Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

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Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part V: Eigenvalues and eigenvectors)

Stand: 28. Nov. 2016

Teaching & learning contents according to the modular description of LV 200 691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Most of this will be discussed in the standard language of the rather old-fashioned linear algebra or matrix algebra which can be found in most textbooks of business mathematics or mathematical economics.

But in this fifth part of the lecture series we will again adopt a more modern view: Eigenvalues and eigenvectors of matrices will be discussed using the mathematical language of Geometric Algebra.

Repetition: Basics of Geometric Algebra

$1 + 3 + 3 + 1 = 2^3 = 8$ different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: $\sigma_x, \sigma_y, \sigma_z$

Three base bivectors: $\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

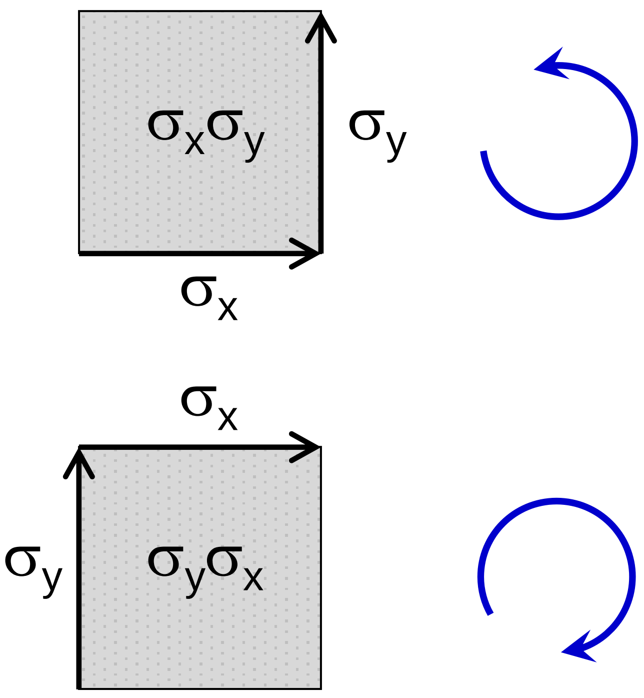
$$1^2 = \sigma_x{}^2 = \sigma_y{}^2 = \sigma_z{}^2 = 1$$

Base bivectors and base trivector square to minus one:

$$(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1$$

Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the resulting area elements.



right-handed orientation (anticlockwise orientation)

positive orientation in a right-handed coordinate system

left-handed orientation (clockwise orientation)

negative orientation in a right-handed coordinate system

Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

$$\sigma_x\sigma_y = -\sigma_y\sigma_x$$

$$\sigma_y\sigma_z = -\sigma_z\sigma_y$$

$$\sigma_z\sigma_x = -\sigma_x\sigma_z$$

Scalars

Scalars are geometric entities without direction. They can be expressed as multiples of the base scalar:

$$k = k\,1$$

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

$$r = x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

$$A = A_{xy}\,\sigma_x\sigma_y + A_{yz}\,\sigma_y\sigma_z + A_{zx}\,\sigma_z\sigma_x$$

Trivectors

Trivectors are oriented volume elements. They can be expressed as multiples of the base tri-vector:

$$V = V_{xyz}\,\sigma_x\sigma_y\sigma_z$$

Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$a\,b = a \bullet b + a \wedge b$$

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

$$a \bullet b = b \bullet a = \frac{1}{2}(a\,b + b\,a)$$

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

$$a \wedge b = -\,b \wedge a = \frac{1}{2}(a\,b - b\,a)$$

Geometric Multiplication of Vectors and Bivectors

The product of a bivector B and a vector a consists of a vector term and a trivector term. As the dimension of bivector B is reduced, the vector term is called inner product (dot product). And as the dimension of bivector B is increased, the trivector term is called outer product (exterior product or wedge product).

$$B\,a = B \bullet a + B \wedge a$$

In contrast to what was said on the last slide, the inner product of a bivector and a vector is an anti-commutative product as a reversion of the order of bivector and vector changes the sign of the inner product:

$$B \bullet a = -\,a \bullet B = \frac{1}{2}(B\,a - a\,B)$$

The outer product of a bivector and a vector is a commutative product as a reversion of the order of bivector and vector does not change it:

$$B \wedge a = a \wedge B = \frac{1}{2}(B\,a + a\,B)$$

Systems of Two Linear Equations

$$\begin{aligned} a_1\, x + b_1\, y &= d_1 \\ a_2\, x + b_2\, y &= d_2 \end{aligned} \quad \Rightarrow \quad a\, x + b\, y = d$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1\, \sigma_x + a_2\, \sigma_y)\, x + (b_1\, \sigma_x + b_2\, \sigma_y)\, y = d_1\, \sigma_x + d_2\, \sigma_y$$

Solutions:

$$x = \frac{1}{a \wedge b}(d \wedge b) = (a \wedge b)^{-1}\, (d \wedge b)$$

$$y = \frac{1}{a \wedge b}\, (a \wedge d) = (a \wedge b)^{-1}\, (a \wedge d)$$

Systems of Three Linear Equations

$$\begin{aligned} a_1\, x + b_1\, y + c_1\, z &= d_1 \\ a_2\, x + b_2\, y + c_2\, z &= d_2 \qquad \Rightarrow \qquad a\, x + b\, y + c\, z = d \\ a_3\, x + b_3\, y + c_3\, z &= d_3 \end{aligned}$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1\, \sigma_x + a_2 \sigma_y + a_3\, \sigma_z)\, x + (b_1\, \sigma_x + b_2\, \sigma_y + b_3\, \sigma_z)\, y$$
$$+ (c_1\, \sigma_x + c_2\, \sigma_y + c_3\, \sigma_z)\, z = d_1\, \sigma_x + d_2\, \sigma_y + d_3\, \sigma_z$$

Solutions: $\quad x = (a \wedge b \wedge c)^{-1} (d \wedge b \wedge c)$

$$y = (a \wedge b \wedge c)^{-1} (a \wedge d \wedge c)$$

$$z = (a \wedge b \wedge c)^{-1} (a \wedge b \wedge d)$$

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of former lessons and in Geometric Algebra books.

Halloween Product Engineering Problem

A firm manufactures two different types of final products P_1 and P_2. To produce these products two different raw materials R_1 and R_2 are required:

To produce one unit of the first final product P_1 30 units of raw material R_1 and 20 units of raw material R_2 are required.

To produce one unit of the second final product P_2 70 units of raw material R_1 and 80 units of raw material R_2 are required.

At Halloween 2017 the CEO of the firm orders his management to consume only quantities of raw materials in the production process which are perfect multiples λ of the production vector:

Production vector, which shows the quantities of final products produced:

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

scalar

Demand vector, which shows the quantities of raw materials required:

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \lambda P_1 \\ \lambda P_2 \end{bmatrix} = \lambda \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \lambda P$$

Find the relation of quantities of final products P_1 and P_2 which will be produced at Halloween 2017.

Solution of the Halloween Product Engineering Problem

Demand matrix, which shows the demand of raw materials to produce one unit of the final products:

$$D = \begin{bmatrix} 30 & 70 \\ 20 & 80 \end{bmatrix}$$

Matrix equation: $D\,P = q$

$$\Rightarrow \qquad D\,P = \lambda\,P \qquad (*)$$

$$\Rightarrow \qquad (D - \lambda\,I)\,P = 0 \qquad (**)$$

These equations are important!

If equation $(*)$ holds, vector P is called **eigenvector** (or characteristic vector, or latent vector) of matrix D.

The scalar λ is then called **eigenvalue** (or characteristic root, or latent root) of matrix D.

And matrix $(D - \lambda\,I)$ is called **characteristic matrix** of matrix D.

As equation $(**)$ equals to 0, the characteristic matrix must be singular.

In Geometric Algebra vectors are expressed as Pauli vectors.

Production vector:

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \longrightarrow P = P_1\,\sigma_x + P_2\,\sigma_y$$

Demand vector:

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \lambda \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \longrightarrow q = q_1\,\sigma_x + q_2\,\sigma_y$$

$$= \lambda\,P_1\,\sigma_x + \lambda\,P_2\,\sigma_y$$

Demand matrix: $D = \begin{bmatrix} 30 & 70 \\ 20 & 80 \end{bmatrix}$

Coefficient vectors:

$$a = a_1\,\sigma_x + a_2\,\sigma_y$$
$$= 30\,\sigma_x + 20\,\sigma_y$$

$$b = b_1\,\sigma_x + b_2\,\sigma_y$$
$$= 70\,\sigma_x + 80\,\sigma_y$$

Matrix equation: $D\,P = \lambda\,P$

Pauli vector equation:

$$a\,P_1 + b\,P_2 = \lambda\,P$$

$$(30\,\sigma_x + 20\,\sigma_y)\,P_1 + (70\,\sigma_x + 80\,\sigma_y)\,P_2 =$$
$$\lambda\,P_1\,\sigma_x + \lambda\,P_2\,\sigma_y$$

Characteristic Coefficient Vectors

The coefficient vectors of the characteristic matrix

$$(D - \lambda\,I) = \begin{bmatrix} 30-\lambda & 70 \\ 20 & 80-\lambda \end{bmatrix}$$

can be called
characteristic coefficient vectors:

$$a - \lambda\,\sigma_x = (30 - \lambda)\,\sigma_x + 20\,\sigma_y$$

$$b - \lambda\,\sigma_y = 70\,\sigma_x + (80 - \lambda)\,\sigma_y$$

Characteristic Outer Product

The Halloween product engineering problem thus asks about finding eigenvalues and eigenvectors of demand matrix D.

As the characteristic matrix

$$(D - \lambda\, I) = \begin{bmatrix} 30-\lambda & 70 \\ 20 & 80-\lambda \end{bmatrix}$$

is singular, its determinant has to be zero.

Therefore the characteristic coefficient vectors have to be linearly dependent and the outer product of the characteristic coefficient vectors vanish.

