Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

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Dr. Horn

Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part VI: Solving systems of linear equations with sandwich products)

Stand: 21. Jan. 2017

Teaching & learning contents according to the modular description of LV 200 691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Starting with a historical review on the solution of systems of linear equations, the strategy of Grassmann using outer products to solve these systems is transformed into strategies using complete geometric products.

Repetition: Basics of Geometric Algebra

$1 + 3 + 3 + 1 = 2^3 = 8$ different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: $\sigma_x, \sigma_y, \sigma_z$

Three base bivectors: $\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

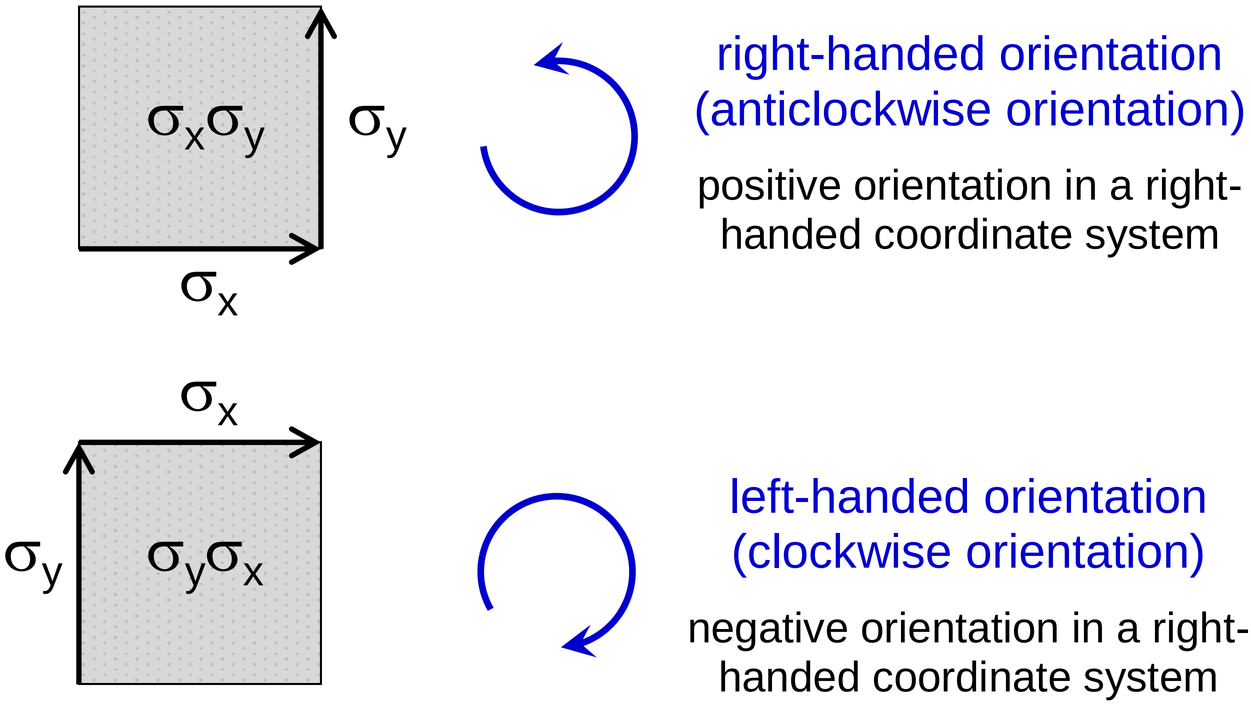
$$1^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

Base bivectors and base trivector square to minus one:

$$(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1$$

Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the resulting area elements.



Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

$$\sigma_x\sigma_y = -\,\sigma_y\sigma_x$$

$$\sigma_y\sigma_z = -\,\sigma_z\sigma_y$$

$$\sigma_z\sigma_x = -\,\sigma_x\sigma_z$$

Scalars

Scalars are geometric entities without direction. They can be expressed as multiples of the base scalar:

$$k = k\,1$$

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

$$r = x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

$$A = A_{xy}\,\sigma_x\sigma_y + A_{yz}\,\sigma_y\sigma_z + A_{zx}\,\sigma_z\sigma_x$$

Trivectors

Trivectors are oriented volume elements. They can be expressed as multiples of the base tri-vector:

$$V = V_{xyz}\,\sigma_x\sigma_y\sigma_z$$

Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$a\,b = a \bullet b + a \wedge b$$

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

$$a \bullet b = b \bullet a = \frac{1}{2}(a\,b + b\,a)$$

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

$$a \wedge b = -\,b \wedge a = \frac{1}{2}(a\,b - b\,a)$$

Geometric Multiplication of Vectors and Bivectors

The product of a bivector B and a vector a consists of a vector term and a trivector term. As the dimension of bivector B is reduced, the vector term is called inner product (dot product). And as the dimension of bivector B is increased, the trivector term is called outer product (exterior product or wedge product).

$$B\,a = B \bullet a + B \wedge a$$

In contrast to what was said on the last slide, the inner product of a bivector and a vector is an anti-commutative product as a reversion of the order of bivector and vector changes the sign of the inner product:

$$B \bullet a = -\,a \bullet B = \frac{1}{2}(B\,a - a\,B)$$

The outer product of a bivector and a vector is a commutative product as a reversion of the order of bivector and vector does not change it:

$$B \wedge a = a \wedge B = \frac{1}{2}(B\,a + a\,B)$$

Systems of Two Linear Equations

$$a_1 x + b_1 y = r_1$$
$$a_2 x + b_2 y = r_2 \quad \Rightarrow \quad a x + b y = r$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1 \sigma_x + a_2 \sigma_y)\, x + (b_1 \sigma_x + b_2 \sigma_y)\, y = r_1 \sigma_x + r_2 \sigma_y$$

Solutions:

$$x = \frac{1}{a \wedge b}(r \wedge b) = (a \wedge b)^{-1}\,(r \wedge b)$$

$$y = \frac{1}{a \wedge b}(a \wedge r) = (a \wedge b)^{-1}\,(a \wedge r)$$

Systems of Three Linear Equations

$$a_1\, x + b_1\, y + c_1\, z = r_1$$
$$a_2\, x + b_2\, y + c_2\, z = r_2 \quad \Rightarrow \quad a\, x + b\, y + c\, z = r$$
$$a_3\, x + b_3\, y + c_3\, z = r_3$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1\, \sigma_x + a_2 \sigma_y + a_3\, \sigma_z)\, x + (b_1\, \sigma_x + b_2\, \sigma_y + b_3\, \sigma_z)\, y$$
$$+ (c_1\, \sigma_x + c_2\, \sigma_y + c_3\, \sigma_z)\, z = r_1\, \sigma_x + r_2\, \sigma_y + r_3\, \sigma_z$$

Solutions:

$$x = (a \wedge b \wedge c)^{-1}\, (r \wedge b \wedge c)$$
$$y = (a \wedge b \wedge c)^{-1}\, (a \wedge r \wedge c)$$
$$z = (a \wedge b \wedge c)^{-1}\, (a \wedge b \wedge r)$$

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of former lessons and in Geometric Algebra books.

History of Solving Systems of Linear Equations

Babylonian Mathematics:

At cuneiform tablets from the Old Babylonian period (2000 – 1600 BCE) the first systems of linear equations written down by mankind had been found.

First VAT 8389 Problem

There are two fields whose total area is 1800 sar.

The rent for one field is 2/3 silà of grain per sar, the rent for the other is 1/2 silà per sar, and the total rent on the first exceeds that on the other by 500 silà.

What is the size of each field?

VAT 8389: This important cuneiform tablet is part of the collection of the “Vorderasiati-sches Museum (VAT)” in Berlin.

First VAT 8389 Problem

The first VAT 8393 problem can be written in modern form as a system of two linear equations:

$$x + y = 1800$$

$$\frac{2}{3}x - \frac{1}{2}y = 500$$

4000 years ago the Babylonians routinely solved such systems of linear equations!

The solutions are: $x = ?$ $y = ?$

$\Rightarrow$ The fields have following sizes: . . .

See: Joseph F. Grcar: Mathematicians of Gaussian Elimination. Notices of the AMS, Vol. 58, No. 6, June/July 2011, pp. 782 – 792.

Joseph F. Grcar: How Ordinary Elimination Became Gaussian Elimination. arXiv:0907.2397v4 [math.HO], 30. Sep. 2010, download at www.arxiv.org.

First VAT 8389 Problem

The first VAT 8389 problem can be written in modern form as a system of two linear equations:

$$x + y = 1800$$

$$\frac{2}{3}x - \frac{1}{2}y = 500$$

4000 years ago the Babylonians routinely solved such systems of linear equations!

The solutions are: $x = 1200$ $y = 600$

$\Rightarrow$ The first field has a size of 1200 sar.
The second field has s size of 600 sar.

See: Joseph F. Grcar: Mathematicians of Gaussian Elimination. Notices of the AMS, Vol. 58, No. 6, June/July 2011, pp. 782 – 792.

Joseph F. Grcar: How Ordinary Elimination Became Gaussian Elimination. arXiv:0907.2397v4 [math.HO], 30. Sep. 2010, download at www.arxiv.org.

History of Solving Systems of Linear Equations

Old China:

The Early Han dynasty (202 BCE – 9 CE) was a period of mathematical creativity in China. At this time a book called

> ***Nine Chapters on the Art of Calculation***

had been written (or compiled from other books burned earlier in the Qin dynasty) by an unknown author or compiler.

"The book certainly existed by the 1st century CE and played a part in the subsequent mathematical culture of China comparable to that played by Euclid's *Elements* in Europe."
(John Derbyshire: Unknown Quantity, p. 163)

Chapter 8 of these ***Nine Chapters*** contains the problem of the following page:

Early Han Dynasty Problem

There are three types of grain.

Three baskets of the first, two of the second, and one of the third weigh 39 measures.

Two baskets of the first, three of the second, and one of the third weigh 34 measures.

And one basket of the first, two of the second, and three of the third weigh 26 measures.

How many measures of grain are contained in one basket of each type?

See: John Derbyshire: Unknown Quantity – A Real and Imaginary History of Algebra. Joseph Henry Press, Washington, DC, 2006.

$\Rightarrow$ Chapter 9: “An Oblong Arrangement of Terms”, (p. 161).

$\Rightarrow$ “Systematic method for solving this and any similar problem, in any number of unknowns – a method that is still taught to beginning students of matrix algebra today. And all this took place over 2,000 years ago!” (p. 162)

$\Rightarrow$ Hans Wußing: “Eine Art Matrizenrechnung!” (p. 57, see next slide)

German Version of the Early Han Dynasty Problem

Jetzt hat man folgendes Problem zur Getreideernte:

Aus 3 Garben einer guten Ernte, 2 Garben einer mittelmäßigen Ernte und 1 Garbe einer schlechten Ernte erhält man den Ertrag von 39 Tou.

Aus 2 Garben einer guten Ernte, 3 Garben einer mittelmäßigen Ernte und 1 Garbe einer schlechten Ernte erhält man den Ertrag von 34 Tou.

Aus 1 Garbe guter Ernte, 2 Garben mittelmäßiger Ernte und 3 Garben schlechter Ernte erhält man den Ertrag von 26 Tou.

Frage: Wieviel ist jedesmal aus 1 Garbe der Ertrag der guten, mittelmäßigen und schlechten Ernte?

See: Hans Wußing: 6000 Jahre Mathematik. Eine kulturgeschichtliche Zeitreise – 1. Von den Anfängen bis Leibniz und Newton. Springer-Verlag, Berlin, Heidelberg 2006.

$\Rightarrow$ Kap. 2.1: Mathematik im alten China (p. 57, 58)

Early Han Dynasty Problem

This problem of the Early Han dynasty can be modelled by the following system of three linear equations:

$$3x + 2y + z = 39$$

$$2x + 3y + z = 34$$

$$x + 2y + 3z = 26$$

with x … first type of grain
(Getreide aus guter Ernte)

y … second type of grain
(Getreide aus mittelmäßiger Ernte)

z … third type of grain
(Getreide aus schlechter Ernte)

Of course you are able to find the solution without any problems:

x = ? y = ? z = ?

