrices

As the second term $(a_2 b_3 - b_2 a_3)\,\sigma_y \sigma_z$ of the outer product $a \wedge b$ is missing, the second check does not give the expected result.

$\Rightarrow$ The generalized matrix inverse A^{-1}

$$A^{-1} = \begin{bmatrix} (a \wedge b)^{-1}(\sigma_x \wedge b) & (a \wedge b)^{-1}(\sigma_y \wedge b) & (a \wedge b)^{-1}(\sigma_z \wedge b) \\ (a \wedge b)^{-1}(a \wedge \sigma_x) & (a \wedge b)^{-1}(a \wedge \sigma_y) & (a \wedge b)^{-1}(a \wedge \sigma_z) \end{bmatrix}$$

$$= \frac{1}{a \wedge b} \begin{bmatrix} (\sigma_x \wedge b) & (\sigma_y \wedge b) & (\sigma_z \wedge b) \\ (a \wedge \sigma_x) & (a \wedge \sigma_y) & (a \wedge \sigma_z) \end{bmatrix}$$

is only a left-sided matrix inverse, which can be pre-multiplied from the left.

$\Rightarrow$ If A^{-1} is treated as a right-sided matrix inverse and is post-multiplied from the right, the result will be erroneous.

And as we are using the mathematics of Pauli algebra, this generalized matrix inverse should be denoted as Pauli algebra generalized matrix inverse.

Summary

If a (3 x 2)-matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

with non-parallel

$$a \wedge b \neq 0$$

coefficient vectors

$$a = a_1\,\sigma_x + a_2\,\sigma_y + a_3\,\sigma_z$$

$$b = b_1\,\sigma_x + b_2\,\sigma_y + b_3\,\sigma_z$$

is given, the (2 x 3)-matrix

$$A^{-1} = \frac{1}{a \wedge b} \begin{bmatrix} (\sigma_x \wedge b) & (\sigma_y \wedge b) & (\sigma_z \wedge b) \\ (a \wedge \sigma_x) & (a \wedge \sigma_y) & (a \wedge \sigma_z) \end{bmatrix}$$

is called Pauli algebra generalized matrix inverse of A.

A^{-1} is a left-sided non-square matrix inverse.

Second Example: Problem

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

3 units of R_1, 5 units of R_2, and 4 units of R_3 to produce 1 unit of P_1

2 units of R_1, 4 units of R_2, and 8 units of R_3 to produce 1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 120 units of the first raw material R_1, 220 units of the second raw material R_2, and 320 units of the third raw material R_3 are consumed in the production process.

$\Rightarrow$ In real life it is quite common that a larger number of raw materials is required to produce a small number of final products.

Business Mathematics Example

This product engineering problem

$$3\,x + 2\,y = 120$$
$$5\,x + 4\,y = 220$$
$$4\,x + 8\,y = 320$$

can be modelled by a simple matrix multipli-cation

$$D\,q = r$$

with the following matrices:

Demand of raw materials to pro-duce one unit of each final pro-duct (matrix of total demand)

$$D = \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 4 & 8 \end{bmatrix}$$

Quantities of final products which are produced (production vector)

$$q = \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

Total demand of raw materials which are consumed (vector of consumption of raw materials)

$$r = \begin{bmatrix} 120 \\ 220 \\ 320 \end{bmatrix}$$

Business Mathematics Example

Scheme of Falk:

$$\begin{array}{c|l}
 & \left.\begin{matrix} x \\ y \end{matrix}\right\} q \\
\hline
D\left\{\begin{matrix} 3 & 2 \\ 5 & 4 \\ 4 & 8 \end{matrix}\right. & \left.\begin{matrix} 120 \\ 220 \\ 320 \end{matrix}\right\} r
\end{array}$$

To find the solution values x and y the generalized inverse of the matrix of total demand D^{-1} will be pre-multiplied from the left:

$$q = D^{-1} r$$

To construct the generalized matrix inverse D^{-1} the outer products of the coefficient vectors

$$a = 3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z$$

$$b = 2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z$$

and the base vectors

$$r_1 = \sigma_x$$

$$r_2 = \sigma_y$$

$$r_3 = \sigma_z$$

are required.