The outer outer product of the characteristic coefficient vectors can be called **characteristic outer product**:

$$(a - \lambda\, \sigma_x) \wedge (b - \lambda\, \sigma_y) = 0$$

It coincides with the determinant of the characteristic matrix:

$$\det(D - \lambda\, I) = 0$$

Characteristic Polynomial and Characteristic Equation

The characteristic outer product equation can be solved for eigenvalues λ.

$$(a - \lambda\,\sigma_x) \wedge (b - \lambda\,\sigma_y) = 0$$

$$\big((30-\lambda)\,\sigma_x + 20\,\sigma_y\big) \wedge \big(70\,\sigma_x + (80-\lambda)\,\sigma_y\big) = 0$$

$$(30-\lambda)\,(80-\lambda)\,\sigma_x\sigma_y + 70\cdot 20\,\sigma_y\sigma_x = 0$$

$$(30-\lambda)\,(80-\lambda) - 70\cdot 20 = 0$$

$$\Rightarrow \qquad \underbrace{\underbrace{\lambda^2 - 110\,\lambda + 1000}_{\text{characteristic polynomial}} = 0}_{\text{characteristic equation of matrix D}}$$

These mathematical objects are called ... **characteristic polynomial**

and ... **characteristic equation of matrix D**

Finding the Eigenvalues

The eigenvalues can be found by solving the characteristic equation for x.

As the characteristic equation of the Halloween product engineering problem is a quadratic equation, there should be two different solutions.

$$\begin{aligned} \lambda^2 - 110\,\lambda + 1000 &= 0 \\ \lambda^2 - 2\cdot 55\,\lambda + 55^2 - 55^2 + 1000 &= 0 \\ \lambda^2 - 2\cdot 55\,\lambda + 55^2 &= 2025 \\ (\lambda - 55)^2 &= 2025 = (\pm 45)^2 \end{aligned}$$

Therefore the two eigenvalues are

$$\lambda_1 = 55 - 45 = \ \ 10$$

$$\lambda_2 = 55 + 45 = 100$$

Short check of results:

The characteristic polynomial can be rewritten as

$$(\lambda - 10)\,(\lambda - 100) = \lambda^2 - 110\,\lambda + 1000$$

showing that the two results are indeed correct.

Finding the Eigenvectors

As a system of two linear equations

$a\,x + b\,y = d$

$$\begin{array}{cc|c} & & x \\ & & y \\ \hline a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{array} \longrightarrow \begin{array}{cc|c} & & P_1 \\ & & P_2 \\ \hline a_1 & b_1 & \lambda P_1 \\ a_2 & b_2 & \lambda P_2 \end{array}$$

can be solved by

$(a \wedge b)\, x = d \wedge b$

$(a \wedge b)\, y = a \wedge d$ (see slide #7),

the Pauli vector equation

$$a\,P_1 + b\,P_2 = \lambda\,P$$

will give the mathematical relations

$$(a \wedge b)\,P_1 = \lambda\,P \wedge b = \lambda\,(P_1\,\sigma_x + P_2\,\sigma_y) \wedge b$$

$$(a \wedge b)\,P_2 = \lambda\,a \wedge P = \lambda\,a \wedge (P_1\,\sigma_x + P_2\,\sigma_y)$$

These two equations show the wanted relation between demand vector coefficients.

Finding the Eigenvectors: Calculation of Outer Products

$a = 30\,\sigma_x + 20\,\sigma_y \qquad b = 70\,\sigma_x + 80\,\sigma_y$

$a \wedge b = 1000\,\sigma_x\sigma_y$

$\sigma_x \wedge b = \quad 80\,\sigma_x\sigma_y \qquad \sigma_y \wedge b = -70\,\sigma_x\sigma_y$

$a \wedge \sigma_x = -20\,\sigma_x\sigma_y \qquad a \wedge \sigma_y = \quad 30\,\sigma_x\sigma_y$

All these outer products represent area elements which are parallel. Therefore the two relations

$$(a \wedge b)\,P_1 = \lambda\,(P_1\,\sigma_x + P_2\,\sigma_y) \wedge b$$

$$\Rightarrow \qquad P_1 = \lambda\,(a \wedge b - \lambda\,\sigma_x \wedge b)^{-1}\,(\sigma_y \wedge b)\,P_2$$

$$\text{or} \qquad P_2 = \lambda^{-1}\,(\sigma_y \wedge b)^{-1}\,(a \wedge b - \lambda\,\sigma_x \wedge b)\,P_1$$

$$(a \wedge b)\,P_2 = \lambda\,a \wedge (P_1\,\sigma_x + P_2\,\sigma_y)$$

$$\Rightarrow \qquad P_1 = \lambda^{-1}\,(a \wedge \sigma_x)^{-1}\,(a \wedge b - \lambda\,a \wedge \sigma_y)\,P_2$$

$$\text{or} \qquad P_2 = \lambda\,(a \wedge b - \lambda\,a \wedge \sigma_y)^{-1}\,(a \wedge \sigma_x)\,P_1$$

have to be identical.

Finding the Eigenvectors

Part I: Eigenvectors of First Eigenvalue λ_1

First eigenvalue: $\lambda_1 = 10$

$$P_2 = \lambda_1^{-1} (\sigma_y \wedge b)^{-1} (a \wedge b - \lambda_1 \sigma_x \wedge b)\, P_1$$

$$= \frac{1}{10} \cdot \frac{1}{70}\, \sigma_x\sigma_y\, (1000\, \sigma_x\sigma_y - 10 \cdot 80\, \sigma_x\sigma_y)\, P_1$$

$$= \frac{200}{700} (\sigma_x\sigma_y)^2\, P_1 = -\,\frac{2}{7}\, P_1$$

or $$P_2 = \lambda_1 (a \wedge b - \lambda_1\, a \wedge \sigma_y)^{-1} (a \wedge \sigma_x)\, P_1$$

$$= 10 \cdot \frac{1}{1000 - 10 \cdot 30} (-\,\sigma_x\sigma_y)\, (-\,20\, \sigma_x\sigma_y)\, P_1$$

$$= \frac{200}{700} (\sigma_x\sigma_y)^2\, P_1 = -\,\frac{2}{7}\, P_1$$

$\Rightarrow$ Every vector $\begin{bmatrix} P_1 \\ -2/7\, P_1 \end{bmatrix}$

or every Pauli vector $P_1\, \sigma_x - \dfrac{2}{7}\, P_1\, \sigma_y$

is eigenvector of matrix D correspond-ing to the first eigenvalue $\lambda_1 = 10$.

Summary: Finding the Eigenvectors
Corresponding to the First Eigenvalue λ_1

The first characteristic matrix which corres-ponds to eigenvalue λ_1 can be evaluated:

$$(D - \lambda_1 \, I) = \begin{bmatrix} 30-\lambda_1 & 70 \\ 20 & 80-\lambda_1 \end{bmatrix} = \begin{bmatrix} 20 & 70 \\ 20 & 70 \end{bmatrix}$$

First characteristic matrix equation
(Scheme of Falk):

$$\begin{array}{cc|c} \multicolumn{2}{l|}{(D - \lambda_1 \, I)\ P = 0} & P_1 \\ & & P_2 \\ \hline 20 & 70 & 20\,P_1 + 70\,P_2 = 0 \\ 20 & 70 & 20\,P_1 + 70\,P_2 = 0 \end{array}$$

$$\left.\begin{array}{l} \Rightarrow \quad P_1 = -\dfrac{70}{20}\,P_2 = -\dfrac{7}{2}\,P_2 \\ \text{or} \quad P_2 = -\dfrac{2}{7}\,P_1 \end{array}\right\} \quad \text{Eigenvectors: } \begin{bmatrix} P_1 \\ -2/7\,P_1 \end{bmatrix}$$

$$\Downarrow$$

$$P_1\,\sigma_x - \frac{2}{7}\,P_1\,\sigma_y$$

Eigenspace of First Eigenvalue λ_1

Different values of P_1 will get different eigen-vectors:

$P_1 = 1$: $\quad r = \begin{bmatrix} 1 \\ -2/7 \end{bmatrix} \longrightarrow r = \sigma_x - \frac{2}{7}\sigma_y$

$P_1 = 2$: $\quad r = \begin{bmatrix} 2 \\ -4/7 \end{bmatrix} \longrightarrow r = 2\sigma_x - \frac{4}{7}\sigma_y$

$P_1 = 3.5$: $\quad r = \begin{bmatrix} 3.5 \\ -1 \end{bmatrix} \longrightarrow r = 3.5\sigma_x - \sigma_y$

$P_1 = 210$: $\quad r = \begin{bmatrix} 210 \\ -60 \end{bmatrix} \longrightarrow r = 210\sigma_x - 60\sigma_y$

The space, which is formed by all these vectors, is called **eigenspace**.

The set of all solutions of linear equation $(*)$

$$D\,P = \lambda_1 P$$

is equivalent to the **eigenspace** of matrix D with respect to eigenvalue λ_1.

Basis of the First Eigenspace

Eigenvectors can be normalized by dividing them by their length:

Eigenvectors: $r = P_1\,\sigma_x - \frac{2}{7}\,P_1\,\sigma_y$

$$r^2 = \left(P_1\,\sigma_x - \frac{2}{7}\,P_1\,\sigma_y\right)^2 = \frac{53}{49}\,P_1^{\;2}$$

Length of eigenvectors:

$$|r| = \sqrt{r^2} = \frac{1}{7}\sqrt{53}\;P_1$$

The normalized eigenvector

$$v_1 = \frac{r}{|r|} = \frac{7}{\sqrt{53}}\,\sigma_x - \frac{2}{\sqrt{53}}\,\sigma_y$$

is a basis of the one-dimensional first eigen-space of matrix D with respect to eigenval-ue λ_1.

> As the next slides will show, a second eigen-space (which corresponds to eigenvalue λ_2) exists.

Finding the Eigenvectors

Part II: Eigenvectors of Second Eigenvalue λ_2

Second eigenvalue: $\lambda_2 = 100$

$$P_2 = \lambda_2^{-1} (\sigma_y \wedge b)^{-1} (a \wedge b - \lambda_2 \sigma_x \wedge b) P_1$$

$$= \frac{1}{100} \cdot \frac{1}{70} \sigma_x\sigma_y (1000 \sigma_x\sigma_y - 100 \cdot 80 \sigma_x\sigma_y) P_1$$

$$= -\frac{7000}{7000} (\sigma_x\sigma_y)^2 P_1 = P_1$$

or $$P_2 = \lambda_2 (a \wedge b - \lambda_2 a \wedge \sigma_y)^{-1} (a \wedge \sigma_x) P_1$$

$$= 100 \cdot \frac{1}{1000 - 100 \cdot 30} (-\sigma_x\sigma_y)(-20 \sigma_x\sigma_y) P_1$$

$$= -\frac{2000}{2000} (\sigma_x\sigma_y)^2 P_1 = P_1$$

$\Rightarrow$ Every vector $\begin{bmatrix} P_1 \\ P_1 \end{bmatrix}$ or every Pauli vector $P_1 \sigma_x + P_1 \sigma_y$ is eigenvector of matrix D corresponding to the second eigenvalue $\lambda_2 = 100$.