Early Han Dynasty Problem

This problem of the Early Han dynasty can be modelled by the following system of three linear equations:

$$
\begin{aligned}
3x + 2y + z &= 39 \\
2x + 3y + z &= 34 \\
x + 2y + 3z &= 26
\end{aligned}
$$

with x … first type of grain
(Getreide aus guter Ernte)

y … second type of grain
(Getreide aus mittelmäßiger Ernte)

z … third type of grain
(Getreide aus schlechter Ernte)

Of course you are able to find the solution without any problems:

$$x = \frac{37}{4} \qquad y = \frac{17}{4} \qquad z = \frac{11}{4}$$

The Father of Algebra

Alexandria / Roman Egypt:

"As long as the Roman Empire showed some stability, Eastern science continued to flourish as a curious blend of Hellenistic and Oriental elements. (…)

Coexistent with the ***pax Romana*** was for some centuries the ***pax Sinensis***; the Eurasian continent in all its history never knew such a period of uninterrupted peace as under the Antonins in Rome and the Han in China. (…)

Hellenistic science continued to flow into China and India and was influenced in its turn by the sciences of those countries. (…)

Alexandria remained the center of antique mathematics. (…) **In Diophantus we find the first systematic use of algebraic symbols**."

See: Dirk J. Struik: A Concise History of Mathematics. Vol. 1, The Beginnings – The Beginnings in Western Europe, Dover Publications, New York 1948, p. 70, 71, 75.

The Father of Algebra

At the time prior to Diophantus Greek mathematics has been mainly geometrical. The school of the Pythagoreans (strongly disturbed by the discovery of the irrationality of $\sqrt{2}$) and others favoured pure geometrical reasoning. Only at the time of Diophantus there was a turn of thought and algebraic thinking evolved.

“So, was Diophantus the father of algebra? I am willing to give him the title just for his literal symbolism – his use of special letter symbols for the unknown and its powers, for subtraction and equality. (…)

It is a shame that we do not know who first used a symbol for the unknown, (…) Probably someone of whom we have no knowledge, nor ever will have any knowledge, was the true father of algebra. Since the title is vacant, though, we may as well attach it to the most worthy name that has survived from antiquity, and that name is surely Diophantus.”

See: John Derbyshire: Unknown Quantity – A Real and Imaginary History of Algebra. Joseph Henry Press, Washington, DC, 2006, p. 32, 34, 41.

The Wonder of Algebra

"We started in 1800 BCE with the Babylonians solving quadratic equations written as word problems, and now here we are 2,600 years later with al-Khwarizmi… solving quadratic equations written as word problems.

It is, I agree, all a bit depressing. Yet it is also inspiring, in a way. The extreme slowness of progress in putting together a symbolic algebra testifies to the very high level at which this subject dwells.

The wonder, to borrow a trope from Dr. Johnson, **is not that it took us so long to learn how to do this stuff; the wonder is that we can do it at all**."

See: John Derbyshire: Unknown Quantity – A Real and Imaginary History of Algebra. Joseph Henry Press, Washington, DC, 2006, p. 51.

Learning is Repeating

Learning is repeating the same stuff over and over again. And learning is rethinking the same stuff from different perspectives over and over again.

Learning is repeating…

Mankind has learned to solve systems of linear equations in the last several thousands of years.

Over and over again Babylonian mathematicians, Egyptian mathematicians, Chinese mathematicians, Indian mathematicians, Greek mathematicians, Arabian mathematicians and many more repeated or newly developed nearly identical strategies to solve systems of linear equations.

The mathematical core of all these strategies has been similar to the elimination method of Gauss. Even the great Newton independently invented Gaussian elimination – only to hide his findings in the drawers of his desk.

Learning is Rethinking

Even the great Newton independently invented Gaussian elimination – only to hide his findings in the drawers of his desk.

So Gaussian elimination was finally named after Carl Friedrich Gauss, although this method has already been used before.

Learning is rethinking…

Learning is not only repeating the same stuff over and over again. Learning also means rethinking the same stuff from different perspectives over and over again.

Such a process of rethinking solution strategies clearly started in 1683:

"It is one of the most remarkable coincidences in the history of mathematics that the discovery of determinants took place ***twice*** in that year. One of these discoveries occurred in the kingdom of Hannover, now part of Germany; the other was in Edo, now known as Tokyo, Japan."

(John Derbyshire: Unknown Quantity, p. 168)

The Theory of Linear Extensions

The discovery of determinants by Gottfried Wilhelm von Leibniz (in Hannover) and Takakazu Seki (in Edo) lays the foundations of solving systems of linear equations by a different method, which is now called Cramer's rule.

This method again was modified and re-arranged by Hermann Günther Grassmann, a schoolmaster in the Prussian city of Stettin, who published his book "Die lineale Ausdehnungslehre" in 1844.

This book is of enormous importance to understanding modern mathematics. Therefore this year 1844 can be compared to the miraculous year ***"annus mirabilis"*** of Leonhard Euler (1744) and the miraculous year of Albert Einstein (1905):

"There occurred another annus mirabilis, 1844, the birthyear of the Ausdehnungslehre, one of the supreme landmarks in the history of the human mind."

(G. Sarton: Grassmann – 1844. Isis, Vol. 35, 1944, pp. 326 – 330)

Die Wissenschaft

der

extensiven Grösse

oder

die Ausdehnungslehre,

eine neue mathematische Disciplin

dargestellt und durch Anwendungen erläutert

von

Hermann Grassmann

Lehrer an der Friedrich - Wilhelm - Schule zu Stettin

Erster Theil,

die **lineale Ausdehnungslehre** enthaltend.

Leipzig, 1844.

Verlag von Otto Wigand

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§ 45. Dass nun die äussere Multiplikation, da sie den Begriff des Verschiedenartigen wesentlich voraussetzt, auf die Zahlenlehre keine so unmittelbare Anwendung findet, wie auf die Geometrie und Mechanik, darf uns freilich nicht wundern, indem die Zahlen ihrem Inhalte nach als gleichartige erscheinen. Aber desto interessanter ist es, zu bemerken, wie in der Algebra, sobald an der Zahl

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noch die Art ihrer Verknüpfung mit anderen Größen festgehalten, und in dieser Hinsicht die eine als von der anderen formell verschiedenartig aufgefasst wird, auch die Anwendbarkeit der äusseren Multiplikation mit einer so schlagenden Entschiedenheit heraustritt, dass ich wohl behaupten darf, es werde durch diese Anwendung auch die Algebra eine wesentlich veränderte Gestalt gewinnen. Um hiervon eine Idee zu geben, will ich n Gleichungen ersten Grades mit n Unbekannten setzen, von der Form

$$\begin{array}{l} a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n == a_0 \\ b_1 x_1 + b_2 x_2 + \ldots\ldots + b_n x_n == b_0 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ s_1 x_1 + s_2 x_2 + \ldots\ldots + s_n x_n == s_0, \end{array}$$

wo $x_1 \ldots . x_n$ die Unbekannten seien. Hier können wir die Zahlenkoefficienten, welche verschiedenen Gleichungen angehören, sofern wir diese Verschiedenheit an ihrem Begriff noch festhalten, als verschiedenartig ansehen, und zwar alle als an sich verschiedenartig, d. h. als unabhängig in dem Sinne unserer Wissenschaft, die einer und derselben Gleichung als unter sich in derselben Beziehung gleichartig. Addiren wir nun in diesem Sinne alle n Gleichungen und bezeichnen die Summe des Verschiedenartigen in dem Sinne unserer Wissenschaft mit dem Verknüpfungszeichen $\dotplus$, indem die gleichen Stellen in den so gebildeten Summenausdrücken immer dem Gleichartigen zukommen sollen, so erhalten wir

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Begriff des
hlehre keine
rie und Me-
n ihrem In-
santer ist es,

… the applicability of outer multiplication emerges with such a striking determination and firmness, that …

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noch die Art ihrer Verknüpfung mit anderen Größen festgehalten, und in dieser Hinsicht die eine als von der anderen formell verschiedenartig aufgefasst wird, auch die Anwendbarkeit der äusseren Multiplikation mit einer so schlagenden Entschiedenheit heraustritt, dass ich wohl behaupten darf, es werde durch diese Anwendung auch die Algebra eine wesentlich veränderte Gestalt gewinnen. Um hiervon eine Idee zu geben, will ich n Gleichungen ersten Grades mit n Unbekannten setzen, von der Form

$$
\begin{array}{l}
a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n == a_0 \\
b_1 x_1 + b_2 x_2 + \ldots\ldots + b_n x_n == b_0 \\
\quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
s_1 x_1 + s_2 x_2 + \ldots\ldots + s_n x_n == s_0,
\end{array}
$$

wo $x_1 \ldots . x_n$ die Unbekann
efficienten, welche verschie
diese Verschiedenheit an ih
denartig ansehen, und zwar
unabhängig in dem Sinne
selben Gleichung als unter

… algebra will gain a substantial different shape.

Addiren wir nun in diesem Sinne alle n Gleichungen und bezeichnen die Summe des Verschiedenartigen in dem Sinne unserer Wissenschaft mit dem Verknüpfungszeichen ∔, indem die gleichen Stellen in den so gebildeten Summenausdrücken immer dem Gleichartigen zukommen sollen, so erhalten wir

$$(a_1 \dotplus b_1 \dotplus . . \dotplus s_1)\, x_1 \dotplus (a_2 \dotplus b_2 \dotplus . . \dotplus s_2)\, x_2 \dotplus . . \dotplus (a_n \dotplus b_n \dotplus . . \dotplus s_n)\, x_n$$
$$== (a_0 \dotplus b_0 \dotplus . . \dotplus s_0),$$

oder bezeichnen wir $(a_1 \dotplus b_1 \dotplus . . . . \dotplus s_1)$ mit p_1 und entsprechend die übrigen Summen, so haben wir

$$p_1 x_1 \dotplus p_2 x_2 \dotplus . . . . \dotplus p_n x_n == p_0.$$

Aus dieser Gleichung, welche die Stelle jener n Gleichungen vertritt, lässt sich nun auf der Stelle jede der Unbekannten, z. B. x_1 finden, wenn wir die beiden Seiten mit dem äusseren Produkte aus den Koefficienten der übrigen Unbekannten äusserlich multipliciren, also hier mit $p_2 \,.\, p_3 \,.\,.\,.\,.\,.\, p_n$. Da nämlich, wenn man die Glieder der linken Seite einzeln multiplicirt, nach dem Begriff des

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äusseren Produktes (§ 31) alle Produkte wegfallen, welche zwei gleiche Faktoren enthalten, so erhält man

$$p_1 \cdot p_2 \cdot p_3 \cdot \cdot \cdot \cdot p_n x_1 == p_0 \cdot p_2 \cdot p_3 \cdot \cdot \cdot \cdot p_n.$$

Also da beide Produkte, als demselben System n-ter Stufe angehörig einander gleichartig sind, so hat man

$$x_1 == \frac{p_0 \cdot p_2 \cdot p_3 \cdot \cdot \cdot \cdot p_n}{p_1 \cdot p_2 \cdot p_3 \cdot \cdot \cdot \cdot p_n} \ *).$$

Also jede Unbekannte ist einem Bruche gleich, dessen Nenner das äussere Produkt der Koefficienten $p_1 \,.\,.\,.\, p_n$ ist, und dessen Zähler man erhält, wenn man in diesem Produkt statt des Koefficienten jener Unbekannten die rechte Seite, nämlich p_0, als Faktor setzt. Alle Unbekannten haben also denselben Nenner, und werden unbestimmt oder unendlich, wenn dieser Nenner null wird, d. h.

$$p_1 \cdot p_2 \cdot \cdot \cdot \cdot \cdot \cdot p_n == 0$$

ist.

Having changed his style of writing, Grassmann described his method 33 years later in the following way:

Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre.

Von H. GRASSMANN in Stettin.

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Nun seien a, b, c drei beliebige Strecken, die nicht einer Ebene angehören, so lässt sich jede andere Strecke d aus ihnen numerisch ableiten. Es sei

$$d = x\,a + y\,b + z\,c\,,$$

so erhält man durch äussere Multiplication mit $[b\,c]$, da $[b\,b\,c]$, $[c\,b\,c]$ null sind, $[d\,b\,c] = x\,[a\,b\,c]$, also $x = \frac{[d\,b\,c]}{[a\,b\,c]}$ und entsprechend für die übrigen, . . .

Stettin, den 25. April 1877.