Business Mathematics Example

Outer products:

$$a \wedge b = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z)$$

$$= 12\,\sigma_x\sigma_y + 24\,\sigma_x\sigma_z + 10\,\sigma_y\sigma_x + 40\,\sigma_y\sigma_z + 8\,\sigma_z\sigma_x + 16\,\sigma_z\sigma_y$$

$$= (12-10)\,\sigma_x\sigma_y + (40-16)\,\sigma_y\sigma_z + (8-24)\,\sigma_z\sigma_x$$

$$= 2\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z - 16\,\sigma_z\sigma_x$$

$$(a \wedge b)^2 = (2\,\sigma_x\sigma_y + 24\,\sigma_x\sigma_z - 16\,\sigma_z\sigma_x)^2$$

$$= (2\,\sigma_x\sigma_y)^2 + (24\,\sigma_y\sigma_z)^2 + (-16\,\sigma_z\sigma_x)^2$$

$$= -4 - 576 - 256$$

$$= -836 \qquad = -2 \cdot 2 \cdot 11 \cdot 19$$

$$(a \wedge b)^{-1} = \frac{a \wedge b}{(a \wedge b)^2}$$

$$= -\frac{1}{836}\,(2\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z - 16\,\sigma_z\sigma_x)$$

$$= -\frac{1}{418}\,(\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)$$

Business Mathematics Example

(Sub-matrix or minor) outer products:

$$\sigma_x \wedge b = \sigma_x \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z)$$
$$= 4\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x$$

$$a \wedge \sigma_x = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge \sigma_x$$
$$= -\,5\,\sigma_x\sigma_y + 4\,\sigma_z\sigma_x$$

$$\sigma_y \wedge b = \sigma_y \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z)$$
$$= -\,2\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z$$

$$a \wedge \sigma_y = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge \sigma_y$$
$$= 3\,\sigma_x\sigma_y - 4\,\sigma_y\sigma_z$$

$$\sigma_z \wedge b = \sigma_z \wedge (2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z)$$
$$= 2\,\sigma_z\sigma_x - 4\,\sigma_y\sigma_z$$

$$a \wedge \sigma_z = (3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \wedge \sigma_z$$
$$= -\,3\,\sigma_z\sigma_x + 5\,\sigma_y\sigma_z$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$x_1 = (a \wedge b)^{-1} (\sigma_x \wedge b)$$

$$= -\frac{1}{418} (\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)\,(4\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x)$$

$$= -\frac{1}{418} (4\,\sigma_x\sigma_y\sigma_x\sigma_y - 8\,\sigma_x\sigma_y\sigma_z\sigma_x + 48\,\sigma_y\sigma_z\sigma_x\sigma_y$$
$$- 96\,\sigma_y\sigma_z\sigma_z\sigma_x - 32\,\sigma_z\sigma_x\sigma_x\sigma_y + 64\,\sigma_z\sigma_x\sigma_z\sigma_x)$$

$$= -\frac{1}{418} (-4 - 8\,\sigma_y\sigma_z + 48\,\sigma_z\sigma_x$$
$$+ 96\,\sigma_x\sigma_y + 32\,\sigma_y\sigma_z - 64)$$

$$= -\frac{1}{418} (-68 + 96\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z + 48\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (68 - 96\,\sigma_x\sigma_y - 24\,\sigma_y\sigma_z - 48\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (68 - 12\,M)$$

$$\uparrow$$

$$M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$x_2 = (a \wedge b)^{-1} (\sigma_y \wedge b)$$

$$= -\frac{1}{418} (\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(-2\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (-2\,\sigma_x\sigma_y\sigma_x\sigma_y + 8\,\sigma_x\sigma_y\sigma_y\sigma_z - 24\,\sigma_y\sigma_z\sigma_x\sigma_y$$
$$+ 96\,\sigma_y\sigma_z\sigma_y\sigma_z + 16\,\sigma_z\sigma_x\sigma_x\sigma_y - 64\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (2 - 8\,\sigma_z\sigma_x - 24\,\sigma_z\sigma_x$$
$$- 96 - 16\,\sigma_y\sigma_z - 64\,\sigma_x\sigma_y)$$

$$= -\frac{1}{418} (-94 - 64\,\sigma_x\sigma_y - 16\,\sigma_y\sigma_z - 32\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (94 + 64\,\sigma_x\sigma_y + 16\,\sigma_y\sigma_z + 32\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (94 + 8\,M)$$