Summary: Finding the Eigenvectors

Corresponding to the Second Eigenvalue λ_2

The second characteristic matrix which corresponds to eigenvalue λ_2 can be evaluated:

$$(D - \lambda_2 I) = \begin{bmatrix} 30-\lambda_2 & 70 \\ 20 & 80-\lambda_2 \end{bmatrix} = \begin{bmatrix} -70 & 70 \\ 20 & -20 \end{bmatrix}$$

Second characteristic matrix equation
(Scheme of Falk):

$$(D - \lambda_2 I)\, P = 0$$

$$\begin{array}{cc|c} & & P_1 \\ & & P_2 \\ \hline -70 & 70 & -70\,P_1 + 70\,P_2 = 0 \\ 20 & -20 & 20\,P_1 - 20\,P_2 = 0 \end{array}$$

$$\left.\begin{array}{ll} & 70\,P_1 = 70\,P_2 \\ \text{or} & 20\,P_1 = 20\,P_2 \\ \Rightarrow & P_1 = P_2 \end{array}\right\}$$

Eigenvectors:

$$\begin{bmatrix} P_1 \\ P_1 \end{bmatrix}$$

$$\downarrow$$

$$P_1\,\sigma_x + P_1\,\sigma_y$$

Eigenspace of Second Eigenvalue λ_2

Different values of P_1 will get different eigen-vectors:

$P_1 = 1$: $\quad r = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow r = \sigma_x + \sigma_y$

$P_1 = 2$: $\quad r = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \longrightarrow r = 2\,\sigma_x + 2\,\sigma_y$

$P_1 = 3.5$: $\quad r = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix} \longrightarrow r = 3.5\,\sigma_x + 3.5\,\sigma_y$

$P_1 = 210$: $\quad r = \begin{bmatrix} 210 \\ 210 \end{bmatrix} \longrightarrow r = 210\,\sigma_x + 210\,\sigma_y$

All these vectors are situated in the eigen-space of matrix D with respect to eigenval-ue λ_2.

The set of all solutions of linear equation $(*)$

$$D\,P = \lambda_2\,P$$

is equivalent to this eigenspace.

Basis of the Second Eigenspace

Eigenvectors can be normalized by dividing them by their length:

Eigenvectors: $r = P_1 \sigma_x + P_1 \sigma_y$

$$r^2 = (P_1 \sigma_x + P_1 \sigma_y)^2 = 2\, P_1^{\;2}$$

Length of eigenvectors:

$$|r| = \sqrt{r^2} = \sqrt{2}\, P_1$$

The normalized eigenvector

$$v_2 = \frac{r}{|r|} = \frac{1}{\sqrt{2}}\, \sigma_x + \frac{1}{\sqrt{2}}\, \sigma_y$$

is a basis of the one-dimensional eigenspace of matrix D with respect to eigenvalue λ_2.

Check of Normalized Eigenvectors

Normalized eigenvector corresponding to first eigenvalue $\lambda_1 = 10$:

$$D\, v_1 = \frac{1}{\sqrt{53}} \begin{bmatrix} 30 & 70 \\ 20 & 80 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{53}} \begin{bmatrix} 70 \\ -20 \end{bmatrix} = 10\, v_1$$

Normalized eigenvector corresponding to second eigenvalue $\lambda_2 = 100$:

$$D\, v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 30 & 70 \\ 20 & 80 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = 100\, v_2$$

Additional Remark

There is always only one eigenvalue λ_i associated to an eigenvector v_i.

Solution of the Halloween Product Engineering Problem

The CEO of the firm ordered his management to consume only quantities of raw materials q in the production process which are perfect multiples λ of the production vector

$$q = \lambda P$$

and asks about the relation of quantities of final products P_1 and P_2 which will be produced.

The possible relations are given by the eigenvalues $\lambda_1 = 10$ and $\lambda_2 = 100$.

The possible production vectors are eigenvectors corresponding to these eigenvalues. As production quantities should be positive, a production vector with negative components (see first eigenvector) does not make sense.

Therefore eigenvectors corresponding to the second eigenvalue can be production vectors at Halloween 2017. The relation between the quantities of final products P_1 and P_2 will be one:

$$\frac{P_1}{P_2} = 1$$

Stochastic Matrices

To analyse the development of different market participants and their market shares of a closed market, stochastic matrices are of special significance.

Definition:

Stochastic matrices are matrices…

… which have non-negative elements only

… whose columns add to 1.

Thus the entries of stochastic matrices can be interpreted as percentages.

The coefficient vectors (i.e. the columns) of stochastic matrices can be called probability vectors.

Short note: Some math books define stochastic matrices as matrices whose rows add to 1.

Stochastic Matrices

To analyse the development of different market participants and their market shares of a closed market, stochastic matrices are of special significance.

Definition:

Stochastic matrices are matrices…

… which have non-negative elements only

… whose columns add to 1.

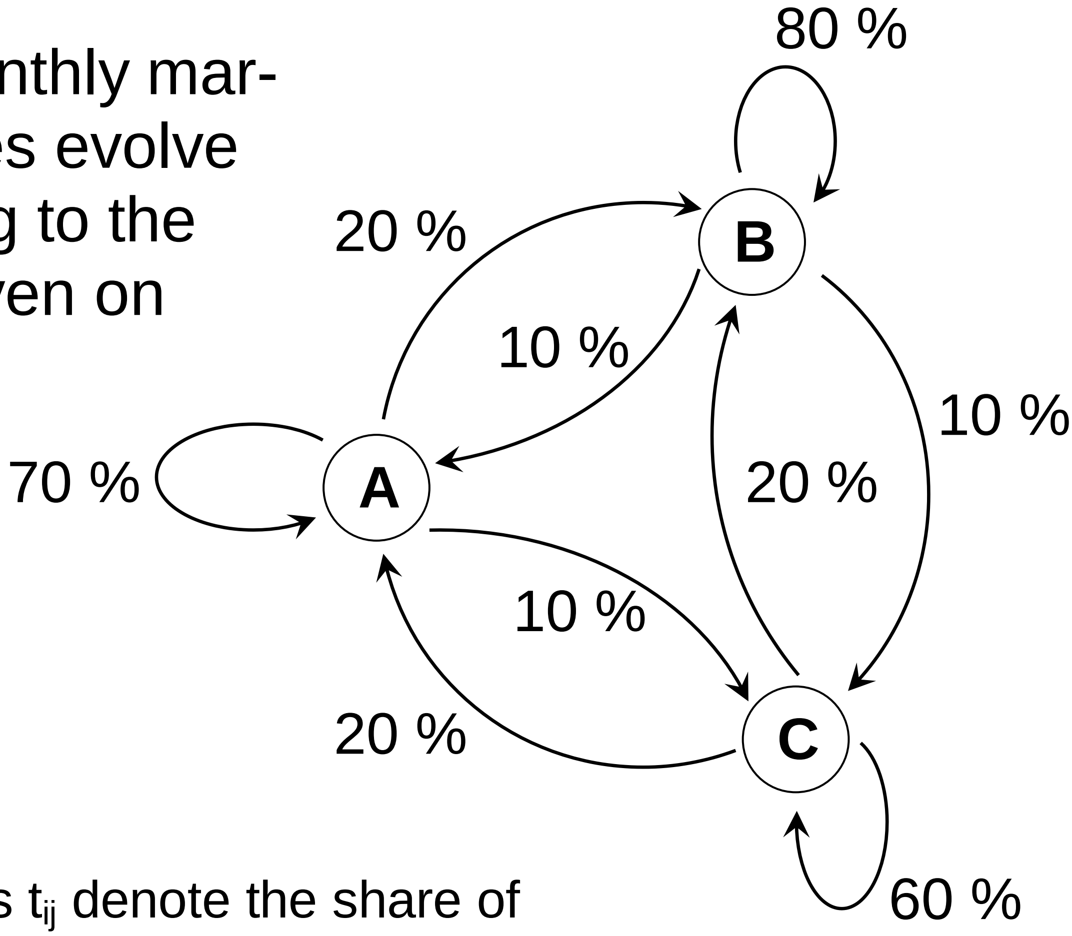
Stochastic matrices have important characteristic properties:

- All of the eigenvalues of stochastic matrices are positive.
- The largest eigenvalue of a stochastic matrix is 1.
- There is only one eigenvector associated with the eigenvalue $\lambda = 1$.

Petrol Station Problem

There are three petrol stations A, B, and C in a small city in the middle of the Australian Desert.

Their monthly market shares evolve according to the graph given on the right:



(The values t_{ij} denote the share of customers of patrol station i which will go to patrol station j next month.)

- Find the transition matrix T.
- Find eigenvalues and eigenvectors of transition matrix T.
- Find the vector v of current market shares which will remain unchanged next month.

Petrol Station Transition Matrix

$$T = \begin{bmatrix} 70\,\% & 10\,\% & 20\,\% \\ 20\,\% & 80\,\% & 20\,\% \\ 10\,\% & 10\,\% & 60\,\% \end{bmatrix}$$

$$= \begin{bmatrix} 0.70 & 0.10 & 0.20 \\ 0.20 & 0.80 & 0.20 \\ 0.10 & 0.10 & 0.60 \end{bmatrix}$$

As all coefficient vectors of the petrol station transition matrix are probability vectors (whose elements add to 100 % = 1), this transition matrix is a stochastic matrix.