Modern Way of Writing Grassmann's Equations

Nowadays outer products usually are written with the wedge symbol $\wedge$. Therefore Grassmann's equations can be transformed into

1844: $$\mathbf{x_1} == \frac{\mathbf{p_0 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_n}}{\mathbf{p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_n}}$$

$$\downarrow$$

2017: $$x_1 = \frac{p_0 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n}{p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n}$$

and

1877: $$x = \frac{[dbc]}{[abc]}$$

$$\downarrow$$

2017: $$x = \frac{d \wedge b \wedge c}{a \wedge b \wedge c}$$

Solving the Early Han Dynasty Problem with the Strategy of Grassmann

The problem of the Early Han dynasty can be modelled by the following system of three linear equations:

$$3x + 2y + z = 39$$

$$2x + 3y + z = 34$$

$$x + 2y + 3z = 26$$

The coefficient vectors then are:

$$a = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \longrightarrow a = 3\,\sigma_x + 2\,\sigma_y + \sigma_z$$

$$b = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \longrightarrow b = 2\,\sigma_x + 3\,\sigma_y + 2\,\sigma_z$$

$$c = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \longrightarrow c = \sigma_x + \sigma_y + 3\,\sigma_z$$

The resulting vector of constant terms is:

$$r = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix} \longrightarrow r = 39\,\sigma_x + 34\,\sigma_y + 26\,\sigma_z$$

The four outer products are three-dimensional oriented volume elements, representing the volumes of the corresponding parallelepipeds:

$$a \wedge b \wedge c = 12\,\sigma_x\sigma_y\sigma_z$$

$$r \wedge b \wedge c = 111\,\sigma_x\sigma_y\sigma_z$$

$$a \wedge r \wedge c = 51\,\sigma_x\sigma_y\sigma_z$$

$$a \wedge b \wedge r = 33\,\sigma_x\sigma_y\sigma_z$$

$$\Rightarrow \quad (a \wedge b \wedge c)^{-1} = \frac{1}{12}\,\sigma_z\sigma_y\sigma_x$$

Solutions:

$$x = \frac{111}{12} = \frac{37}{4} = 9.25$$

$$y = \frac{51}{12} = \frac{17}{4} = 4.25$$

$$z = \frac{33}{12} = \frac{11}{4} = 2.75$$

Conceptual Problem of Grassmann

In 1844 Grassmann had only invented the outer product. He did not realize, that a combination of outer product $a \wedge b$ and inner product $a \bullet b$ results in the powerful geometric product

$$a\,b = a \bullet b + a \wedge b$$

Even after having defined geometric products in his 1877 paper, he stuck at his strategy of solving systems of linear equations with the help of outer products.

And so we do today. We do not solve systems of linear equations by using the geometric product, because the history of mathematics led us (and chained us mentally) to the solution strategy of Grassmann.

$\Rightarrow$ All this has severe didactical consequences.

Mathematical Viruses

David Hestenes: “My purpose here is to call your attention to another kind of virus – one which can infect the mind – the mind of anyone doing mathematics, from young student to professional mathematician.

(…) A mathematical virus (MV) is a preconception about the structure, function or method of mathematics which impairs one's ability to do mathematics. Just as a computer virus (CV) is program which impairs the operating system of a computer, an MV is an idea which impairs the conceptualization of mathematics in the mind.”

⇒ **MV/G:** “I call this the Grassmann Virus, because it is a distortion of Grassmann's own view.”

See: David Hestenes: Mathematical Viruses. In: A. Micali et al.: Clifford Algebras and their Applications in Mathematical Physics. Kluwer Academic Publishers, Dordrecht 1992, pp. 3 – 16.

Grassmann Virus

MV/G: Grassmann Algebra is more fundamental than Clifford Algebra.

This virus has infected mathematicians who think that calculations using only outer products are more fundamental than calculations using the geometric product.

David Hestenes: "I call this the Grassmann Virus, because it is a distortion of Grassmann's own view. (…) Some mathematicians regard this as a mapping of Grassmann Algebra into Clifford Algebra and insist on regarding Grassmann Algebra as a separate entity. Evidently this violates Occam's stricture against unnecessary assumptions, unless it can be given stronger justification than the historical accident that Grassmann Algebra has been developed independently of Clifford Algebra."

Grassmann Virus

MV/G: Grassmann Algebra is more fundamental than Clifford Algebra.

Anti-MV/G: To prevent students from believing that calculations using only outer products are more fundamental than calculations using the geometric product, it is helpful to discuss the solution of systems of linear equations using the geometric product.

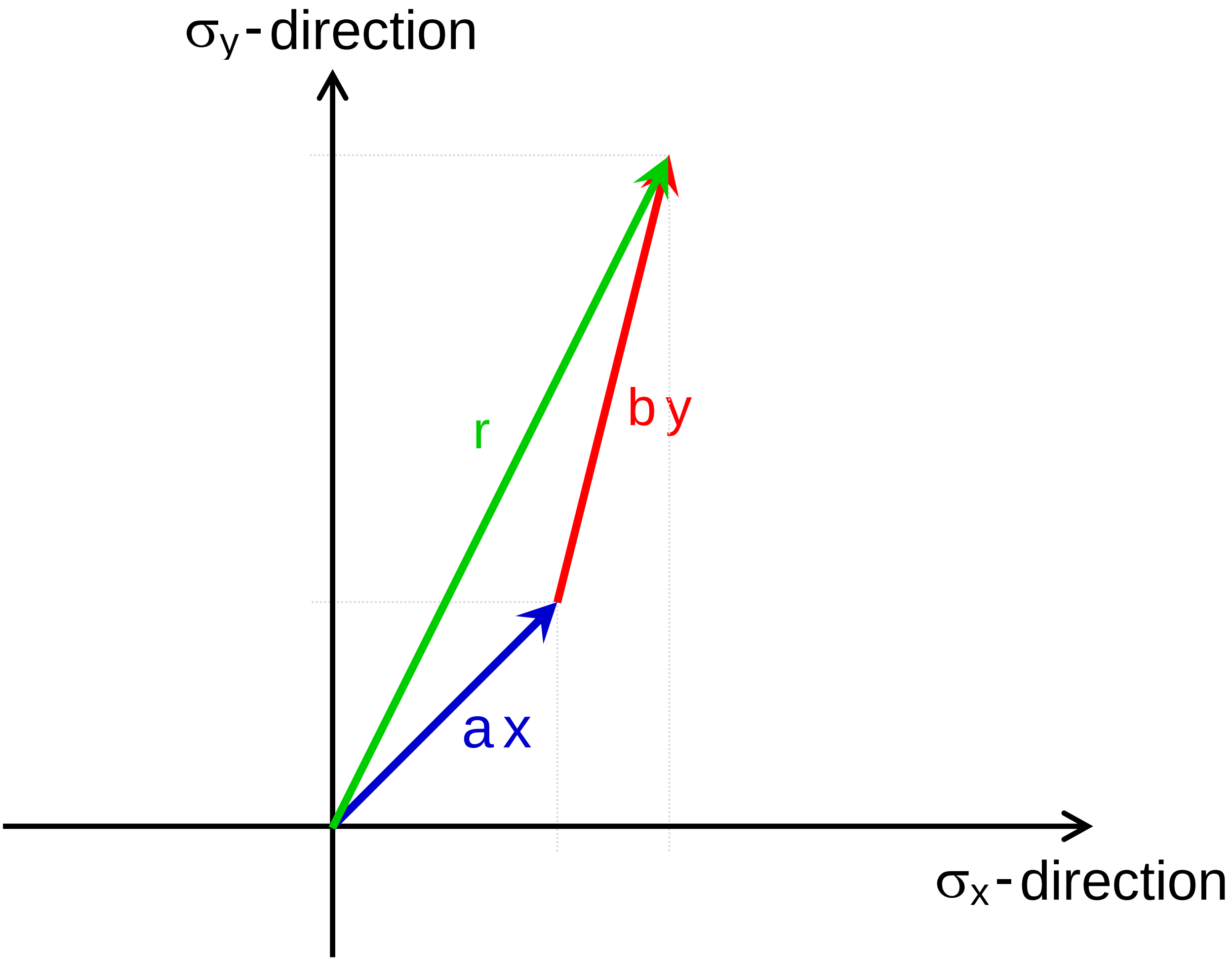
$\Rightarrow$ Such strategies to solve systems of linear equations with geometric products can be found by applying the sandwich product.

$\Rightarrow$ As pure sandwich products model reflections, this strategy can be understood geometrically.

$\Rightarrow$ And as mixed sandwich products model the exchange of two axis, this strategy can be understood geometrically, too.

Geometric Representation of a System of Linear Equations

$a\,x + b\,y = r$



The resulting vector of constant terms r (green) is composed of vectors a x (blue) and b y (red).

These vectors are now reflected at an axis pointing into the direction of a x.

Repetition: The Sandwich Product

(⇒ see slide 21 of part I)

If vector r is multiplied by another vector from the left and from the right in a sand-wich-like manner, vector r will be mapped to a resulting vector r':

$$r' = \sigma_x\, r\, \sigma_x = \; x\, \sigma_x - y\, \sigma_y - z\, \sigma_z$$

$$r'' = \sigma_y\, r\, \sigma_y = -x\, \sigma_x + y\, \sigma_y - z\, \sigma_z$$

$$r''' = \sigma_z\, r\, \sigma_z = -x\, \sigma_x - y\, \sigma_y + z\, \sigma_z$$

These formulae describe reflections!

If r is reflected at a vector pointing into the direction of the x-axis, the x-coordinate is unchanged while the y- and z-coordinates will change their signs. Thus r is mapped to r' by a reflection at the x-axis.

r is mapped to r'' by a reflection at the y-axis.
r is mapped to r''' by a reflection at the z-axis.

The sandwich product of a vector with a unit vector results in a reflection.

Repetition: Reflections

(⇒ see slide 22 of part I)

The sandwich product of a vector with an arbitrary vector n (which is no unit vector) results in a reflection **and** a dilation.

To suppress the dilation and to get a pure reflection, it is necessary to divide by n^2. Thus a reflection at an axis which points into the direction of vector n, has to be written as

$$r_{ref} = \frac{1}{n^2}\, n\, r\, n$$

or

$$r_{ref} = n\, r\, n^{-1}$$

Reflections are important operations, as they conserve the length of vectors:

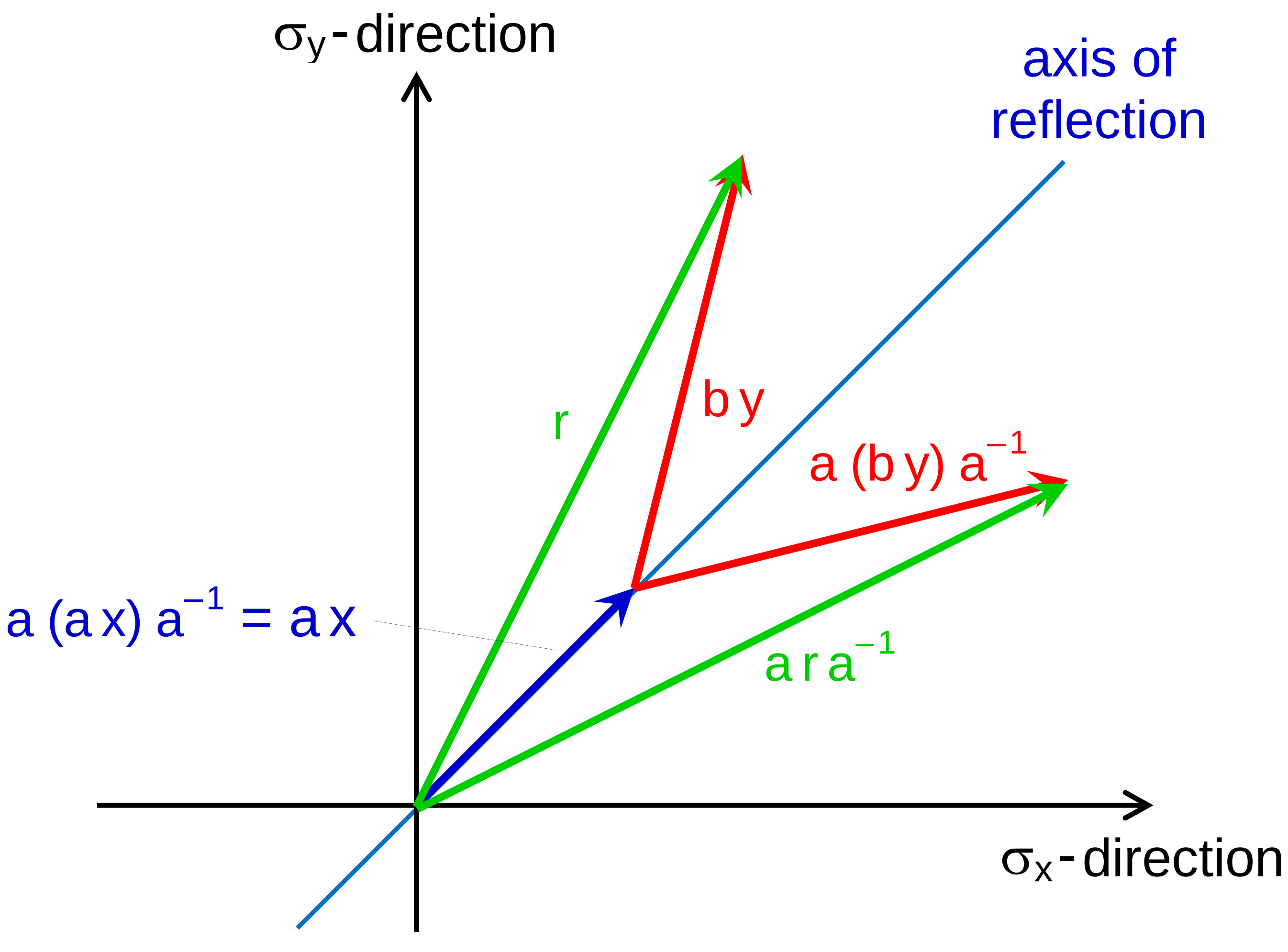
$$|r_{ref}| = |r|$$

Lasenby, Doran: “This formula is already more compact than can be written down without the geometric product … The compression afforded by the geometric product becomes increasingly impressive as reflections are compounded together.”