$$\uparrow$$

$$M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$x_3 = (a \wedge b)^{-1} (\sigma_z \wedge b)$$

$$= -\frac{1}{418} (\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)\,(2\,\sigma_z\sigma_x - 4\,\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (2\,\sigma_x\sigma_y\sigma_z\sigma_x - 4\,\sigma_x\sigma_y\sigma_y\sigma_z + 24\,\sigma_y\sigma_z\sigma_z\sigma_x$$
$$- 48\,\sigma_y\sigma_z\sigma_y\sigma_z - 16\,\sigma_z\sigma_x\sigma_z\sigma_x + 32\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x - 24\,\sigma_x\sigma_y$$
$$+ 48 + 16 + 32\,\sigma_x\sigma_y)$$

$$= -\frac{1}{418} (64 + 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (-\,64 - 8\,\sigma_x\sigma_y - 2\,\sigma_y\sigma_z - 4\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (-\,64 - M)$$

$$\uparrow$$

$$M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_1 = (a \wedge b)^{-1} (a \wedge \sigma_x)$$

$$= -\frac{1}{418} (\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)\,(-5\,\sigma_x\sigma_y + 4\,\sigma_z\sigma_x)$$

$$= -\frac{1}{418} (-5\,\sigma_x\sigma_y\sigma_x\sigma_y + 4\,\sigma_x\sigma_y\sigma_z\sigma_x - 60\,\sigma_y\sigma_z\sigma_x\sigma_y$$
$$+ 48\,\sigma_y\sigma_z\sigma_z\sigma_x + 40\,\sigma_z\sigma_x\sigma_x\sigma_y - 32\,\sigma_z\sigma_x\sigma_z\sigma_x)$$

$$= -\frac{1}{418} (5 + 4\,\sigma_y\sigma_z - 60\,\sigma_z\sigma_x$$
$$- 48\,\sigma_x\sigma_y - 40\,\sigma_y\sigma_z + 32)$$

$$= -\frac{1}{418} (37 - 48\,\sigma_x\sigma_y - 36\,\sigma_y\sigma_z - 60\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (-37 + 48\,\sigma_x\sigma_y + 36\,\sigma_y\sigma_z + 60\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (-37 + 12\,N)$$

$$\uparrow$$

$$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_2 = (a \wedge b)^{-1} (a \wedge \sigma_y)$$

$$= -\frac{1}{418} (\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)\,(3\,\sigma_x\sigma_y - 4\,\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (3\,\sigma_x\sigma_y\sigma_x\sigma_y - 4\,\sigma_x\sigma_y\sigma_y\sigma_z + 36\,\sigma_y\sigma_z\sigma_x\sigma_y$$

$$- 48\,\sigma_y\sigma_z\sigma_y\sigma_z - 24\,\sigma_z\sigma_x\sigma_x\sigma_y + 32\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (-3 + 4\,\sigma_z\sigma_x + 36\,\sigma_z\sigma_x$$

$$+ 48 + 24\,\sigma_y\sigma_z + 32\,\sigma_x\sigma_y)$$

$$= -\frac{1}{418} (45 + 32\,\sigma_x\sigma_y + 24\,\sigma_y\sigma_z + 40\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (-45 - 32\,\sigma_x\sigma_y - 24\,\sigma_y\sigma_z - 40\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (-45 - 8\,N)$$

$$\uparrow$$

$$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$

Business Mathematics Example

Elements of the generalized matrix inverse:

$$D^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_3 = (a \wedge b)^{-1} (a \wedge \sigma_z)$$

$$= -\frac{1}{418} (\sigma_x\sigma_y + 12\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)(-3\,\sigma_z\sigma_x + 5\,\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (-3\,\sigma_x\sigma_y\sigma_z\sigma_x + 5\,\sigma_x\sigma_y\sigma_y\sigma_z - 36\,\sigma_y\sigma_z\sigma_z\sigma_x$$
$$+ 60\,\sigma_y\sigma_z\sigma_y\sigma_z + 24\,\sigma_z\sigma_x\sigma_z\sigma_x - 40\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= -\frac{1}{418} (-\,3\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x + 36\,\sigma_x\sigma_y$$
$$-\,60 - 24 - 40\,\sigma_x\sigma_y)$$