Characteristic Matrix of the Petrol Station Problem

$$(T - \lambda\, I) = \begin{bmatrix} 0.70-\lambda & 0.10 & 0.20 \\ 0.20 & 0.80-\lambda & 0.20 \\ 0.10 & 0.10 & 0.60-\lambda \end{bmatrix}$$

Characteristic Coefficient Vectors

$$a - \lambda\, \sigma_x = (0.70 - \lambda)\, \sigma_x + 0.20\, \sigma_y + 0.10\, \sigma_z$$

$$b - \lambda\, \sigma_y = 0.10\, \sigma_x + (0.80 - \lambda)\, \sigma_y + 0.10\, \sigma_z$$

$$c - \lambda\, \sigma_z = 0.20\, \sigma_x + 0.20\, \sigma_y + (0.60 - \lambda)\, \sigma_z$$

Characteristic Outer Product

$$\begin{aligned}(a - \lambda\, \sigma_x) \wedge (b - \lambda\, \sigma_y) &= (\lambda^2 - 1.50\, \lambda + 0.54)\, \sigma_x\sigma_y \\ &\quad + (0.10\, \lambda - 0.06)\, \sigma_y\sigma_z \\ &\quad + (0.10\, \lambda - 0.06)\, \sigma_z\sigma_x\end{aligned}$$

$$\begin{aligned}(a - \lambda\, \sigma_x) \wedge (b - \lambda\, \sigma_y) \wedge (c - \lambda\, \sigma_z) &= (-\lambda^3 + 2.10\, \lambda^2 - 1.40\, \lambda + 0.30)\, \sigma_x\sigma_y\sigma_z\end{aligned}$$

Characteristic Polynomial

$$(a - \lambda\, \sigma_x) \wedge (b - \lambda\, \sigma_y) \wedge (c - \lambda\, \sigma_z)\ \sigma_z\sigma_y\sigma_x$$
$$= -\lambda^3 + 2.10\, \lambda^2 - 1.40\, \lambda + 0.30$$

Characteristic Equation

$$-\lambda^3 + 2.10\, \lambda^2 - 1.40\, \lambda + 0.30 = 0$$

As one of the eigenvalues equals 1

$$\lambda_1 = 1$$

the characteristic equation can be transformed into

$$(\lambda - 1)\ \frac{-\lambda^3 + 2.10\,\lambda^2 - 1.40\lambda + 0.30}{\lambda - 1} = 0$$

$$\Rightarrow \qquad (\lambda - 1)\ \left(-\lambda^2 + 1.10\, \lambda - 0.30\right) = 0$$

$$(\lambda - 1)\ \underbrace{\left(\lambda^2 - 1.10\, \lambda + 0.30\right)}_{\lambda^2 - 1.10\, \lambda + 0.30\ = 0} = 0$$

Finding the Eigenvalues

$$\Rightarrow \qquad (\lambda - 1)\,\underbrace{(\lambda^2 - 1.10\,\lambda + 0.30)}\ = 0$$

$$\lambda^2 - 1.10\,\lambda + 0.30 \ = 0$$

$$\lambda^2 - 2 \cdot 0.55\,\lambda + 0.55^2 - 0.55^2 + 0.30 = 0$$

$$\lambda^2 - 2 \cdot 0.55\,\lambda + 0.55^2 = 0.0025$$

$$(\lambda - 0.55)^2 = (\pm 0.05)^2$$

Therefore the three eigenvalues are

$\lambda_1 = 1$

$\lambda_2 = 0.55 + 0.05 = 0.60$

$\lambda_3 = 0.55 - 0.05 = 0.50$

Short check of results:

The characteristic polynomial can be rewritten as

$$-(\lambda - 1)\,(\lambda - 0.60)\,(\lambda - 0.50)$$
$$= -\lambda^3 + 2.10\,\lambda^2 - 1.40\,\lambda + 0.30$$

showing that the three results are correct.

Finding the Eigenvectors

As a system of three linear equations

$$a\,x + b\,y + c\,z = d$$

can be solved by

$$(a \wedge b \wedge c)\,x = (d \wedge b \wedge c)$$

$$(a \wedge b \wedge c)\,y = (a \wedge d \wedge c)$$

$$(a \wedge b \wedge c)\,z = (a \wedge b \wedge d)$$

(see repetition slide #8),

$x = q_1$
$y = q_2$
$z = q_3$
$d = \lambda\, q$

the Pauli vector equation

$$a\,q_1 + b\,q_2 + b\,q_3 = \lambda\, q$$

will give the mathematical relations

$$(a \wedge b \wedge c)\,q_1 = \lambda\,(q \wedge b \wedge c)$$

$$(a \wedge b \wedge c)\,q_2 = \lambda\,(a \wedge q \wedge c)$$

$$(a \wedge b \wedge c)\,q_3 = \lambda\,(a \wedge b \wedge q)$$

With $q = q_1\,\sigma_x + q_2\,\sigma_y + q_3\,\sigma_z$

these equations show the mathematical relations between the eigenvector coefficients.

$$q = q_1\,\sigma_x + q_2\,\sigma_y + q_3\,\sigma_z$$

$$\downarrow$$

$$(a \wedge b \wedge c)\, q_1 = \lambda\, (q \wedge b \wedge c)$$

$$(a \wedge b \wedge c)\, q_2 = \lambda\, (a \wedge q \wedge c)$$

$$(a \wedge b \wedge c)\, q_3 = \lambda\, (a \wedge b \wedge q)$$

$\Rightarrow$ Mathematical relations between eigen-vector coefficients:

$$\big((a \wedge b \wedge c) - \lambda\, (\sigma_x \wedge b \wedge c)\big)\, q_1 = \lambda\, (\sigma_y \wedge b \wedge c)\, q_2 + \lambda\, (\sigma_z \wedge b \wedge c)\, q_3$$

$$\big((a \wedge b \wedge c) - \lambda\, (a \wedge \sigma_y \wedge c)\big)\, q_2 = \lambda\, (a \wedge \sigma_x \wedge c)\, q_1 + \lambda\, (a \wedge \sigma_z \wedge c)\, q_3$$

$$\big((a \wedge b \wedge c) - \lambda\, (a \wedge b \wedge \sigma_z)\big)\, q_3 = \lambda\, (a \wedge b \wedge \sigma_x)\, q_1 + \lambda\, (a \wedge b \wedge \sigma_y)\, q_2$$

Finding the Eigenvectors: Calculation of Outer Products

$a = 0.70\,\sigma_x + 0.20\,\sigma_y + 0.10\,\sigma_z$

$b = 0.10\,\sigma_x + 0.80\,\sigma_y + 0.10\,\sigma_z$

$c = 0.20\,\sigma_x + 0.20\,\sigma_y + 0.60\,\sigma_z$

$a \wedge b \wedge c = 0.30\ \sigma_x\sigma_y\sigma_z$

$\sigma_x \wedge b \wedge c = \quad 0.46\ \sigma_x\sigma_y\sigma_z$

$a \wedge \sigma_x \wedge c = -0.10\ \sigma_x\sigma_y\sigma_z$

$a \wedge b \wedge \sigma_x = -0.06\ \sigma_x\sigma_y\sigma_z$

$\sigma_y \wedge b \wedge c = -0.04\ \sigma_x\sigma_y\sigma_z$

$a \wedge \sigma_y \wedge c = \quad 0.40\ \sigma_x\sigma_y\sigma_z$

$a \wedge b \wedge \sigma_y = -0.06\ \sigma_x\sigma_y\sigma_z$

$\sigma_z \wedge b \wedge c = -0.14\ \sigma_x\sigma_y\sigma_z$

$a \wedge \sigma_z \wedge c = -0.10\ \sigma_x\sigma_y\sigma_z$

$a \wedge b \wedge \sigma_z = \quad 0.54\ \sigma_x\sigma_y\sigma_z$

All these outer products represent volume elements which are parallel to the same three-dimensional space.

Finding the Eigenvectors

Part I: Eigenvectors of First Eigenvalue λ_1

First eigenvalue: $\lambda_1 = 1$

$$(0.30 - 1 \cdot 0.46)\, q_1 = 1 \cdot (-0.04)\, q_2 + 1 \cdot (-0.14)\, q_3$$

$$\Rightarrow \quad -0.16\, q_1 = -0.04\, q_2 - 0.14\, q_3$$

$$\Rightarrow \quad 8\, q_1 = 2\, q_2 + 7\, q_3$$

$$(0.30 - 1 \cdot 0.40)\, q_2 = 1 \cdot (-0.10)\, q_1 + 1 \cdot (-0.10)\, q_3$$

$$\Rightarrow \quad -0.10\, q_2 = -0.10\, q_1 - 0.10\, q_3$$

$$\Rightarrow \quad q_2 = q_1 + q_3$$

$$\Rightarrow \quad 8\, q_1 = 2\,(q_1 + q_3) + 7\, q_3$$

$$\Rightarrow \quad 6\, q_1 = 9\, q_3 \quad \Rightarrow \quad q_1 = \frac{3}{2}\, q_3$$

e.g.: $q_3 = 2 \;\rightarrow\; q_1 = 3 \;\rightarrow\; q_3 = 5$

Check of result:

$$(0.30 - 1 \cdot 0.54)\, q_3 = 1 \cdot (-0.06)\, q_1 + 1 \cdot (-0.06)\, q_2$$

$$\Rightarrow \quad -0.24\, q_3 = -0.06\, q_1 - 0.06\, q_2$$

$$\Rightarrow \quad 4\, q_3 = q_1 + q_2$$

$$4 \cdot 2 = 3 + 5 = 8$$

Summary: Finding the Eigenvectors
Corresponding to the First Eigenvalue λ_1

The first characteristic matrix which corres-ponds to eigenvalue λ_1 can be evaluated:

$$(T - \lambda_1 I) = \begin{bmatrix} -0.30 & 0.10 & 0.20 \\ 0.20 & -0.20 & 0.20 \\ 0.10 & 0.10 & -0.40 \end{bmatrix}$$

First characteristic matrix equation
(Scheme of Falk):

$$\begin{array}{ccc|c} \multicolumn{3}{l|}{(T - \lambda_1 I)\, q = 0} & q_1 \\ & & & q_2 \\ & & & q_3 \\ \hline -0.30 & 0.10 & 0.20 & -0.30\, q_1 + 0.10\, q_2 + 0.20\, q_3 = 0 \\ 0.20 & -0.20 & 0.20 & 0.20\, q_1 - 0.20\, q_2 + 0.20\, q_3 = 0 \\ 0.10 & 0.10 & -0.40 & 0.10\, q_1 + 0.10\, q_2 - 0.40\, q_3 = 0 \end{array}$$

row 2 – row 1: $\quad 0.50\, q_1 - 0.30\, q_2 = 0$

$$q_1 = 0.60\, q_2$$

substitute q_1 in one of the rows: $\quad q_3 = 0.40\, q_2$

e.g.: $\quad q_2 = 1 \;\rightarrow\; q_1 = 0.60 \;\rightarrow\; q_3 = 0.40$

Different values of eigenvector coefficients will get different eigenvectors, e.g.

$$q_2 = 1: \qquad r = \begin{bmatrix} 0.60 \\ 1 \\ 0.40 \end{bmatrix} \longrightarrow r = 0.60\,\sigma_x + \sigma_y + 0.40\,\sigma_z$$

$$q_3 = 2: \qquad r = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \longrightarrow r = 3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z$$

Normalized First Eigenvector

Eigenvectors: $r = 0.6\, q_2\, \sigma_x + q_2\, \sigma_y + 0.4\, q_2\, \sigma_z$

$$r^2 = 1.52\, q_2^{\,2} = \frac{38}{25}\, q_2^{\,2}$$

Length of eigenvectors:

$$|r| = \sqrt{r^2} = \frac{1}{5}\sqrt{38}\; q_2$$

The normalized eigenvector

$$v_1 = \frac{r}{|r|} = \frac{3}{\sqrt{38}}\,\sigma_x + \frac{5}{\sqrt{38}}\,\sigma_y + \frac{2}{\sqrt{38}}\,\sigma_z$$

is base vector of the one-dimensional first eigenspace of matrix D with respect to eigenvalue $\lambda_1 = 1$.