Geometric Representation of the Reflected System of Linear Equations

$$\frac{1}{a^2}\,(a^3 x + a\,b\,a\,y) = \frac{1}{a^2}\,(a\,r\,a)$$

$$a\,x + \frac{a\,b\,a}{a^2}\,y = \frac{a\,r\,a}{a^2}$$

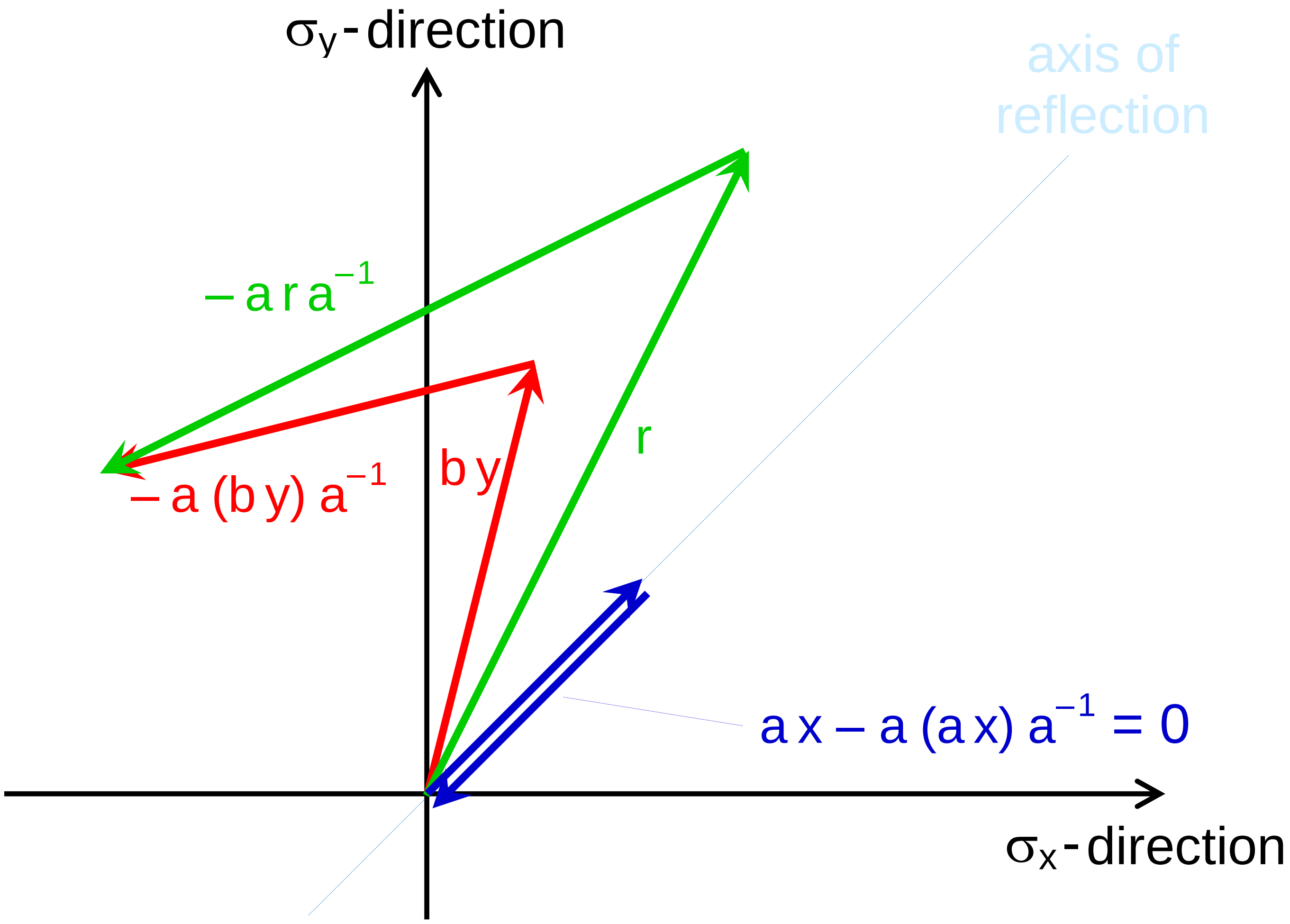


Vector $a\,x$ (blue) remains unchanged. Therefore this vector (and thus variable x) will disappear, if the original and reflected systems of linear equations are subtracted.

Subtraction of Original and Reflected Systems of Linear Equations

$$a\,x + b\,y - \left(a\,x + \frac{1}{a^2}\,a\,b\,a\,y \right) = r - \frac{1}{a^2}\,a\,r\,a$$

$$b\,y - \frac{1}{a^2}\,a\,b\,a\,y = r - \frac{1}{a^2}\,a\,r\,a$$

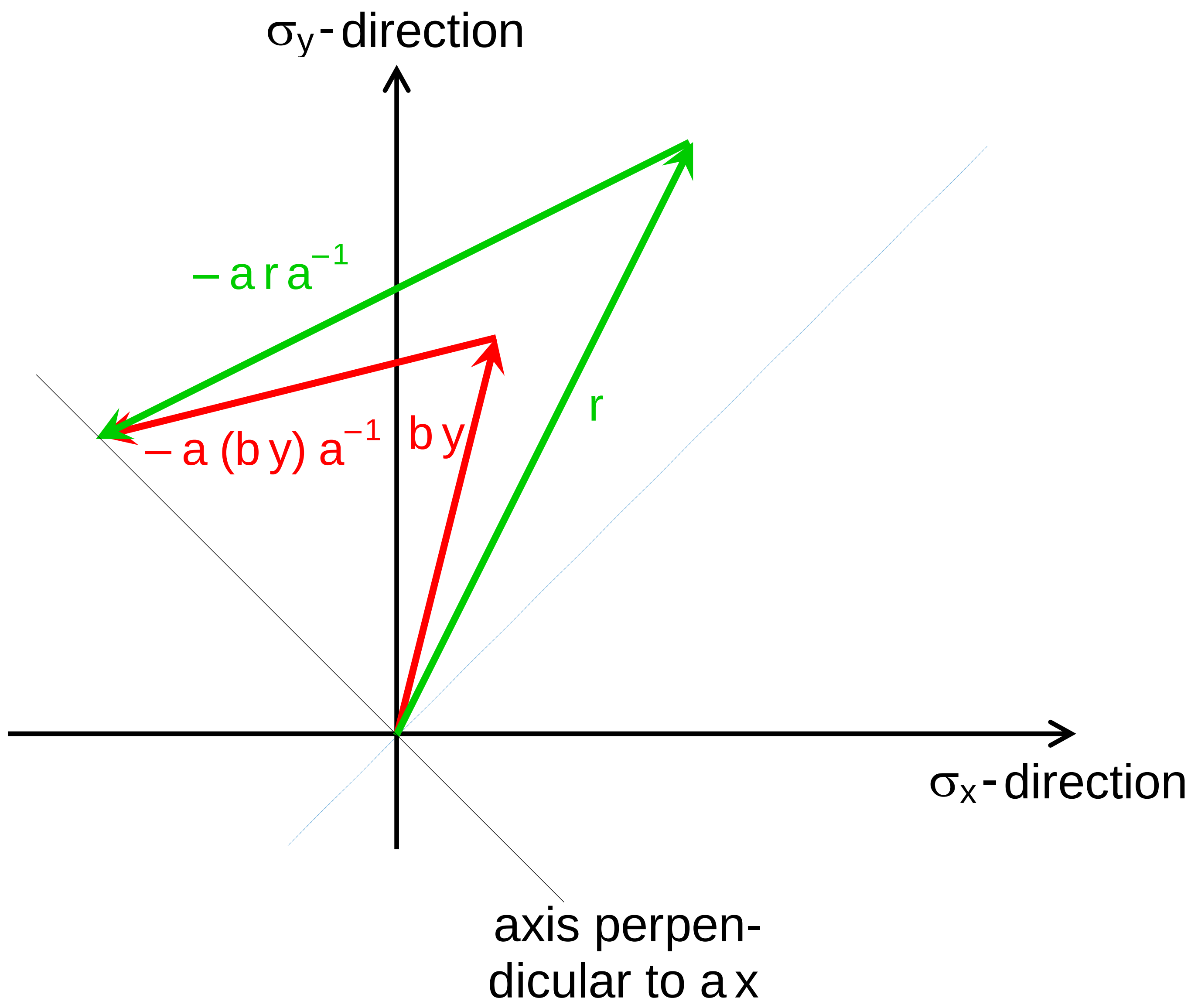


We have now got an equation with only one unknown variable y, which can be found by solving this equation for y.

Subtraction of Original and Reflected Systems of Linear Equations

$$\left(b - \frac{1}{a^2}\, a\, b\, a\right) y = r - \frac{1}{a^2}\, a\, r\, a$$

$$y = (a^2\, b - a\, b\, a)^{-1}\, (a^2\, r - a\, r\, a)$$



Solving Systems of Two Linear Equations With Sandwich Products

The system of two linear equations

$$a\,x + b\,y = r$$

can be solved for the unknown variables with pure sandwich products:

$$x = (b^2\,a - b\,a\,b)^{-1}\,(b^2\,r - b\,r\,b)$$

$$y = (a^2\,b - a\,b\,a)^{-1}\,(a^2\,r - a\,r\,a)$$

As $b^{-1}b = 0,$ these equations coincide with the original outer product equations of Grassmann:

$$x = b^{-1}\,(b\,a - a\,b)^{-1}\,b\,(b\,r - r\,b)$$

$$= b^{-1}\,(b \wedge a)^{-1}\,b\,(b \wedge r)$$

$$= (b \wedge a)^{-1}\,(b \wedge r)$$

$$= (a \wedge b)^{-1}\,(r \wedge b)$$

$$y = a^{-1}\,(a\,b - b\,a)^{-1}\,a\,(a\,r - r\,a)$$

$$= (a \wedge b)^{-1}\,(a \wedge r)$$

Solving Systems of Three Linear Equations With Sandwich Products

The system of three linear equations

$$a\,x + b\,y + c\,z = r$$

can be solved by two successive reflections at axes pointing into directions of the coefficient vectors.

Finding z:

- Sandwiching the system of linear equations by vector a from left and right

$$a\,(a\,x + b\,y + c\,z)\,a = a\,r\,a$$

$$\Rightarrow \quad a^3\,x + a\,b\,a\,y + a\,c\,a\,z = a\,r\,a$$

  which represents a reflection at the axis pointing into the direction of a x and a dilation by a^2.

- Subtracting the reflected system of linear equations a r a from the original system of linear equations dilated by a^2:

$$a^2\,r - a\,r\,a = (a^2\,b - a\,b\,a)\,y + (a^2\,c - a\,c\,a)\,z$$

Solving Systems of Three Linear Equations With Sandwich Products

The system of three linear equations

$$a\,x + b\,y + c\,z = r$$

with three unknown variables has now become a system of linear equations with only two unknown variables, which can be divided by vector a:

$$a^2\,r - a\,r\,a = (a^2\,b - a\,b\,a)\,y + (a^2\,c - a\,c\,a)\,z$$

$$a\,r - r\,a = (a\,b - b\,a)\,y + (a\,c - c\,a)\,z$$

- Sandwiching this modified system of linear equations by vector b from left and right

$$b\,(a\,r - r\,a)\,b = b\,\big((a\,b - b\,a)\,y + (a\,c - c\,a)\,z\big)\,b$$

$$\Rightarrow \quad b\,a\,r\,b - b\,r\,a\,b$$

$$= (b\,a\,b^2 - b^2\,a\,b)\,y + (b\,a\,c\,b - b\,c\,a\,b)\,z$$

$$= b^2\,(-a\,b + b\,a)\,y + (b\,a\,c\,b - b\,c\,a\,b)\,z$$

The last step can be made because b^2 is a scalar and commutes with vectors.