$$= -\frac{1}{418} (-\,84 - 4\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (84 + 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x)$$

$$= \frac{1}{418} (84 + N)$$

$$N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$$

Business Mathematics Example

The generalized matrix inverse D^{-1} then is:

$$D^{-1} = \frac{1}{418}\begin{bmatrix} 68-12\,M & 94+8\,M & -64-M \\ -37+12\,N & -45-8\,N & 84+N \end{bmatrix}$$

with $\quad M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$

$\quad N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$

Left-sided check of generalized matrix inverse D^{-1}

$$\begin{array}{ccc|cc} & & & 3 & 2 \\ & & & 5 & 4 \\ & & & 4 & 8 \\ \hline 68-12\,M & 94+8\,M & -64-M & 418 & 0 \\ -37+12\,N & -45-8\,N & 84+N & 0 & 418 \end{array}$$

Result of Second Example

$q = D^{-1}\, r$

$$\begin{array}{ccc|c} & & & 120 \\ & & & 220 \\ & & & 320 \\ \hline 68-12\,M & 94+8\,M & -64-M & 8360 \\ -37+12\,N & -45-8\,N & 84+N & 12540 \end{array}$$

$$\Downarrow$$

$$x = \frac{8360}{418} = 20 \qquad y = \frac{12540}{418} = 30$$

$\Rightarrow$ If exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Check of result:

$$\begin{array}{cc|c} & & 20 \\ & & 30 \\ \hline 3 & 2 & 120 \\ 5 & 4 & 220 \\ 4 & 8 & 320 \end{array}$$

Addendum: The Dual

The coefficient vectors of the second example problem

$$a = 3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z$$

$$b = 2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z$$

are one-dimensional objects in a three-dimensional space. Therefore two planes will exist which are perpendicular to the coefficient vectors. These planes are called the duals of the given vectors.

The bivectors which represent these planes can be found by a pre-multiplication of the unit volume element

$$I = \sigma_x\sigma_y\sigma_z$$

$$\begin{aligned} I\,a &= \sigma_x\sigma_y\sigma_z\,(3\,\sigma_x + 5\,\sigma_y + 4\,\sigma_z) \\ &= 3\,\sigma_x\sigma_y\sigma_z\sigma_x + 5\,\sigma_x\sigma_y\sigma_z\sigma_y + 4\,\sigma_x\sigma_y\sigma_z\sigma_z \\ &= 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x = N \end{aligned}$$

$$\begin{aligned} I\,b &= \sigma_x\sigma_y\sigma_z\,(2\,\sigma_x + 4\,\sigma_y + 8\,\sigma_z) \\ &= 2\,\sigma_x\sigma_y\sigma_z\sigma_x + 4\,\sigma_x\sigma_y\sigma_z\sigma_y + 8\,\sigma_x\sigma_y\sigma_z\sigma_z \\ &= 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x = M \end{aligned}$$

$\Rightarrow$ The bivectors terms of the generalized matrix inverse are parallel to the duals of a and b.

Moore-Penrose Generalized Matrix Inverses

More and more introductory business mathematics textbooks discuss Moore-Penrose generalized matrix inverses as elementary part of the foundations of matrix algebra.

Some universities in Germany routinely present Moore-Penrose in introductory courses and apply them routinely to solve business math problems and problems of mathematical economics, e.g. see:

> Karsten Schmidt, Götz Trenkler: Einführung in die Moderne Matrix-Algebra. Mit Anwendungen in der Statistik. Dritte Auflage, Springer/Gabler, Berlin, Heidelberg 2015.

→ *Der vermittelte Stoff soll aktuell und modern sein. Deshalb bedienen wir uns der in letzter Zeit immer populärer gewordenen Hilfsmittel wie verallgemeinerte Inversen und Moore-Penrose-Inverse von Matrizen und ihrer Anwendung zur Lösung linearer Gleichungssysteme.* (Preface of Schmidt & Trenkler)

→ *Moore-Penrose generalized matrix inverses are relevant and they are of topical interest.*

Moore-Penrose Generalized Matrix Inverses

There are two possible ways to define Moore-Penrose generalized matrix inverses (denoted by A^+).