Finding the Eigenvectors

Part II: Eigenvectors of Second Eigenvalue λ_2

Second eigenvalue: $\lambda_2 = 0.60$

$$(0.30 - 0.60 \cdot 0.46)\, q_1 = 0.60 \cdot (-0.04)\, q_2 + 0.60 \cdot (-0.14)\, q_3$$

$$\Rightarrow \quad 0.024\, q_1 = -0.024\, q_2 - 0.084\, q_3$$

$$\Rightarrow \quad 2\, q_1 = -2\, q_2 - 7\, q_3$$

$$(0.30 - 0.60 \cdot 0.40)\, q_2 = 0.60 \cdot (-0.10)\, q_1 + 0.60 \cdot (-0.10)\, q_3$$

$$\Rightarrow \quad 0.06\, q_2 = -0.06\, q_1 - 0.06\, q_3$$

$$\Rightarrow \quad q_2 = -q_1 - q_3$$

$$\Rightarrow \quad 2\, q_1 = -2\,(-q_1 - q_3) - 7\, q_3$$

$$\Rightarrow \quad 7\, q_3 = 0 \quad \Rightarrow \quad q_3 = 0$$

$$\Rightarrow \quad q_2 = -q_1$$

e.g.: $q_3 = 0 \;\rightarrow\; q_1 = 1 \;\rightarrow\; q_2 = -1$

Check of result:

$$(0.30 - 0.60 \cdot 0.54)\, q_3 = 0.60 \cdot (-0.06)\, q_1 + 0.60 \cdot (-0.06)\, q_2$$

$$\Rightarrow \quad -0.024\, q_3 = -0.036\, q_1 - 0.036\, q_2$$

$$\Rightarrow \quad 2\, q_3 = 3\, q_1 + 3\, q_2$$

$$2 \cdot 0 = 3 \cdot 1 + 3 \cdot (-1) = 0$$

Summary: Finding the Eigenvectors

Corresponding to the Second Eigenvalue λ_2

The second characteristic matrix which corresponds to eigenvalue λ_2 can be evaluated:

$$(T - \lambda_2 \mathrm{I}) = \begin{bmatrix} 0.10 & 0.10 & 0.20 \\ 0.20 & 0.20 & 0.20 \\ 0.10 & 0.10 & 0 \end{bmatrix}$$

First characteristic matrix equation

(Scheme of Falk):

$$\begin{array}{ccc|l} \multicolumn{3}{l|}{(T - \lambda_2 \mathrm{I})\, q = 0} & q_1 \\ & & & q_2 \\ & & & q_3 \\ \hline 0.10 & 0.10 & 0.20 & 0.10\, q_1 + 0.10\, q_2 + 0.20\, q_3 = 0 \\ 0.20 & 0.20 & 0.20 & 0.20\, q_1 + 0.20\, q_2 + 0.20\, q_3 = 0 \\ 0.10 & 0.10 & 0 & 0.10\, q_1 + 0.10\, q_2 \qquad\qquad\quad = 0 \end{array}$$

row 2 – 2 · row 1: $\quad -0.20\, q_3 = 0$

$$q_3 = 0$$

row 3: $\quad q_1 = -q_2$

e.g.: $\quad q_1 = 1 \;\rightarrow\; q_2 = -1 \;\rightarrow\; q_3 = 0$

Different values of eigenvector coefficients will get different eigenvectors, e.g.

$$q_2 = -1: \quad r = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \longrightarrow r = \sigma_x - \sigma_y$$

$$q_2 = 5: \quad r = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} \longrightarrow r = -5\,\sigma_x + 5\,\sigma_y$$

Normalized Second Eigenvector

Eigenvectors: $\quad r = q_1\,\sigma_x - q_1\,\sigma_y$

$$r^2 = 2\,q_1^{\,2}$$

Length of eigenvectors:

$$|r| = \sqrt{r^2} = \sqrt{2}\;q_1$$

The normalized eigenvector

$$v_2 = \frac{r}{|r|} = \frac{1}{\sqrt{2}}\,\sigma_x - \frac{1}{\sqrt{2}}\,\sigma_y$$

is base vector of the one-dimensional second eigenspace of matrix D with respect to eigen-value $\lambda_2 = 0.60$.

Finding the Eigenvectors

Part III: Eigenvectors of Third Eigenvalue λ_3

Third eigenvalue: $\lambda_3 = 0.50$

$$(0.30 - 0.50 \cdot 0.46)\, q_1 = 0.50 \cdot (-0.04)\, q_2 + 0.50 \cdot (-0.14)\, q_3$$

$$\Rightarrow \quad 0.07\, q_1 = -0.02\, q_2 - 0.07\, q_3$$

$$\Rightarrow \quad 7\, q_1 = -2\, q_2 - 7\, q_3$$

$$(0.30 - 0.50 \cdot 0.40)\, q_2 = 0.50 \cdot (-0.10)\, q_1 + 0.50 \cdot (-0.10)\, q_3$$

$$\Rightarrow \quad 0.10\, q_2 = -0.05\, q_1 - 0.05\, q_3$$

$$\Rightarrow \quad 2\, q_2 = -q_1 - q_3$$

$$\Rightarrow \quad 7\, q_1 = -(-q_1 - q_3) - 7\, q_3$$

$$\Rightarrow \quad 8\, q_1 = -8\, q_3 \quad \Rightarrow \quad q_1 = -q_3$$

$$\Rightarrow \quad q_2 = 0$$

e.g.: $q_2 = 0 \;\rightarrow\; q_1 = 1 \;\rightarrow\; q_3 = -1$

Check of result:

$$(0.30 - 0.50 \cdot 0.54)\, q_3 = 0.50 \cdot (-0.06)\, q_1 + 0.50 \cdot (-0.06)\, q_2$$

$$\Rightarrow \quad 0.03\, q_3 = -0.03\, q_1 - 0.03\, q_2$$

$$\Rightarrow \quad q_3 = -q_1 - q_2$$

$$-1 = -1 - 0$$

Summary: Finding the Eigenvectors
Corresponding to the Third Eigenvalue λ_3

The third characteristic matrix which corres-ponds to eigenvalue λ_3 can be evaluated:

$$(T - \lambda_3 \mathrm{I}) = \begin{bmatrix} 0.20 & 0.10 & 0.20 \\ 0.20 & 0.30 & 0.20 \\ 0.10 & 0.10 & 0.10 \end{bmatrix}$$

First characteristic matrix equation
(Scheme of Falk):

$$\begin{array}{ccc|c} \multicolumn{3}{l|}{(T - \lambda_3 \mathrm{I})\, q = 0} & q_1 \\ & & & q_2 \\ & & & q_3 \\ \hline 0.20 & 0.10 & 0.20 & 0.20\, q_1 + 0.10\, q_2 + 0.20\, q_3 = 0 \\ 0.20 & 0.30 & 0.20 & 0.20\, q_1 + 0.30\, q_2 + 0.20\, q_3 = 0 \\ 0.10 & 0.10 & 0.10 & 0.10\, q_1 + 0.10\, q_2 + 0.10\, q_3 = 0 \end{array}$$

row 2 – row 1: $0.20\, q_2 = 0$

$$q_2 = 0$$

$$q_1 = -q_3$$

e.g.: $q_2 = 0 \;\rightarrow\; q_3 = 5 \;\rightarrow\; q_1 = -5$

Different values of eigenvector coefficients will get different eigenvectors, e.g.

$$q_1 = 1: \qquad r = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \longrightarrow r = \sigma_x - \sigma_z$$

$$q_1 = -5: \qquad r = \begin{bmatrix} -5 \\ 0 \\ 5 \end{bmatrix} \longrightarrow r = -5\,\sigma_x + 5\,\sigma_z$$

Normalized Third Eigenvector

Eigenvectors: $r = q_1\,\sigma_x - q_1\,\sigma_z$

$$r^2 = 2\,q_1^{\,2}$$

Length of eigenvectors:

$$|r| = \sqrt{r^2} = \sqrt{2}\;q_1$$

The normalized eigenvector

$$v_3 = \frac{r}{|r|} = \frac{1}{\sqrt{2}}\,\sigma_x - \frac{1}{\sqrt{2}}\,\sigma_z$$

is base vector of the one-dimensional third eigenspace of matrix D with respect to eigenvalue $\lambda_3 = 0.50$.

Solution of the Petrol Station Problem

We wanted to find the vector v of current market shares which will remain unchanged next month:

$\lambda = 1$

$$T\,v = v = 1\,v$$

The “state of the market” is supposed to be constant. Therefore vectors like v are sometimes called **state vectors**.

Comparing the matrix equation of the constant state vector with the eigenvector matrix equation

$$T\,v = \lambda\,v$$

it can be seen that the eigenvalue has to be one. Thus the state vector v of unchanged market shares should be an eigenvector which corresponds to the first eigenvalue

$$\lambda_1 = 1$$

These eigenvectors are given on slide #40:

$$r = 0.60\,q_2\,\sigma_x + 1.00\,q_2\,\sigma_y + 0.40\,q_2\,\sigma_z$$

But state vectors are supposed to indicate the state of the market in percentages. They are probability vectors with components which sum to one:

$$r = 0.60\, q_2\, \sigma_x + 1.00\, q_2\, \sigma_y + 0.40\, q_2\, \sigma_z$$

$$0.60\, q_2 + 1.00\, q_2 + 0.40\, q_2 = 2.00\, q_2 = 1$$

$$\Rightarrow \qquad q_2 = \frac{1}{2}$$

The vector v of current market shares which remain unchanged next month therefore is:

$$v = 0.30\, \sigma_x + 0.50\, \sigma_y + 0.20\, \sigma_z$$
$$= 30\,\%\, \sigma_x + 50\,\%\, \sigma_y + 20\,\%\, \sigma_z$$

This solution can also be given in conventional column vector notation:

$$v = \begin{bmatrix} 0.30 \\ 0.50 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 30\,\% \\ 50\,\% \\ 20\,\% \end{bmatrix}$$

Check of Solution

$T\,v = v$			0.30
			0.50
			0.20
0.70	0.10	0.20	0.30
0.20	0.80	0.20	0.50
0.10	0.10	0.60	0.20

Analyzing the Meaning of the Solution

- What is the meaning of this solution?
- And what is the significance of this solution for real market situations?