Solving Systems of Three Linear Equations With Sandwich Products

Please compare the modified system of linear equations with the reflected modified system of linear equations. As the signs of the y-term are already reversed, we now have to add these two systems of linear equations.

- Adding the reflected modified system of linear equations $b\,a\,r\,b - b\,r\,a\,b$ to the modified system of linear equations dilated by b^2:

$$b^2\,(a\,r - r\,a) + b\,a\,r\,b - b\,r\,a\,b$$
$$= b^2\,(a\,c - c\,a)\,z + (b\,a\,c\,b - b\,c\,a\,b)\,z$$

The system of linear equations with two unknown variables has now become a system of linear equations with only one unknown variable, which can be divided by vector b:

$$b\,a\,r - b\,r\,a + a\,r\,b - r\,a\,b$$
$$= (b\,a\,c - b\,c\,a + a\,c\,b - c\,a\,b)\,z$$

This equation can be solved for z.

Solving Systems of Three Linear Equations With Sandwich Products

In a similar way, the other two unknown variables can be found. Therefore the system of three linear equations

$$a\,x + b\,y + c\,z = r$$

can be solved for the unknown variables with sandwich products:

$$x = (cba - cab + bac - abc)^{-1}(cbr - crb + brc - rbc)$$

$$y = (acb - abc + cba - bca)^{-1}(acr - arc + cra - rca)$$

$$z = (bac - bca + acb - cab)^{-1}(bar - bra + arb - rab)$$

Of course, these equations again coincide with the original outer product equations of Grassmann:

$$z = (b(a\wedge c) + (a\wedge c)b)^{-1}(b(a\wedge r) + (a\wedge r)b)$$

$$= (b \wedge a \wedge c)^{-1}\,(b \wedge a \wedge r)$$

$$= (a \wedge b \wedge c)^{-1}\,(a \wedge b \wedge r)$$

etc…

Solving the Early Han Dynasty Problem With Sandwich Products

As already given on slides 29 & 30 the co-efficient vectors and the resulting vector of constant terms of the Early Han dynasty problem are:

$a = 3\ \sigma_x + 2\ \sigma_y + \ \ \sigma_z$

$b = 2\ \sigma_x + 3\ \sigma_y + 2\ \sigma_z$

$c = \ \ \sigma_x + \ \ \sigma_y + 3\ \sigma_z$

$r = 39\ \sigma_x + 34\ \sigma_y + 26\ \sigma_z$

Intermediate twofold geometric products:

$a\,b = 14 + 5\ \sigma_x\sigma_y + 1\ \sigma_y\sigma_z - 4\ \sigma_z\sigma_x$
$b\,a = 14 - 5\ \sigma_x\sigma_y - 1\ \sigma_y\sigma_z + 4\ \sigma_z\sigma_x$

$b\,c = 11 - 1\ \sigma_x\sigma_y + 7\ \sigma_y\sigma_z - 4\ \sigma_z\sigma_x$
$c\,b = 11 + 1\ \sigma_x\sigma_y - 7\ \sigma_y\sigma_z + 4\ \sigma_z\sigma_x$

$c\,a = \ \ 8 - 1\ \sigma_x\sigma_y - 5\ \sigma_y\sigma_z + 8\ \sigma_z\sigma_x$
$a\,c = \ \ 8 + 1\ \sigma_x\sigma_y + 5\ \sigma_y\sigma_z - 8\ \sigma_z\sigma_x$

Threefold geometric products:

$$a\,b\,c = 31\,\sigma_x + 12\,\sigma_y + 37\,\sigma_z + 12\,\sigma_x\sigma_y\sigma_z$$
$$c\,b\,a = 31\,\sigma_x + 12\,\sigma_y + 37\,\sigma_z - 12\,\sigma_x\sigma_y\sigma_z$$

Check of result: $a^2 b^2 c^2 = 14 \cdot 17 \cdot 11 = 31^2 + 12^2 + 37^2 + 12^2 = 2618$

$$b\,c\,a = 35\,\sigma_x + 32\,\sigma_y - 15\,\sigma_z + 12\,\sigma_x\sigma_y\sigma_z$$
$$a\,c\,b = 35\,\sigma_x + 32\,\sigma_y - 15\,\sigma_z - 12\,\sigma_x\sigma_y\sigma_z$$

Check of result: $a^2 b^2 c^2 = 14 \cdot 17 \cdot 11 = 35^2 + 32^2 + 15^2 + 12^2 = 2618$

$$c\,a\,b = -3\,\sigma_x + 16\,\sigma_y + 47\,\sigma_z + 12\,\sigma_x\sigma_y\sigma_z$$
$$b\,a\,c = -3\,\sigma_x + 16\,\sigma_y + 47\,\sigma_z - 12\,\sigma_x\sigma_y\sigma_z$$

Check of result: $a^2 b^2 c^2 = 14 \cdot 17 \cdot 11 = 3^2 + 16^2 + 47^2 + 12^2 = 2618$

$$a\,b\,r = 820\,\sigma_x + 307\,\sigma_y + 174\,\sigma_z + 33\,\sigma_x\sigma_y\sigma_z$$
$$r\,b\,a = 820\,\sigma_x + 307\,\sigma_y + 174\,\sigma_z - 33\,\sigma_x\sigma_y\sigma_z$$

Check of result: $a^2 b^2 r^2 = 14 \cdot 17 \cdot 3353$
$= 820^2 + 307^2 + 174^2 + 33^2 = 798014$

$$b\,c\,r = 499\,\sigma_x + 595\,\sigma_y - 108\,\sigma_z + 111\,\sigma_x\sigma_y\sigma_z$$
$$r\,c\,b = 499\,\sigma_x + 595\,\sigma_y - 108\,\sigma_z - 111\,\sigma_x\sigma_y\sigma_z$$

Check of result: $b^2 c^2 r^2 = 17 \cdot 11 \cdot 3353$
$= 499^2 + 595^2 + 108^2 + 111^2 = 627011$

$$c\,a\,r = 70\,\sigma_x + 181\,\sigma_y + 690\,\sigma_z + 51\,\sigma_x\sigma_y\sigma_z$$
$$r\,a\,c = 70\,\sigma_x + 181\,\sigma_y + 690\,\sigma_z - 51\,\sigma_x\sigma_y\sigma_z$$

Check of result: $c^2 a^2 r^2 = 11 \cdot 14 \cdot 3353$
$= 70^2 + 181^2 + 690^2 + 51^2 = 516362$

$b\,a\,r = 272\,\sigma_x + 645\,\sigma_y + 554\,\sigma_z - 33\,\sigma_x\sigma_y\sigma_z$
$r\,a\,b = 272\,\sigma_x + 645\,\sigma_y + 554\,\sigma_z + 33\,\sigma_x\sigma_y\sigma_z$

Check of result: $b^2 a^2 r^2 = 17 \cdot 14 \cdot 3353$
$= 272^2 + 645^2 + 554^2 + 33^2 = 798014$

$c\,b\,r = 359\,\sigma_x + 153\,\sigma_y + 680\,\sigma_z - 111\,\sigma_x\sigma_y\sigma_z$
$r\,b\,c = 359\,\sigma_x + 153\,\sigma_y + 680\,\sigma_z + 111\,\sigma_x\sigma_y\sigma_z$

Check of result: $c^2 b^2 r^2 = 11 \cdot 17 \cdot 3353$
$= 359^2 + 153^2 + 680^2 + 111^2 = 627011$

$a\,c\,r = 554\,\sigma_x + 363\,\sigma_y - 274\,\sigma_z - 51\,\sigma_x\sigma_y\sigma_z$
$r\,c\,a = 554\,\sigma_x + 363\,\sigma_y - 274\,\sigma_z + 51\,\sigma_x\sigma_y\sigma_z$

Check of result: $a^2 c^2 r^2 = 14 \cdot 11 \cdot 3353$
$= 554^2 + 363^2 + 274^2 + 51^2 = 516362$

More intermediate twofold geometric products:

$a\,r = 211 + 24\,\sigma_x\sigma_y + 18\,\sigma_y\sigma_z - 39\,\sigma_z\sigma_x$
$r\,a = 211 - 24\,\sigma_x\sigma_y - 18\,\sigma_y\sigma_z + 39\,\sigma_z\sigma_x$

$b\,r = 232 - 49\,\sigma_x\sigma_y + 10\,\sigma_y\sigma_z + 26\,\sigma_z\sigma_x$
$r\,b = 232 + 49\,\sigma_x\sigma_y - 10\,\sigma_y\sigma_z - 26\,\sigma_z\sigma_x$

$c\,r = 151 - 5\,\sigma_x\sigma_y - 76\,\sigma_y\sigma_z + 91\,\sigma_z\sigma_x$
$r\,c = 151 + 5\,\sigma_x\sigma_y + 76\,\sigma_y\sigma_z - 91\,\sigma_z\sigma_x$

More threefold geometric products:

$$a\,r\,c = 352\,\sigma_x + 241\,\sigma_y + 576\,\sigma_z + 51\,\sigma_x\sigma_y\sigma_z$$
$$c\,r\,a = 352\,\sigma_x + 241\,\sigma_y + 576\,\sigma_z - 51\,\sigma_x\sigma_y\sigma_z$$

Check of result: $a^2 r^2 c^2 = 14 \cdot 3353 \cdot 11$
$= 352^2 + 241^2 + 576^2 + 51^2 = 516362$

$$b\,r\,a = 572\,\sigma_x + 621\,\sigma_y + 290\,\sigma_z + 33\,\sigma_x\sigma_y\sigma_z$$
$$a\,r\,b = 572\,\sigma_x + 621\,\sigma_y + 290\,\sigma_z - 33\,\sigma_x\sigma_y\sigma_z$$

Check of result: $b^2 r^2 a^2 = 17 \cdot 3353 \cdot 14$
$= 572^2 + 621^2 + 290^2 + 33^2 = 798014$

$$c\,r\,b = 105\,\sigma_x + 311\,\sigma_y + 712\,\sigma_z + 111\,\sigma_x\sigma_y\sigma_z$$
$$b\,r\,c = 105\,\sigma_x + 311\,\sigma_y + 712\,\sigma_z - 111\,\sigma_x\sigma_y\sigma_z$$

Check of result: $c^2 r^2 b^2 = 11 \cdot 3353 \cdot 17$
$= 105^2 + 311^2 + 712^2 + 111^2 = 627011$

Please be patient!

As all these calculations are rather time-consuming the use of computer programs (as a substitute for non-existing Geometric Algebra pocket calculators) will be discussed in business math courses of following semesters.

$\Rightarrow$ GAALOP (Geometric Algebra algorithms optimizer), see URL: www.gaalop.de

Solving the Early Han Dynasty Problem With Sandwich Products

The solution of the Early Han dynasty problem now is:

$$x = (cba - cab + bac - abc)^{-1}(cbr - crb + brc - rbc)$$

$$= \big(31\,\sigma_x + 12\,\sigma_y + 37\,\sigma_z - 12\,\sigma_x\sigma_y\sigma_z$$
$$- (-3\,\sigma_x + 16\,\sigma_y + 47\,\sigma_z + 12\,\sigma_x\sigma_y\sigma_z)$$
$$+ (-3\,\sigma_x + 16\,\sigma_y + 47\,\sigma_z - 12\,\sigma_x\sigma_y\sigma_z)$$
$$- (31\,\sigma_x + 12\,\sigma_y + 37\,\sigma_z + 12\,\sigma_x\sigma_y\sigma_z)\big)^{-1}$$
$$\big(359\,\sigma_x + 153\,\sigma_y + 680\,\sigma_z - 111\,\sigma_x\sigma_y\sigma_z$$
$$- (105\,\sigma_x + 311\,\sigma_y + 712\,\sigma_z + 111\,\sigma_x\sigma_y\sigma_z)$$
$$+ (105\,\sigma_x + 311\,\sigma_y + 712\,\sigma_z - 111\,\sigma_x\sigma_y\sigma_z)$$
$$- (359\,\sigma_x + 153\,\sigma_y + 680\,\sigma_z + 111\,\sigma_x\sigma_y\sigma_z)\big)$$

$$= (-48\,\sigma_x\sigma_y\sigma_z)^{-1}\,(-444\,\sigma_x\sigma_y\sigma_z)$$

$$= \frac{444}{48} = \frac{37}{4} = 9.25$$

$$y = (acb - abc + cba - bca)^{-1}(acr - arc + cra - rca)$$

$$= \frac{204}{48} = \frac{17}{4} = 4.25$$

$$z = (bac - bca + acb - cab)^{-1}(bar - bra + arb - rab)$$

$$= \frac{132}{48} = \frac{11}{4} = 2.75$$

Pure and Mixed Sandwich Products

Please note:

We started to apply pure sandwich products to model reflections and ended with equations, which consist of mixed sandwich products.