The first definition is based on the four Moore-Penrose conditions:

$$A\,A^+ A = A$$

$$A^+ A\,A^+ = A^+$$

$$(A\,A^+)^T = A\,A^+$$

$$(A^+ A)^T = A^+ A$$

The Moore-Penrose generalized matrix inverse is then uniquely defined as

$$A^+ = (A^T A)^{-1} A^T$$

If you are interested in this definition and its consequences please have a look at:

Kamran Dadkhah: Foundations of Mathematical and Computational Economics. 2nd ed., Springer, Heidelberg, Dordrecht 2011, p. 147, eq. 7.14.

Moore-Penrose Generalized Matrix Inverses

The second definition simply states that the Moore-Penrose generalized matrix inverse is the scalar part of the Pauli algebra generalized matrix inverse.

Thus the Moore-Penrose generalized matrix inverse of a (3 x 2)-matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

will be

$$A^+ = \begin{bmatrix} \left\langle (a\wedge b)^{-1}(\sigma_x \wedge b)\right\rangle_0 & \left\langle (a\wedge b)^{-1}(\sigma_y \wedge b)\right\rangle_0 & \left\langle (a\wedge b)^{-1}(\sigma_z \wedge b)\right\rangle_0 \\ \left\langle (a\wedge b)^{-1}(a \wedge \sigma_x)\right\rangle_0 & \left\langle (a\wedge b)^{-1}(a \wedge \sigma_y)\right\rangle_0 & \left\langle (a\wedge b)^{-1}(a \wedge \sigma_z)\right\rangle_0 \end{bmatrix}$$

Forget about higher-dimensional terms: Only scalar terms are taken into account.

If $a \wedge b \neq 0$, the matrix equation $A\,q = r$ can be solved by $q = A^+ r$

Second Example Revisited

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

3 units of R_1, 5 units of R_2, and 4 units of R_3 to produce 1 unit of P_1

2 units of R_1, 4 units of R_2, and 8 units of R_3 to produce 1 unit of P_2

Find the Moore-Penrose generalized matrix inverse and find the quantities of final products P_1 and P_2 which will be produced, if exactly 120 units of the first raw material R_1, 220 units of the second raw material R_2, and 320 units of the third raw material R_3 are consumed in the production process.

Check the Moore-Penrose conditions.

Moore-Penrose Generalized Matrix Inverses of Second Example

The Pauli algebra generalized matrix inverse D^{-1} of this problem has been:

$$D^{-1} = \frac{1}{418}\begin{bmatrix} 68-12\,M & 94+8\,M & -64-M \\ -37+12\,N & -45-8\,N & 84+N \end{bmatrix}$$

with $\quad M = 8\,\sigma_x\sigma_y + 2\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x$

$\quad N = 4\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_z + 5\,\sigma_z\sigma_x$

Thus the Moore-Penrose generalized matrix inverse will be

$$D^{+} = \frac{1}{418}\begin{bmatrix} 68 & 94 & -64 \\ -37 & -45 & 84 \end{bmatrix}$$

because Moore and Penrose have decided to neglect all bivector terms: $\quad M = 0$

$N = 0$

Result of Second Example Revisited

$q = D^{+} r$			120 220 320
$\frac{68}{418}$	$\frac{94}{418}$	$-\frac{64}{418}$	20
$-\frac{37}{418}$	$-\frac{45}{418}$	$\frac{84}{418}$	30

$\Rightarrow$ If exactly 120 units of the first raw material R_1 and 220 units of the second raw material R_2 are consumed, 20 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Checking the Moore-Penrose Conditions

$$D = \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 4 & 8 \end{bmatrix} \qquad D^{+} = \frac{1}{418}\begin{bmatrix} 68 & 94 & -64 \\ -37 & -45 & 84 \end{bmatrix}$$

Third Moore-Penrose condition:

$$\begin{array}{cc|ccc} \multicolumn{2}{c|}{D\,D^{+}} & \frac{68}{418} & \frac{94}{418} & -\frac{64}{418} \\ & & -\frac{37}{418} & -\frac{45}{418} & \frac{84}{418} \\ \hline 3 & 2 & \frac{130}{418} & \frac{192}{418} & -\frac{24}{418} \\ 5 & 4 & \frac{192}{418} & \frac{290}{418} & \frac{16}{418} \\ 4 & 8 & -\frac{24}{418} & \frac{16}{418} & \frac{416}{418} \end{array}$$