To find out whether there is an outstanding importance of this solution, which is connected with the very special eigenvalue of 1, we will now analyze the long-term development of the petrol market.

Markov Chains

To model the long-term development of a market we simply have to calculate state vectors of market shares after several periods.

Initial state vector:	x_0
	$\downarrow$
State vector after 1 period:	$x_1 = T\, x_0$
	$\downarrow$
State vector after 2 periods:	$x_2 = T\, x_1$
	$\downarrow$
State vector after 3 periods:	$x_3 = T\, x_2$
	$\downarrow$
State vector after 4 periods:	$x_4 = T\, x_3$
	$\downarrow$
State vector after n periods:	$x_n = T\, x_{n-1}$

This very simple mathematical model is called **Markov chain**.

Markov Chains

Markov chains are sequences of probability vectors which are generated by pre-multiplicating a stochastic matrix T.

They can be written as the following first order linear difference equation:

$$x_n = T\, x_{n-1}$$

Markov chains are used in many fields to analyse the long-term development of probabilities, of customers' fluctuations, or of market shares in closed systems.

As an example, the development of the petrol market of the Australian city will be analyzed in the following, if the initial state vector x_0 is the vector of equal market shares:

$$x_0 = \tfrac{1}{3}\,\sigma_x + \tfrac{1}{3}\,\sigma_y + \tfrac{1}{3}\,\sigma_z$$

$$\approx 33.33\,\%\,\sigma_x + 33.33\,\%\,\sigma_y + 33.33\,\%\,\sigma_z$$

At the beginning every petrol station will have a third of all customers.

Petrol Market Shares after One Period

After the first month has passed the initial state vector x_0 of equal market shares will have transformed into the state vector x_1:

$$x_1 = T\,x_0$$

$T\,x_0 = x_1$			
			0.3333
			0.3333
			0.3333
0.70	0.10	0.20	0.3333
0.20	0.80	0.20	0.4000
0.10	0.10	0.60	0.2667

The market shares one month later thus are

$$x_1 \approx 33.33\,\%\ \sigma_x + 40.00\,\%\ \sigma_y + 26.67\,\%\ \sigma_z$$

Obviously customers of petrol station C have been most unhappy with service or petrol prices and changed with a higher rate to other petrol stations.

Petrol Market Shares after Two Periods

After two months the initial state vector x_0 of equal market shares will have transformed into the state vector x_2:

$$x_2 = T\, x_1 = T\,(T\, x_0) = T^2\, x_0$$

$T\, x_1 = x_2$			0.3333
			0.4000
			0.2667
0.70	0.10	0.20	0.3267
0.20	0.80	0.20	0.4400
0.10	0.10	0.60	0.2333

The market shares two months later thus are

$$x_2 \approx 32.67\,\%\ \sigma_x + 44.00\,\%\ \sigma_y + 23.33\,\%\ \sigma_z$$

Obviously customers of petrol station B have been satisfied with service and petrol prices to a greater extent compared with customers of other petrol stations.

Petrol Market Shares after Three Periods

After three months the initial state vector x_0 of equal market shares will have transformed into the state vector x_3:

$$x_3 = T\,x_2 = T\,(T^2\,x_0) = T^3\,x_0$$

$T\,x_2 = x_3$			
			0.3267
			0.4400
			0.2333
0.70	0.10	0.20	0.3193
0.20	0.80	0.20	0.4640
0.10	0.10	0.60	0.2167

The market shares three months later are

$$x_3 \approx 31.93\,\%\,\sigma_x + 46.40\,\%\,\sigma_y + 21.67\,\%\,\sigma_z$$

Petrol Market Shares after Four Periods

After four months the initial state vector x_0 of equal market shares will have transformed into the state vector x_4:

$$x_4 = T\, x_3 = T\,(T^3\, x_0) = T^4\, x_0$$

$T\, x_3 = x_4$			
			0.3193
			0.4640
			0.2167
0.70	0.10	0.20	0.3133
0.20	0.80	0.20	0.4784
0.10	0.10	0.60	0.2083

The market shares four months later now are

$$x_4 \approx 31.33\,\%\ \sigma_x + 47.84\,\%\ \sigma_y + 20.83\,\%\ \sigma_z$$

Petrol Market Shares after Five Periods

After five months the initial state vector x_0 of equal market shares will have transformed into the state vector x_5:

$$x_5 = T\,x_4 = T\,(T^4\,x_0) = T^5\,x_0$$

$T\,x_4 = x_5$			
			0.3133
			0.4784
			0.2083
0.70	0.10	0.20	0.3088
0.20	0.80	0.20	0.4870
0.10	0.10	0.60	0.2042

The market shares five months later are

$$x_5 \approx 30.88\,\%\ \sigma_x + 48.70\,\%\ \sigma_y + 20.42\,\%\ \sigma_z$$

Petrol Market Shares after Six Periods

After six months the initial state vector x_0 of equal market shares will have transformed into the state vector x_6:

$$x_6 = T\,x_5 = T\,(T^5\,x_0) = T^6\,x_0$$

$T\,x_5 = x_6$			
			0.3088
			0.4870
			0.2042
0.70	0.10	0.20	0.3057
0.20	0.80	0.20	0.4922
0.10	0.10	0.60	0.2021

The market shares six months later are

$$x_6 \approx 30.57\,\%\ \sigma_x + 49.22\,\%\ \sigma_y + 20.21\,\%\ \sigma_z$$

Petrol Market Shares after Seven Periods

After seven months the initial state vector x_0 of equal market shares will have transformed into the state vector x_7:

$$x_7 = T\, x_6 = T\,(T^6\, x_0) = T^7\, x_0$$

$T\, x_6 = x_7$			
			0.3057
			0.4922
			0.2021
0.70	0.10	0.20	0.3036
0.20	0.80	0.20	0.4953
0.10	0.10	0.60	0.2010

The market shares seven months later are

$$x_7 \approx 30.36\,\%\ \sigma_x + 49.53\,\%\ \sigma_y + 20.10\,\%\ \sigma_z$$

Petrol Market Shares after Eight Periods

After eight months the initial state vector x_0 of equal market shares will have transformed into the state vector x_8:

$$x_8 = T\, x_7 = T\,(T^7\, x_0) = T^8\, x_0$$

$T\, x_7 = x_8$

			0.3036
			0.4953
			0.2010
0.70	0.10	0.20	0.3022
0.20	0.80	0.20	0.4972
0.10	0.10	0.60	0.2005

The market shares eight months later are

$$x_8 \approx 30.22\,\%\ \sigma_x + 49.72\,\%\ \sigma_y + 20.05\,\%\ \sigma_z$$

This indicates, that the state vectors approach a long-term equilibrium state vector, which is identical to the state vector of unchanged market shares (i.e. to the eigenvector).

Long-Term Development of Closed Markets

The development of petrol market shares

$$
\begin{aligned}
x_0 &\approx 33.33\,\%\ \sigma_x + 33.33\,\%\ \sigma_y + 33.33\,\%\ \sigma_z\\
x_1 &\approx 33.33\,\%\ \sigma_x + 40.00\,\%\ \sigma_y + 26.67\,\%\ \sigma_z\\
x_2 &\approx 32.67\,\%\ \sigma_x + 44.00\,\%\ \sigma_y + 23.33\,\%\ \sigma_z\\
x_3 &\approx 31.93\,\%\ \sigma_x + 46.40\,\%\ \sigma_y + 21.67\,\%\ \sigma_z\\
x_4 &\approx 31.33\,\%\ \sigma_x + 47.84\,\%\ \sigma_y + 20.83\,\%\ \sigma_z\\
x_5 &\approx 30.88\,\%\ \sigma_x + 48.70\,\%\ \sigma_y + 20.42\,\%\ \sigma_z\\
x_6 &\approx 30.57\,\%\ \sigma_x + 49.22\,\%\ \sigma_y + 20.21\,\%\ \sigma_z\\
x_7 &\approx 30.36\,\%\ \sigma_x + 49.53\,\%\ \sigma_y + 20.10\,\%\ \sigma_z\\
x_8 &\approx 30.22\,\%\ \sigma_x + 49.72\,\%\ \sigma_y + 20.05\,\%\ \sigma_z
\end{aligned}
$$

$$v = x_\infty \approx 30.00\,\%\ \sigma_x + 50.00\,\%\ \sigma_y + 20.00\,\%\ \sigma_z$$

indicates, that state vectors approach a long-term equilibrium state vector, which is identical to the state vector of unchanged market shares v.

Thus they approach an eigenvector which corresponds to eigenvalue $\lambda_1 = 1$.

Long-Term Development of Closed Markets

If the development of a closed market can be described by a Markov chain, the equilibrium market shares after a long time will be identical to the market shares given by the eigenvector corresponding to the eigenvalue 1.

This can be shown mathematically by splitting the initial state vector x_0 into components pointing into the direction of the different eigenvectors.

Then state vectors are linear combinations of eigenvectors, and the long-term development of closed markets can be understood mathematically as the long-term development of eigenvalues and eigenvectors.

Splitting the Initial State Vector into Eigenvector Components

As an example, the vector of equal market shares of the petrol station problem can be written as linear combination of eigenvectors in the following way:

$$x_0 = \begin{bmatrix} 33.33\,\% \\ 33.33\,\% \\ 33.33\,\% \end{bmatrix}$$

$$= \begin{bmatrix} 30\,\% \\ 50\,\% \\ 20\,\% \end{bmatrix} + \begin{bmatrix} 16.67\,\% \\ -16.67\,\% \\ 0\,\% \end{bmatrix} - \begin{bmatrix} 13.33\,\% \\ 0\,\% \\ -13.33\,\% \end{bmatrix}$$

Or written in Pauli vector notation:

$$x_0 = \frac{1}{3}\,\sigma_x + \frac{1}{3}\,\sigma_y + \frac{1}{3}\,\sigma_z$$

$$= \left(\frac{9}{30}+\frac{5}{30}-\frac{4}{30}\right)\sigma_x + \left(\frac{15}{30}-\frac{5}{30}\right)\sigma_y + \left(\frac{6}{30}+\frac{4}{30}\right)\sigma_z$$

$$= \frac{3}{10}\,\sigma_x + \frac{1}{2}\,\sigma_y + \frac{1}{5}\,\sigma_z + \frac{1}{6}\,(\sigma_x - \sigma_y)$$

$$- \frac{2}{15}\,(\sigma_x - \sigma_z)$$

Splitting the Initial State Vector into Eigenvector Components

With the initial state vector

$$x_0 = \frac{1}{3}\,\sigma_x + \frac{1}{3}\,\sigma_y + \frac{1}{3}\,\sigma_z$$

and the eigenvectors

$$v = \frac{3}{10}\,\sigma_x + \frac{5}{10}\,\sigma_y + \frac{2}{10}\,\sigma_z = \frac{1}{10}\sqrt{38}\;v_1$$

$$v' = \frac{1}{6}\,\sigma_x - \frac{1}{6}\,\sigma_y = \frac{1}{6}\sqrt{2}\;v_2$$

$$v'' = -\frac{2}{15}\,\sigma_x + \frac{2}{15}\,\sigma_z = -\frac{2}{15}\sqrt{2}\;v_3$$

the initial state vector can now be written as

$$x_0 = v + v' + v''$$

normalized eigenvectors

In general:

$$x_0 = c_1\,v_1 + c_2\,v_2 + c_3\,v_3 + \ldots + c_n\,v_n$$

scalar coefficients

Eigenvalues and Eigenvectors of Powers of Matrices

To understand the long-term development of eigenvalues and eigenvectors mathematically, eigenvalues and eigenvectors of powers of matrices are required.