So let's try to start with mixed sandwich products right at the beginning!

Pure Sandwich Products:

Vector r is sandwiched by the same vector a from left and right which results in a reflection at an axis pointing into the direction of vector a

$$r_{ref} = a\ r\ a$$

Mixed sandwich products:

Vector r is sandwiched by two different vectors a and b

$$r_{inter} = a\ r\ b \qquad \text{or} \qquad r'_{inter} = b\ r\ a$$

The Mixed Sandwich Product

If vector r

$$r = x\ \sigma_x + y\ \sigma_y + z\ \sigma_z$$

is multiplied by base vector σ_x from the left and by base vector σ_y from the right, vector r will be mapped into the resulting vector r_{inter}:

$$\begin{aligned} r_{inter} &= \sigma_x\ r\ \sigma_y \\ &= \sigma_x\ x\ \sigma_x\sigma_y + \sigma_x\ y\ \sigma_y\sigma_y + \sigma_x\ z\ \sigma_z\sigma_y \\ &= x\ \sigma_y + y\ \sigma_x + z\ \sigma_x\sigma_z\sigma_y \\ &= y\ \sigma_x + x\ \sigma_y - z\ \sigma_x\sigma_y\sigma_z \end{aligned}$$

The mixed sandwich product $\sigma_x\ r\ \sigma_y$ interchanges x-axis and y-axis.

A similar interchange of axes takes place, if the sandwich product is reversed.

Reversed Mixed Sandwich Product

If vector r

$$r = x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

is multiplied by base vector σ_y from the left and by base vector σ_x from the right, vector r will be mapped into the resulting vector r'_{inter}:

$$\begin{aligned} r'_{inter} &= \sigma_y\, r\, \sigma_x \\ &= \sigma_y\, x\, \sigma_x\sigma_x + \sigma_y\, y\, \sigma_y\sigma_x + \sigma_y\, z\, \sigma_z\sigma_x \\ &= x\,\sigma_y + y\,\sigma_x + z\,\sigma_y\sigma_z\sigma_x \\ &= y\,\sigma_x + x\,\sigma_y + z\,\sigma_x\sigma_y\sigma_z \end{aligned}$$

The reversed mixed sandwich product $\sigma_y\, r\, \sigma_x$ interchanges x-axis and y-axis.

$\Rightarrow$ The two sandwich products $\sigma_x\, r\, \sigma_y$ and product $\sigma_y\, r\, \sigma_x$ both interchange the x-axis with the y-axis.

$\Rightarrow$ The only difference between the two sandwich products is the orientation of the third component which has been transformed into an oriented volume element.

Linear Combinations of Mixed Sandwich Products

If the mixed sandwich products

$$r_{inter} = \sigma_x \, r \, \sigma_y \quad \text{and} \quad r'_{inter} = \sigma_y \, r \, \sigma_x$$

are subtracted, only the third component (which now is an oriented volume element) will survive.

⇒ This property of mixed sandwich products can be used to solve systems of linear equations.

If the mixed sandwich products

$$r_{inter} = \sigma_x \, r \, \sigma_y \quad \text{and} \quad r'_{inter} = \sigma_y \, r \, \sigma_x$$

are added, the third components (which have been transformed into oriented volume elements) cancel and only the interchanged x and y components will survive.

⇒ That's the way some of the matrices of Gell-Mann act on vectors.

Side Remark: Gell-Mann Matrices

The following three matrices are part of the set of eight 3 x 3 – matrices which are called Gell-Mann matrices:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

These three Gell-Mann matrices interchange two axes of a coordinate system and cancel the third:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ 0 \end{bmatrix}$$

Interchange of x-axis with y-axis

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ x \end{bmatrix}$$

Interchange of z-axis with x-axis

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ y \end{bmatrix}$$

Interchange of y-axis with z-axis

Side Remark: Gell-Mann Matrices

Thus the effect of Gell-Mann matrix λ_1

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

on a vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} y \\ x \\ 0 \end{bmatrix}$$

$$x\,\sigma_x + y\,\sigma_y + z\,\sigma_z \longrightarrow y\,\sigma_x + x\,\sigma_y$$

can be mimicked in Geometric Algebra by the sandwich product operation

$$\lambda_1(r) = \frac{1}{2}(\sigma_x\, r\, \sigma_y + \sigma_y\, r\, \sigma_x)$$

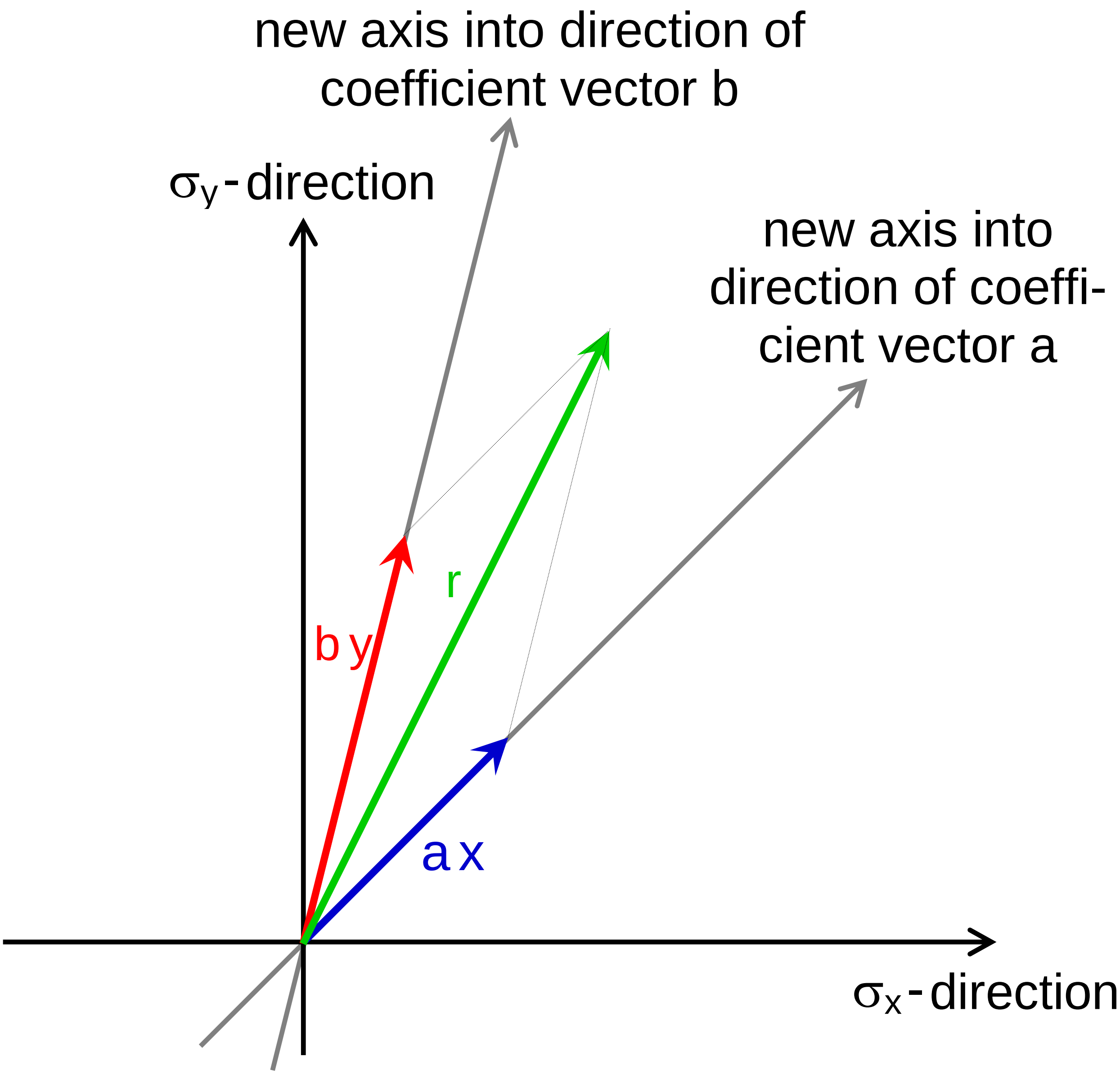
In a similar way there is

$$\lambda_4(r) = \frac{1}{2}(\sigma_z\, r\, \sigma_x + \sigma_x\, r\, \sigma_z)$$

$$\lambda_6(r) = \frac{1}{2}(\sigma_y\, r\, \sigma_z + \sigma_z\, r\, \sigma_y)$$

Reminder: Geometric Representation of System of Two Linear Equations

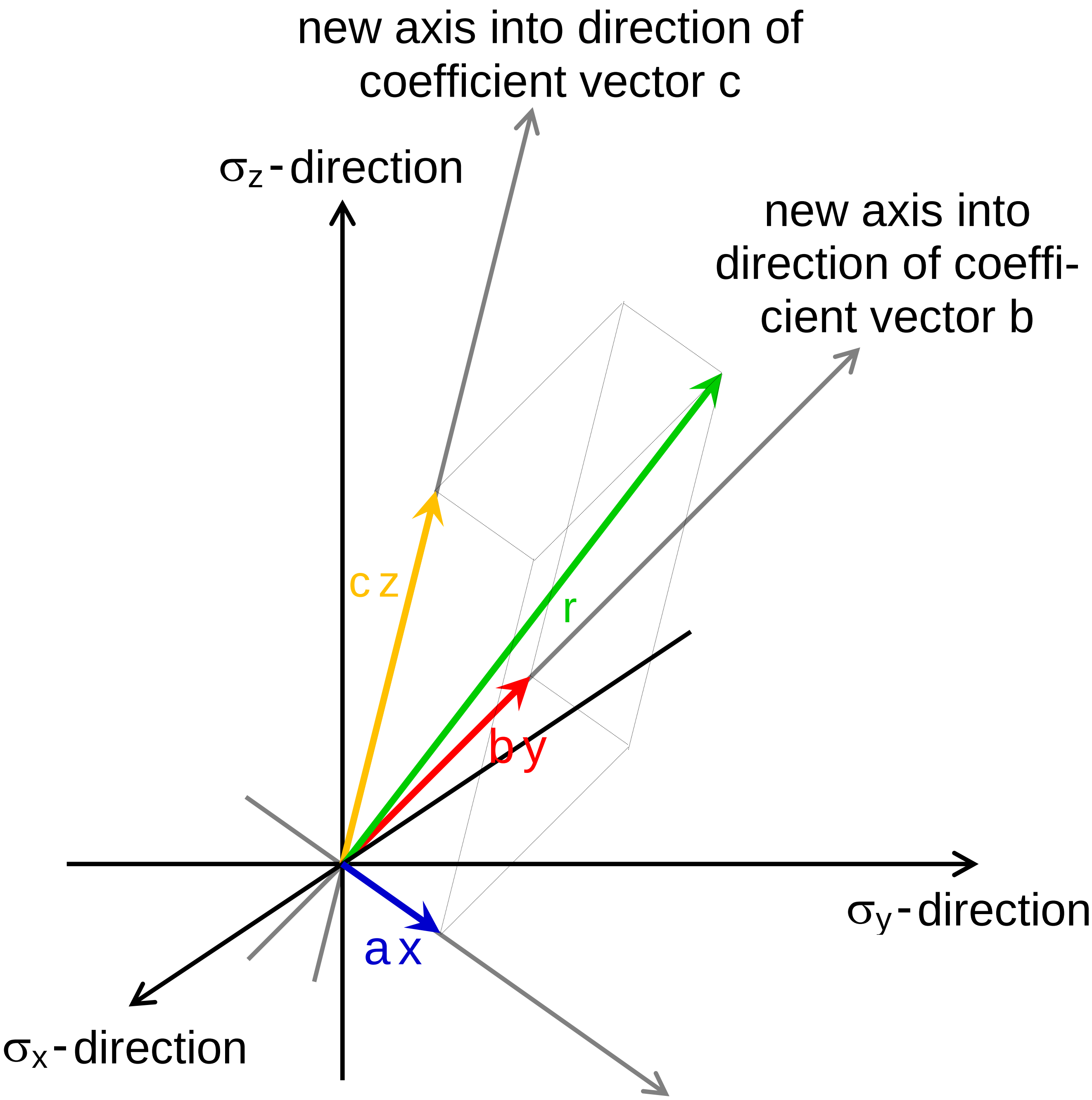
$a\,x + b\,y = r$



The method of Gauss can be interpreted as a transformation of coordinates (see part IV).