$\Rightarrow$ The matrix product $D\,D^{+}$ is symmetric:

$$(D\,D^{+})^{T} = D\,D^{+}$$

Checking the Moore-Penrose Conditions

$$D = \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 4 & 8 \end{bmatrix} \qquad D^+ = \frac{1}{418}\begin{bmatrix} 68 & 94 & -64 \\ -37 & -45 & 84 \end{bmatrix}$$

Fourth Moore-Penrose condition:

$$\begin{array}{ccc|cc}
 & D^+ D & & 3 & 2 \\
 & & & 5 & 4 \\
 & & & 4 & 8 \\
\hline
\frac{68}{418} & \frac{94}{418} & -\frac{64}{418} & 1 & 0 \\
-\frac{37}{418} & -\frac{45}{418} & \frac{84}{418} & 0 & 1
\end{array}$$

$\Rightarrow$ The matrix product D^+D equals the identity matrix I, which of course is symmetric, too:

$$(D^+ D)^T = D^+ D$$

Checking the Moore-Penrose Conditions

$$D\,D^{+} = \frac{1}{418}\begin{bmatrix}130 & 192 & -24\\ 192 & 290 & 16\\ -24 & 16 & 416\end{bmatrix} \qquad D = \begin{bmatrix}3 & 2\\ 5 & 4\\ 4 & 8\end{bmatrix}$$

First Moore-Penrose condition:

$$\begin{array}{ccc|cc}
D\,D^{+}D & & & 3 & 2\\
 & & & 5 & 4\\
 & & & 4 & 8\\
\hline
\frac{130}{418} & \frac{192}{418} & -\frac{24}{418} & 3 & 2\\
\frac{192}{418} & \frac{290}{418} & \frac{16}{418} & 5 & 4\\
-\frac{24}{418} & \frac{16}{418} & \frac{416}{418} & 4 & 8
\end{array}$$

$$\Rightarrow \qquad D\,D^{+}D = D\,I = D$$

Checking the Moore-Penrose Conditions

$$D\,D^+ = \frac{1}{418}\begin{bmatrix} 130 & 192 & -24 \\ 192 & 290 & 16 \\ -24 & 16 & 416 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 4 & 8 \end{bmatrix}$$

Second Moore-Penrose condition:

$$D^+ D\, D^+ = D^+$$

$$\begin{array}{c|c} & \begin{matrix} \frac{130}{418} & \frac{192}{418} & -\frac{24}{418} \\ \frac{192}{418} & \frac{290}{418} & \frac{16}{418} \\ -\frac{24}{418} & \frac{16}{418} & \frac{416}{418} \end{matrix} \\ \hline \begin{matrix} \frac{68}{418} & \frac{94}{418} & -\frac{64}{418} \\ -\frac{37}{418} & -\frac{45}{418} & \frac{84}{418} \end{matrix} & \begin{matrix} \frac{68}{418} & \frac{94}{418} & -\frac{64}{418} \\ -\frac{37}{418} & -\frac{45}{418} & \frac{84}{418} \end{matrix} \end{array}$$

$$\Rightarrow \qquad D^+ D\, D^+ = \mathrm{I}\, D^+ = D^+$$

Outlook: Generalized Matrix Inverses of Higher-Dimensional Non-Square Matrices

Pauli algebra generalized matrix inverses and Moore-Penrose generalized matrix inverses of higher-dimensional non-square matrices can be found in a similar way.

The elements of the generalized matrix inverses then are:

$$x_i = (a \wedge b \wedge c \wedge d \wedge \ldots)^{-1} (\sigma_i \wedge b \wedge c \wedge d \wedge \ldots)$$

$$y_i = (a \wedge b \wedge c \wedge d \wedge \ldots)^{-1} (a \wedge \sigma_i \wedge c \wedge d \wedge \ldots)$$

$$z_i = (a \wedge b \wedge c \wedge d \wedge \ldots)^{-1} (a \wedge b \wedge \sigma_i \wedge d \wedge \ldots)$$

$$v_i = (a \wedge b \wedge c \wedge d \wedge \ldots)^{-1} (a \wedge b \wedge c \wedge \sigma_i \wedge \ldots)$$

etc…