If eigenvalues λ and associated eigenvectors v of a matrix T

$$T\,v = \lambda\,v$$

are given, every eigenvector v will also be an eigenvector of any power of matrix T^n.

$$T^2\,v = T\,(T\,v) = T\,(\lambda\,v) = \lambda\,(T\,v) = \lambda^2\,v$$

$$T^3\,v = T\,(T^2\,v) = T\,(\lambda^2\,v) = \lambda^2\,(T\,v) = \lambda^3\,v$$

$$T^4\,v = T\,(T^3\,v) = T\,(\lambda^3\,v) = \lambda^3\,(T\,v) = \lambda^4\,v$$

etc…

In general:

$$T^n\,v = \lambda^n\,v \qquad (n \in \mathbb{N})$$

Eigenvalues and Eigenvectors of Powers of Matrices

And the powers of eigenvalues λ^n will be eigenvalues of matrix T^n.

In general:

$$T^n v = \lambda^n v \qquad (n \in \mathbb{N})$$

n^{th} power of matrix T

eigenvalue of matrix T^n

eigenvector of matrix T^n

Long-Term Development of Closed Markets

Now the long-term development of the petrol market can be evaluated:

$$x_0 = v + v' + v''$$

$$
\begin{aligned}
x_\infty &= \lim_{n \to \infty} (T^n x_0) \\
&= \lim_{n \to \infty} (T^n (v + v' + v'')) \\
&= \lim_{n \to \infty} (T^n v + T^n v' + T^n v'') \\
&= \lim_{n \to \infty} (\lambda_1{}^n v + \lambda_2{}^n v' + \lambda_3{}^n v'')
\end{aligned}
$$

$$\lambda_1 = 1 \qquad \lambda_2 = 0.60 \qquad \lambda_3 = 0.50$$

$$
\begin{aligned}
&= \lim_{n \to \infty} (1^n v + 0.60^n v' + 0.50^n v'') \\
&= \underbrace{\lim_{n \to \infty} 1^n}_{1} v + \underbrace{\lim_{n \to \infty} 0.60^n}_{0} v' + \underbrace{\lim_{n \to \infty} 0.50^n}_{0} v'' \\
&= v
\end{aligned}
$$

QED (quod erat demonstrandum)

Long-Term Development of Closed Markets

And the long-term development of closed markets with n market participants will be

$$x_0 = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n$$

$$x_\infty = \lim_{n \to \infty} (T^n x_0)$$

$$= \lim_{n \to \infty} (T^n (c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n))$$

$$= \lim_{n \to \infty} (T^n c_1 v_1 + T^n c_2 v_2 + T^n c_3 v_3 + \dots + T^n c_n v_n)$$

$$= c_1 \lim_{n \to \infty} (T^n v_1) + c_2 \lim_{n \to \infty} (T^n v_2) + c_3 \lim_{n \to \infty} (T^n v_3) + \dots + c_n \lim_{n \to \infty} (T^n v_n)$$

$$= c_1 \lim_{n \to \infty} \lambda_1^n v_1 + c_2 \lim_{n \to \infty} \lambda_2^n v_2 + c_3 \lim_{n \to \infty} \lambda_3^n v_3 + \dots + c_n \lim_{n \to \infty} \lambda_n^n v_n$$

$$= c_1 \cdot 1 \cdot v_1 + c_2 \cdot 0 \cdot v_2 + c_3 \cdot 0 \cdot v_3 + \dots + c_n \cdot 0 \cdot v_n$$

$$= c_1 v_1$$

if $\lambda_1 = 1$ and $\lambda_2, \lambda_3, \dots, \lambda_n < 1$

Conclusion & Summary

If closed markets can be modelled by Markov chains, state vectors will be linear combinations of eigenvectors.

Components of state vectors parallel to eigenvectors corresponding to eigenvalues smaller than one will vanish on the long term.

On the long term only the component of the state vector which is parallel to the eigenvector corresponding to the eigenvalue 1 will survive.

Thus the equilibrium market shares after a long time will be identical to market shares given by the eigenvector corresponding to the eigenvalue 1.

> You will learn more about Markov chains at the statistics course next semester or later on at advanced business mathematics courses and advanced courses of mathematics of economics.

Finding Eigenvalues and Eigenvectors of Higher-Dimensional Matrices

Solving the characteristic equation for all values of λ is sometimes difficult or even impossible to do by hand:

- The characteristic equation of a 2 x 2 matrix is quadratic and can be solved rather easily by applying the quadratic formula.
- There are strategies to solve the characteristic equations of 3 x 3 or of 4 x 4 matrices, which are cubic or quartic. But these cubic and quartic formulas are long-winded and cumbersome to apply.
- It is not possible to solve the characteristic equation of a 5 x 5 matrix (or any higher-dimensional matrix) in a general way, as no solution strategy exists to solve quintic equations – an astonishing fact, which was proven by 20 year old Évariste Galois shortly before his untimely death in 1811.

Therefore only eigenvalues of special higher-dimensional matrices (like triangular matrices, see following slides) can be found in a straightforward way.

Eigenvalues of Triangular Matrices

If matrix A is a triangular matrix, the characteristic matrix $(A - \lambda I)$ will also be a triangular matrix.

Then the diagonal elements of matrix A are the eigenvalues of this triangular matrix.

Triangular Matrix Problem

Find eigenvalues and associated eigenvectors of the following matrix A:

$$A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Solution of the Triangular Matrix Problem

Matrix A:

$$A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Characteristic matrix $(A - \lambda\, I)$:

$$A - \lambda\, I = \begin{bmatrix} 1-\lambda & 2 & 4 & 7 \\ 0 & 3-\lambda & 5 & 8 \\ 0 & 0 & 6-\lambda & 9 \\ 0 & 0 & 0 & 10-\lambda \end{bmatrix}$$

Characteristic coefficient vectors:

$$a - \lambda\, \sigma_1 = (1 - \lambda)\, \sigma_1$$

$$b - \lambda\, \sigma_2 = 2\, \sigma_1 + (3 - \lambda)\, \sigma_2$$

$$c - \lambda\, \sigma_3 = 4\, \sigma_1 + 5\, \sigma_2 + (6 - \lambda)\, \sigma_3$$

$$d - \lambda\, \sigma_4 = 7\, \sigma_1 + 8\, \sigma_2 + 9\, \sigma_3 + (10 - \lambda)\, \sigma_4$$

Solution of the Triangular Matrix Problem

Characteristic outer product:

- Intermediate steps

$$(a - \lambda\,\sigma_1) \wedge (b - \lambda\,\sigma_2)$$
$$= (1 - \lambda)\,\sigma_1 \wedge (2\,\sigma_1 + (3 - \lambda)\,\sigma_2)$$
$$= (1 - \lambda)\,(3 - \lambda)\,\sigma_1\sigma_2$$

$$(a - \lambda\,\sigma_1) \wedge (b - \lambda\,\sigma_2) \wedge (c - \lambda\,\sigma_3)$$
$$= (1 - \lambda)\,(3 - \lambda)\,\sigma_1\sigma_2 \wedge (4\,\sigma_1 + 5\,\sigma_2 + (6 - \lambda)\,\sigma_3)$$
$$= (1 - \lambda)\,(3 - \lambda)\,(6 - \lambda)\,\sigma_1\sigma_2\sigma_3$$

- Final step

$$(a - \lambda\,\sigma_1) \wedge (b - \lambda\,\sigma_2) \wedge (c - \lambda\,\sigma_3) \wedge (d - \lambda\,\sigma_4)$$
$$= (1 - \lambda)\,(3 - \lambda)\,(6 - \lambda)\,\sigma_1\sigma_2\sigma_3 \wedge (7\,\sigma_1 + 8\,\sigma_2 + 9\,\sigma_3 + (10 - \lambda)\,\sigma_4)$$
$$= (1 - \lambda)\,(3 - \lambda)\,(6 - \lambda)\,(10 - \lambda)\,\sigma_1\sigma_2\sigma_3\sigma_4$$

Characteristic polynomial:

$$(a - \lambda\,\sigma_1) \wedge (b - \lambda\,\sigma_2) \wedge (c - \lambda\,\sigma_3) \wedge (d - \lambda\,\sigma_4)\,\sigma_4\sigma_3\sigma_2\sigma_1$$
$$= (1 - \lambda)\,(3 - \lambda)\,(6 - \lambda)\,(10 - \lambda)$$
$$= \lambda^4 - 20\,\lambda^3 + 127\,\lambda^2 - 288\,\lambda + 180$$

Solution of the Triangular Matrix Problem

Characteristic equation:

$$\lambda^4 - 20\,\lambda^3 + 127\,\lambda^2 - 288\,\lambda + 180 = 0$$

or $$(1 - \lambda)\,(3 - \lambda)\,(6 - \lambda)\,(10 - \lambda) = 0$$

Eigenvalues: $\lambda_1 = 1$

$\lambda_2 = 3$

$\lambda_3 = 6$

$\lambda_4 = 10$

Associated eigenvectors:

$r_1 = \sigma_1$

$r_2 = \sigma_1 + \sigma_2$

$r_3 = 22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3$

$r_4 = 86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4$

Solution of the Triangular Matrix Problem

Check of results:

$A\ r_1 = 1\ r_1$

				1
				0
				0
				0
1	2	4	7	1
0	3	5	8	0
0	0	6	9	0
0	0	0	10	0

$A\ r_2 = 3\ r_2$

				1
				1
				0
				0
1	2	4	7	3
0	3	5	8	3
0	0	6	9	0
0	0	0	10	0

$A\ r_3 = 6\ r_3$

				22
				25
				15
				0
1	2	4	7	132
0	3	5	8	150
0	0	6	9	90
0	0	0	10	0

$A\ r_4 = 10\ r_4$

				86
				99
				81
				36
1	2	4	7	860
0	3	5	8	990
0	0	6	9	810
0	0	0	10	360

Solution of the Triangular Matrix Problem

Normalized eigenvectors:

- Eigenvector corresponding to first eigenvalue $\lambda_1 = 1$

$$v_1 = \sigma_1$$

- Eigenvector corresponding to second eigenvalue $\lambda_2 = 3$

$$v_2 = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_2)$$

- Eigenvector corresponding to third eigenvalue $\lambda_3 = 6$

$$v_3 = \frac{1}{\sqrt{1334}} (22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3)$$

- Eigenvector corresponding to fourth eigenvalue $\lambda_4 = 10$

$$v_4 = \frac{1}{\sqrt{25054}} (86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4)$$

The Mathematical Significance of Eigenvalues and Eigenvectors

Why have mathematicians invented (or tried hard to discover) the mathematics of eigenvalues and eigenvectors?