Reminder: Geometric Representation of Systems of Three Linear Equations

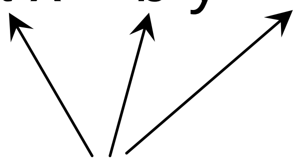
$a\,x + b\,y + c\,z = r$



Transformation of Coordinates

The method of Gauss can be interpreted as a transformation of coordinates (see part IV).

$$a\,x + b\,y + c\,z = r$$



The coefficient vectors a, b, c are now base vectors of the new coordinate system.

Therefore the mixed sandwich products of the resulting vector r, sandwiched by two different coefficient vectors, are required to solve this system of three different linear equations.

These sandwich products are then linear combinations of vectors and an oriented volume element (a trivector).

Finding y (and x, z)

To find the value of the unknown variable y, the following mixed sandwich products can be formed:

$$a\,r\,c = a\,(a\,x + b\,y + c\,z)\,c$$
$$= a^2 c\,x + a b c\,y + a c^2\,z$$

$$c\,r\,a = c\,(a\,x + b\,y + c\,z)\,a$$
$$= a^2 c\,x + c b a\,y + a c^2\,z$$

First and last terms are identical, because a^2 and b^2 are scalars, which commute with vectors. Thus they cancel, if the two mixed sandwich products are subtracted:

$$a r c - c r a = (a b c - c b a)\,y$$

$$\Rightarrow \qquad y = (a b c - c b a)^{-1}\,(a r c - c r a)$$

In a similar way the other two unknowns can be found:

$$x = (c a b - b a c)^{-1}\,(c r b - b r c)$$

$$z = (b c a - a c b)^{-1}\,(b r a - a r b)$$

Solving the Early Han Dynasty Problem With Sandwich Products

As all required mixed sandwich products have already been calculated earlier (see slides # 46 – 49), the results of the Early Han dynasty problem are:

$$x = (c\,a\,b - b\,a\,c)^{-1}\,(c\,r\,b - b\,r\,c)$$

$$= (24\ \sigma_x\sigma_y\sigma_z)^{-1}\,(222\ \sigma_x\sigma_y\sigma_z)$$

$$= \frac{222}{24} = \frac{37}{4} = 9.25$$

$$y = (a\,b\,c - c\,b\,a)^{-1}\,(a\,r\,c - c\,r\,a)$$

$$= (24\ \sigma_x\sigma_y\sigma_z)^{-1}\,(102\ \sigma_x\sigma_y\sigma_z)$$

$$= \frac{102}{24} = \frac{17}{4} = 4.25$$

$$z = (b\,c\,a - a\,c\,b)^{-1}\,(b\,r\,a - a\,r\,b)$$

$$= (24\ \sigma_x\sigma_y\sigma_z)^{-1}\,(66\ \sigma_x\sigma_y\sigma_z)$$

$$= \frac{66}{24} = \frac{11}{4} = 2.75$$

Solving Systems of Three Linear Equations With Sandwich Products

Of course the mixed sandwich product solutions

$$x = (c\,a\,b - b\,a\,c)^{-1}\,(c\,r\,b - b\,r\,c)$$

$$y = (a\,b\,c - c\,b\,a)^{-1}\,(a\,r\,c - c\,r\,a)$$

$$z = (b\,c\,a - a\,c\,b)^{-1}\,(b\,r\,a - a\,r\,b)$$

are identical to Grassmann's outer product solutions:

$$x = (a \wedge b \wedge c)^{-1}\,(r \wedge b \wedge c)$$

$$y = (a \wedge b \wedge c)^{-1}\,(a \wedge r \wedge c)$$

$$z = (a \wedge b \wedge c)^{-1}\,(a \wedge b \wedge r)$$

because the difference between a sandwich product and its reverse results in a pure tri-vector while the vector parts cancel:

$$a\,b\,c - c\,b\,a = b\,c\,a - a\,c\,b = c\,a\,b - b\,a\,c$$

$$= 2\,a \wedge b \wedge c$$

This property of sandwich products is also required to solve systems of more than three linear equations.

Another Alternative of

Solving Systems of Three Linear Equations With Sandwich Products

The system of three linear equations

$$a\,x + b\,y + c\,z = r$$

can be sandwiched by the geometric products $a\,b$ and $b\,a$ to get

$$a\,b\,(a\,x + b\,y + c\,z)\,a\,b = a\,b\,r\,a\,b$$

$$a^2\,b^2\,a\,x + a^2\,b^2\,b\,y + a\,b\,c\,a\,b\,z = a\,b\,r\,a\,b$$

and

$$b\,a\,(a\,x + b\,y + c\,z)\,b\,a = b\,a\,r\,b\,a$$

$$a^2\,b^2\,a\,x + a^2\,b^2\,b\,y + b\,a\,c\,b\,a\,z = b\,a\,r\,b\,a$$

These sandwich products might be considered (in a formal way) as reflections at the parallelograms $a\,b$ and $b\,a$.

Another Alternative of

Solving Systems of Three Linear Equations With Sandwich Products

The difference between the two pure sandwich products then delivers a solution formula for the unknown variable z:

$$a\,b\,r\,a\,b - b\,a\,r\,b\,a = (a\,b\,c\,a\,b - b\,a\,c\,b\,a)\,z$$

$$\Rightarrow \quad z = (a\,b\,c\,a\,b - b\,a\,c\,b\,a)^{-1}(a\,b\,r\,a\,b - b\,a\,r\,b\,a)$$

Analogously the other two unknown variables can be found:

$$x = (b\,c\,a\,b\,c - c\,b\,a\,c\,b)^{-1}(b\,c\,r\,b\,c - c\,b\,r\,c\,b)$$

$$y = (c\,a\,b\,c\,a - a\,c\,b\,a\,c)^{-1}(c\,a\,r\,c\,a - a\,c\,r\,a\,c)$$

Thus sandwich products of three factors might easily be replaced by more complicated sandwich products of five factors to get the same results.

Another Alternative of

Solving the Early Han Dynasty Problem With Sandwich Products

The intermediate results of slides 46 – 49 can be used again to solve the Early Han dynasty problem with the strategy of the previous slide:

$$b\,c\,a\,b\,c = (b\,c\,a)\,(b\,c) = 393\,\sigma_x + 470\,\sigma_y + 211\,\sigma_z + 264\,\sigma_x\sigma_y\sigma_z$$

$$c\,b\,a\,c\,b = 393\,\sigma_x + 470\,\sigma_y + 211\,\sigma_z - 264\,\sigma_x\sigma_y\sigma_z$$

$$b\,c\,a\,b\,c - c\,b\,a\,c\,b = 528\,\sigma_x\sigma_y\sigma_z$$

$$b\,c\,r\,b\,c = (b\,c\,r)\,(b\,c) = 5739\,\sigma_x + 7246\,\sigma_y + 5084\,\sigma_z + 2442\,\sigma_x\sigma_y\sigma_z$$

$$c\,b\,r\,c\,b = 5739\,\sigma_x + 7246\,\sigma_y + 5084\,\sigma_z - 2442\,\sigma_x\sigma_y\sigma_z$$

$$b\,c\,r\,b\,c - c\,b\,r\,c\,b = 4884\,\sigma_x\sigma_y\sigma_z$$

$$\Rightarrow\; x = (b\,c\,a\,b\,c - c\,b\,a\,c\,b)^{-1}\,(b\,c\,r\,b\,c - c\,b\,r\,c\,b)$$

$$= (528\,\sigma_x\sigma_y\sigma_z)^{-1}\,(2442\,\sigma_x\sigma_y\sigma_z)$$

$$= \frac{4884}{528} = \frac{37}{4} = 9.25$$

Another Alternative of

Solving the Early Han Dynasty Problem With Sandwich Products

$$\mathrm{c\,a\,b\,c\,a} = (\mathrm{c\,a\,b})(\mathrm{c\,a})$$
$$= 428\,\sigma_x + 270\,\sigma_y + 332\,\sigma_z + 192\,\sigma_x\sigma_y\sigma_z$$

$$\mathrm{a\,c\,b\,a\,c} = 428\,\sigma_x + 270\,\sigma_y + 332\,\sigma_z - 192\,\sigma_x\sigma_y\sigma_z$$

$$\mathrm{c\,a\,b\,c\,a} - \mathrm{a\,c\,b\,a\,c} = 384\,\sigma_x\sigma_y\sigma_z$$

$$\mathrm{c\,a\,r\,c\,a} = (\mathrm{c\,a\,r})(\mathrm{c\,a})$$
$$= 6516\,\sigma_x + 4420\,\sigma_y + 4106\,\sigma_z + 816\,\sigma_x\sigma_y\sigma_z$$

$$\mathrm{a\,c\,r\,a\,c} = 6516\,\sigma_x + 4420\,\sigma_y + 4106\,\sigma_z - 816\,\sigma_x\sigma_y\sigma_z$$

$$\mathrm{c\,a\,r\,c\,a} - \mathrm{a\,c\,r\,a\,c} = 1632\,\sigma_x\sigma_y\sigma_z$$

$$\Rightarrow\ y = (\mathrm{c\,a\,b\,c\,a} - \mathrm{a\,c\,b\,a\,c})^{-1}(\mathrm{c\,a\,r\,c\,a} - \mathrm{a\,c\,r\,a\,c})$$
$$= (384\,\sigma_x\sigma_y\sigma_z)^{-1}(1632\,\sigma_x\sigma_y\sigma_z)$$
$$= \frac{1632}{384} = \frac{17}{4} = 4.25$$

Another Alternative of

Solving the Early Han Dynasty Problem With Sandwich Products

$$a\,b\,c\,a\,b = (a\,b\,c)\,(a\,b)$$
$$= 214\,\sigma_x + 334\,\sigma_y + 594\,\sigma_z + 336\,\sigma_x\sigma_y\sigma_z$$

$$b\,a\,c\,b\,a = 214\,\sigma_x + 334\,\sigma_y + 594\,\sigma_z - 336\,\sigma_x\sigma_y\sigma_z$$

$$a\,b\,c\,a\,b - b\,a\,c\,b\,a = 672\,\sigma_x\sigma_y\sigma_z$$

$$a\,b\,r\,a\,b = (a\,b\,r)\,(a\,b)$$
$$= 9216\,\sigma_x + 8356\,\sigma_y + 5858\,\sigma_z + 924\,\sigma_x\sigma_y\sigma_z$$

$$b\,a\,r\,b\,a = 9216\,\sigma_x + 8356\,\sigma_y + 5858\,\sigma_z - 924\,\sigma_x\sigma_y\sigma_z$$

$$a\,b\,r\,a\,b - b\,a\,r\,b\,a = 1848\,\sigma_x\sigma_y\sigma_z$$

$$\Rightarrow\; z = (a\,b\,c\,a\,b - b\,a\,c\,b\,a)^{-1}(a\,b\,r\,a\,b - b\,a\,r\,b\,a)$$

$$= (672\,\sigma_x\sigma_y\sigma_z)^{-1}(1632\,\sigma_x\sigma_y\sigma_z)$$

$$= \frac{1848}{672} = \frac{11}{4} = 2.75$$

Attachment:

Solving Systems of Five Linear Equations With Sandwich Products

The following slides show, how systems of linear equations with five unknown variables can be solved by using sandwich products.

This solution strategy is presented to indicate, that in principle sandwich products can be used successfully in higher-dimensional situations. All this is shown for didactical reasons.

As calculations of different sandwich products seem to be redundant and in a way confusing, linear equations with more than three unknown variables might be solved by using outer products in most situations.

Solving Systems of Five Linear Equations With Sandwich Products

A system of five linear equations with five unknown variables x_i and $i \in \{1, 2, 3, 4, 5\}$ will now be written as

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5 = r$$

There are five different coefficient vectors a_i with $i \in \{1, 2, 3, 4, 5\}$.

As a product of two different coefficient vectors a_i, a_j is a linear combination of scalar and bivector terms

$$a_i a_j = \langle a_i a_j \rangle_0 + \langle a_i a_j \rangle_2 = a_i \bullet a_j + a_i \wedge a_j$$

a product of three coefficient vectors a_i, a_j, a_k must be a linear combination of vector and trivector terms:

$$a_i a_j a_k = \langle a_i a_j a_k \rangle_1 + \langle a_i a_j a_k \rangle_3$$

↖ $a_i \wedge a_j \wedge a_k$

Reversion

A mathematical reversion reverses the order of base vectors in any product. They are symbolized by a tilde ~.

Scalars, vectors and quadvectors are invariant under reversion:

$$\tilde{\sigma_x} = \sigma_x$$

$$(\sigma_w\sigma_x\sigma_y\sigma_z)^\sim = \sigma_z\sigma_y\sigma_x\sigma_w = \sigma_w\sigma_x\sigma_y\sigma_z$$

Bivectors and trivectors change their orienttation under reversion:

$$(\sigma_x\sigma_y)^\sim = \sigma_y\sigma_x = -\sigma_x\sigma_y$$

$$(\sigma_x\sigma_y\sigma_z)^\sim = \sigma_z\sigma_y\sigma_x = -\sigma_x\sigma_y\sigma_z$$

The reversed product of three coefficient vectors is a linear combination of reversed vectors and reversed trivectors

$$(a_i\, a_j\, a_k)^\sim = a_k\, a_j\, a_i = \langle a_k\, a_j\, a_i\rangle_1 + \langle a_k\, a_j\, a_i\rangle_3$$

$$= \langle a_i\, a_j\, a_k\rangle_1 - \langle a_i\, a_j\, a_k\rangle_3$$

Reversed vectors are identical to the original vectors.

Reversed trivectors are negatives of the original trivectors.

Reversion

Thus the difference between original and reversed sandwich products

$$a_i\, a_j\, a_k - (a_i\, a_j\, a_k)^{\sim}$$

$$= \langle a_i\, a_j\, a_k \rangle_1 + \langle a_i\, a_j\, a_k \rangle_3 - \langle a_i\, a_j\, a_k \rangle_1 + \langle a_i\, a_j\, a_k \rangle_3$$

$$= 2\, \langle a_i\, a_j\, a_k \rangle_3$$

$$= 2\, a_i \wedge a_j \wedge a_k$$

is a pure trivector.

Consequently the square of this difference

$$\left(a_i\, a_j\, a_k - (a_i\, a_j\, a_k)^{\sim}\right)^2$$

$$= 4\, \left(\langle a_i\, a_j\, a_k \rangle_3\right)^2$$

$$= 4\, (a_i \wedge a_j \wedge a_k)^2$$

must be a scalar which commutes with any other mathematical object.

Solving Systems of Five Linear Equations With Sandwich Products

The system of five linear equations with five unknown variables x_1, x_2, x_3, x_4, x_5

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5 = r$$

can be solved for x_3 by first sandwiching the system of linear equations by coefficient vectors a_1 and a_5

$$\begin{aligned} a_1 r a_5 &= a_1 (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5) a_5 \\ &= a_1^2 a_5 x_1 + a_1 a_2 a_5 x_2 + a_1 a_3 a_5 x_3 \\ &\qquad + a_1 a_4 a_5 x_4 + a_1 a_5^2 x_5 \end{aligned}$$

$$\begin{aligned} a_5 r a_1 &= a_5 (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5) a_1 \\ &= a_1^2 a_5 x_1 + a_5 a_2 a_1 x_2 + a_5 a_3 a_1 x_3 \\ &\qquad + a_5 a_4 a_1 x_4 + a_1 a_5^2 x_5 \end{aligned}$$

If these two mixed sandwich products are subtracted, the first and last terms cancel:

$$\begin{aligned} a_1 r a_5 - a_5 r a_1 = \; & (a_1 a_2 a_5 - a_5 a_2 a_1) x_2 \\ & + (a_1 a_3 a_5 - a_5 a_3 a_1) x_3 \\ & + (a_1 a_4 a_5 - a_5 a_4 a_1) x_4 \end{aligned}$$

There are now three unknown variables only.

Solving Systems of Five Linear Equations With Sandwich Products

Now there are only three unknown variables.

$$a_1 r a_5 - a_5 r a_1 = (a_1 a_2 a_5 - a_5 a_2 a_1)\, x_2 + (a_1 a_3 a_5 - a_5 a_3 a_1)\, x_3 + (a_1 a_4 a_5 - a_5 a_4 a_1)\, x_4$$

As a second step the sandwich products with the coefficients of x_2 and x_4 are formed:

$$\begin{aligned}(a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 r a_5 - a_5 r a_1)(a_1 a_4 a_5 - a_5 a_4 a_1) \\ = \quad & (a_1 a_2 a_5 - a_5 a_2 a_1)^2 (a_1 a_4 a_5 - a_5 a_4 a_1)\, x_2 \\ & + (a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 a_3 a_5 - a_5 a_3 a_1)(a_1 a_4 a_5 - a_5 a_4 a_1)\, x_3 \\ & + (a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 a_4 a_5 - a_5 a_4 a_1)^2\, x_4\end{aligned}$$

and …

Solving Systems of Five Linear Equations With Sandwich Products

… and

$$
\begin{aligned}
&(a_1 a_4 a_5 - a_5 a_4 a_1)\,(a_1 r a_5 - a_5 r a_1)\,(a_1 a_2 a_5 - a_5 a_2 a_1) \\
&\quad = (a_1 a_4 a_5 - a_5 a_4 a_1)\,(a_1 a_2 a_5 - a_5 a_2 a_1)^2\, x_2 \\
&\qquad + (a_1 a_4 a_5 - a_5 a_4 a_1)\,(a_1 a_3 a_5 - a_5 a_3 a_1)\,(a_1 a_2 a_5 - a_5 a_2 a_1)\, x_3 \\
&\qquad + (a_1 a_4 a_5 - a_5 a_4 a_1)^2\,(a_1 a_2 a_5 - a_5 a_2 a_1)\, x_4 \\
&\quad = (a_1 a_2 a_5 - a_5 a_2 a_1)^2\,(a_1 a_4 a_5 - a_5 a_4 a_1)\, x_2 \\
&\qquad + (a_1 a_4 a_5 - a_5 a_4 a_1)\,(a_1 a_3 a_5 - a_5 a_3 a_1)\,(a_1 a_2 a_5 - a_5 a_2 a_1)\, x_3 \\
&\qquad + (a_1 a_2 a_5 - a_5 a_2 a_1)\,(a_1 a_4 a_5 - a_5 a_4 a_1)^2\, x_4
\end{aligned}
$$

These squares are scalars which commute with any other mathematical object. Therefore again first and last terms of the sandwich products cancel when subtracted.

Solving Systems of Five Linear Equations With Sandwich Products

In the end the final sandwich product difference

$$\begin{aligned}
&(a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 r a_5 - a_5 r a_1)(a_1 a_4 a_5 - a_5 a_4 a_1) \\
&\quad - (a_1 a_4 a_5 - a_5 a_4 a_1)(a_1 r a_5 - a_5 r a_1)(a_1 a_2 a_5 - a_5 a_2 a_1) \\
&\qquad = \big((a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 a_3 a_5 - a_5 a_3 a_1)(a_1 a_4 a_5 - a_5 a_4 a_1) \\
&\qquad\qquad - (a_1 a_4 a_5 - a_5 a_4 a_1)(a_1 a_3 a_5 - a_5 a_3 a_1)(a_1 a_2 a_5 - a_5 a_2 a_1)\big)\, x_3
\end{aligned}$$

can be solved for x_3:

$$\begin{aligned}
x_3 = &\big((a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 a_3 a_5 - a_5 a_3 a_1)(a_1 a_4 a_5 - a_5 a_4 a_1) \\
&\qquad - (a_1 a_4 a_5 - a_5 a_4 a_1)(a_1 a_3 a_5 - a_5 a_3 a_1)(a_1 a_2 a_5 - a_5 a_2 a_1)\big)^{-1} \\
&\big((a_1 a_2 a_5 - a_5 a_2 a_1)(a_1 r a_5 - a_5 r a_1)(a_1 a_4 a_5 - a_5 a_4 a_1) \\
&\qquad - (a_1 a_4 a_5 - a_5 a_4 a_1)(a_1 r a_5 - a_5 r a_1)(a_1 a_2 a_5 - a_5 a_2 a_1)\big)
\end{aligned}$$

Solving Systems of Five Linear Equations With Sandwich Products

With cyclic permutation of indices the formulas of other unknown variables can be derived:

x_3	a_3	a_4	a_5	a_1	a_2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
x_4	a_4	a_5	a_1	a_2	a_3
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
x_5	a_5	a_1	a_2	a_3	a_4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
x_1	a_1	a_2	a_3	a_4	a_5
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
x_2	a_2	a_3	a_4	a_5	a_1

Solving Systems of Five Linear Equations With Sandwich Products

And of course these mixed sandwich product solutions are again identical to Grassmann's outer product solutions:

$$x_1 = \big((a_4a_5a_3 - a_3a_5a_4)(a_4a_1a_3 - a_3a_1a_4)(a_4a_2a_3 - a_3a_2a_4) - (a_4a_2a_3 - a_3a_2a_4)(a_4a_1a_3 - a_3a_1a_4)(a_4a_5a_3 - a_3a_5a_4)\big)^{-1}$$
$$\big((a_4a_5a_3 - a_3a_5a_4)(a_4ra_3 - a_3ra_4)(a_4a_2a_3 - a_3a_2a_4) - (a_4a_2a_3 - a_3a_2a_4)(a_4ra_3 - a_3ra_4)(a_4a_5a_3 - a_3a_5a_4)\big)$$
$$= (a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5)^{-1}(r \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5)$$

$$x_2 = \big((a_5a_1a_4 - a_4a_1a_5)(a_5a_2a_4 - a_4a_2a_5)(a_5a_3a_4 - a_4a_3a_5) - (a_5a_3a_4 - a_4a_3a_5)(a_5a_2a_4 - a_4a_2a_5)(a_5a_1a_4 - a_4a_1a_5)\big)^{-1}$$
$$\big((a_5a_1a_4 - a_4a_1a_5)(a_5ra_4 - a_4ra_5)(a_5a_3a_4 - a_4a_3a_5) - (a_5a_3a_4 - a_4a_3a_5)(a_5ra_4 - a_4ra_5)(a_5a_1a_4 - a_4a_1a_5)\big)$$
$$= (a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5)^{-1}(a_2 \wedge r \wedge a_3 \wedge a_4 \wedge a_5)$$

$$x_3 = \big((a_1a_2a_5 - a_5a_2a_1)(a_1a_3a_5 - a_5a_3a_1)(a_1a_4a_5 - a_5a_4a_1) - (a_1a_4a_5 - a_5a_4a_1)(a_1a_3a_5 - a_5a_3a_1)(a_1a_2a_5 - a_5a_2a_1)\big)^{-1}$$
$$\big((a_1a_2a_5 - a_5a_2a_1)(a_1ra_5 - a_5ra_1)(a_1a_4a_5 - a_5a_4a_1) - (a_1a_4a_5 - a_5a_4a_1)(a_1ra_5 - a_5ra_1)(a_1a_2a_5 - a_5a_2a_1)\big)$$
$$= (a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5)^{-1}(a_1 \wedge a_2 \wedge r \wedge a_4 \wedge a_5)$$

$$x_4 = \big((a_2a_3a_1 - a_1a_3a_2)(a_2a_4a_1 - a_1a_4a_2)(a_2a_5a_1 - a_1a_5a_2) - (a_2a_5a_1 - a_1a_5a_2)(a_2a_4a_1 - a_1a_4a_2)(a_2a_3a_1 - a_1a_3a_2)\big)^{-1}$$
$$\big((a_2a_3a_1 - a_1a_3a_2)(a_2ra_1 - a_1ra_2)(a_2a_5a_1 - a_1a_5a_2) - (a_2a_5a_1 - a_1a_5a_2)(a_2ra_1 - a_1ra_2)(a_2a_3a_1 - a_1a_3a_2)\big)$$
$$= (a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5)^{-1}(a_1 \wedge a_2 \wedge a_3 \wedge r \wedge a_5)$$

$$x_5 = \big((a_3a_4a_2 - a_2a_4a_3)(a_3a_5a_2 - a_2a_5a_3)(a_3a_1a_2 - a_2a_1a_3) - (a_3a_1a_2 - a_2a_1a_3)(a_3a_5a_2 - a_2a_5a_3)(a_3a_4a_2 - a_2a_4a_3)\big)^{-1}$$
$$\big((a_3a_4a_2 - a_2a_4a_3)(a_3ra_2 - a_2ra_3)(a_3a_1a_2 - a_2a_1a_3) - (a_3a_1a_2 - a_2a_1a_3)(a_3ra_2 - a_2ra_3)(a_3a_4a_2 - a_2a_4a_3)\big)$$
$$= (a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5)^{-1}(a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge r)$$

Outlook

In an analogous way systems of linear equations with more variables can be solved. But present-day mathematics is not as complete as most of us might perhaps think:

> “I think we are still lacking a good understanding of which kind of methods we should use in relation (…) to problems depending on a medium sized number of variables.
>
> We have the machinery for a small number of variables and we have probability for a large number of variables.
>
> But we don‘t even know which questions to ask, much less which methods to use, when we have ten variables or twenty variables.“

Abel Prize winner Lennart Carleson after being asked about the most challenging and exciting area of mathematics that will be explored in the 21st century.
In: Newsletter of the European Mathematical Society, Issue 61, Sept. 2006, pp. 31 – 36.