The starting point was matrix algebra: We want to analyze and understand, how a vector x changes or transforms and becomes a new vector y:

$$x \longrightarrow y$$

This transformation can (often) be modeled by a matrix pre-multiplication:

$$A\,x = y$$

Therefore we (and other mathematicians) are interested in the action of matrices on vectors.

But if we know all of the eigenvalue and eigenvector information about a matrix, we are able to determine its full behavior on any vector without knowing the matrix.

The Mathematical Significance of Eigenvalues and Eigenvectors

That's the main point:

> If all of the eigenvalue and eigenvector information about a matrix is known, it is possible to determine its full behavior on any vector.

Whatever can be done mathematically by a matrix can be done without this matrix by using its eigenvalues and eigenvectors.

If a mathematician does not like matrices, he or she simply shifts to eigenvectors and eigenvalues. He or she can do all matrix calculations without matrices by using the eigenvalue and eigenvector information only.

In addition, the mathematical beauty and strength of eigenvector and eigenvalue calculations is convincing: Many calculations are less complicated if eigenvectors and eigenvalues are applied.

Matrix Multiplications Without Matrices

As every vector x can be split into eigenvector components

normalized eigenvectors

$$x = c_1\, v_1 + c_2\, v_2 + c_3\, v_3 + \ldots + c_n\, v_n$$

scalar coefficients

the action $A\,x = y$ of matrix A on vector x can be rewritten. The resulting vector y will be

$$\begin{aligned} y &= A\,x \\ &= A\,(c_1\, v_1 + c_2\, v_2 + c_3\, v_3 + \ldots + c_n\, v_n) \\ &= c_1\,\lambda_1\, v_1 + c_2\,\lambda_2\, v_2 + c_3\,\lambda_3\, v_3 + \ldots + c_n\,\lambda_n\, v_n \end{aligned}$$

Thus the new vector y can be stated as a linear combination of the eigenvectors, provided eigenvalues λ_i, eigenvectors v_i and the coefficients c_i of the original vector x are known.

Finding the Eigenvector Coefficients of a Vector

If eigenvalues and eigenvectors are known, the only remaining problem should be to find the scalar coefficients c_i of vector x.

This can be done by solving the system of n linear equations

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n = x$$

for these scalar coefficients.

As discussed in part II and III of this Geometric Algebra crash course (see repetition slides # 7 and # 8), this can be done by comparing the relevant outer products.

Finding the Eigenvector Coefficients of a Two-dimensional Vector

The scalar coefficients of a two-dimensional vector

$$c_1\, v_1 + c_2\, v_2 = x$$

are generated by the following wedge product multiplications

$$c_1\, v_1 \wedge v_2 + c_2\, \underbrace{v_2 \wedge v_2}_{0} = x \wedge v_2$$

$$\Rightarrow \qquad c_1 = (v_1 \wedge v_2)^{-1}\, (x \wedge v_2)$$

and

$$c_1\, \underbrace{v_1 \wedge v_1}_{0} + c_2\, v_1 \wedge v_2 = v_1 \wedge x$$

$$\Rightarrow \qquad c_2 = (v_1 \wedge v_2)^{-1}\, (v_1 \wedge x)$$

Finding the Eigenvector Coefficients of a Three-dimensional Vector

The scalar coefficients of a three-dimensional vector

$$c_1\, v_1 + c_2\, v_2 + c_3\, v_3 = x$$

are generated by the following wedge product multiplications

$$c_1\, v_1 \wedge v_2 \wedge v_3 + c_2\, \underbrace{v_2 \wedge v_2}_{0} \wedge v_3 + c_3\, \overbrace{v_3 \wedge v_2 \wedge v_3}^{0}$$
$$= x \wedge v_2 \wedge v_3$$

$$\Rightarrow \qquad c_1 = (v_1 \wedge v_2 \wedge v_3)^{-1}\, (x \wedge v_2 \wedge v_3)$$

and

$$c_1\, \underbrace{v_1 \wedge v_1}_{0} \wedge v_3 + c_2\, v_1 \wedge v_2 \wedge v_3 + c_3\, v_1 \wedge \overbrace{v_3 \wedge v_3}^{0}$$
$$= v_1 \wedge x \wedge v_3$$

$$\Rightarrow \qquad c_2 = (v_1 \wedge v_2 \wedge v_3)^{-1}\, (v_1 \wedge x \wedge v_3)$$

and

$$c_1\, \underbrace{v_1 \wedge v_2 \wedge v_1}_{0} + c_2\, v_1 \wedge \overbrace{v_2 \wedge v_2}^{0} + c_3\, v_1 \wedge v_2 \wedge v_3$$
$$= v_1 \wedge v_2 \wedge x$$

$$\Rightarrow \qquad c_3 = (v_1 \wedge v_2 \wedge v_3)^{-1}\, (v_1 \wedge v_2 \wedge x)$$

Finding the Eigenvector Coefficients of a Four-dimensional Vector

The scalar coefficients of a four-dimensional vector

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = x$$

are generated by the following wedge product multiplications

$$\begin{aligned} &c_1 v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &\qquad + c_2 v_2 \wedge v_2 \wedge v_3 \wedge v_4 \\ &\qquad + c_3 v_3 \wedge v_2 \wedge v_3 \wedge v_4 \\ &\qquad + c_4 v_4 \wedge v_2 \wedge v_3 \wedge v_4 = x \wedge v_2 \wedge v_3 \wedge v_4 \end{aligned}$$

$$\Rightarrow \quad c_1 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1} (x \wedge v_2 \wedge v_3 \wedge v_4)$$

and in a similar way

$$c_2 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1} (v_1 \wedge x \wedge v_3 \wedge v_4)$$

$$c_3 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1} (v_1 \wedge v_2 \wedge x \wedge v_4)$$

$$c_4 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1} (v_1 \wedge v_2 \wedge v_3 \wedge x)$$

And eigenvector coefficients of higher-dimensional vectors are constructed analogously.

Product Engineering Problem II

A firm manufactures two different types of final products P_1 and P_2. To produce these products two different raw materials R_1 and R_2 are required.

The eigenvalues und normalized eigenvectors of the demand matrix, which shows the demand of raw materials to produce one unit of the final products, are:

$$\lambda_1 = 10 \qquad v_1 = \frac{7}{\sqrt{53}}\,\sigma_x - \frac{2}{\sqrt{53}}\,\sigma_y$$

$$\lambda_2 = 100 \qquad v_2 = \frac{1}{\sqrt{2}}\,\sigma_x + \frac{1}{\sqrt{2}}\,\sigma_y$$

The day after Halloween 50 units of the first final product P_1 and 90 units of the second final product were manufactured.

Find the quantities of raw materials R_1 and R_2 which had been required in the production process. Please use only the eigenvalue and eigenvector information to find the result.

Check your result by comparing it with the result of a matrix multiplication (see demand matrix of the Halloween product engineering problem).

Solution of Product Engineering Problem II

Production vector, which shows the quantities of final products:

$P = 50\,\sigma_x + 90\,\sigma_y$

Outer products:

$$v_1 \wedge v_2 = \frac{1}{\sqrt{53}\cdot\sqrt{2}}\,(7\,\sigma_x - 2\,\sigma_y) \wedge (\sigma_x + \sigma_y) = \frac{9}{\sqrt{106}}\,\sigma_x\sigma_y$$

$$P \wedge v_2 = \frac{1}{\sqrt{2}}\,(50\,\sigma_x + 90\,\sigma_y) \wedge (\sigma_x + \sigma_y) = -\,20\cdot\sqrt{2}\;\sigma_x\sigma_y$$

$$v_1 \wedge P = \frac{1}{\sqrt{53}}\,(7\,\sigma_x - 2\,\sigma_y) \wedge (50\,\sigma_x + 90\,\sigma_y) = \frac{730}{\sqrt{53}}\,\sigma_x\sigma_y$$

Eigenvector coefficients of production vector:

$$c_1 = (v_1 \wedge v_2)^{-1}\,(P \wedge v_2) = -\,\frac{\sqrt{106}}{9}\cdot 20\cdot\sqrt{2} = -\,\frac{40}{9}\cdot\sqrt{53}$$

$$c_2 = (v_1 \wedge v_2)^{-1}\,(v_1 \wedge P) = \frac{\sqrt{106}}{9}\cdot\frac{730}{\sqrt{53}} = \frac{730}{9}\cdot\sqrt{2}$$

Solution of Product Engineering Problem II

Check of intermediate result:

Reformulation of production vector

$$P = c_1\, v_1 + c_2\, v_2$$

$$= -\frac{40}{9}\cdot\sqrt{53}\cdot\frac{1}{\sqrt{53}}\,(7\,\sigma_x - 2\,\sigma_y) + \frac{730}{9}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{2}}\,(\sigma_x + \sigma_y)$$

$$= 50\,\sigma_x + 90\,\sigma_y$$

Final solution:

Demand vector, which shows the quantities of raw materials required

$$q = c_1\,\lambda_1\, v_1 + c_2\,\lambda_2\, v_2$$

$$= -\frac{40}{9}\cdot\sqrt{53}\cdot 10\cdot\frac{1}{\sqrt{53}}\,(7\,\sigma_x - 2\,\sigma_y)$$

$$+\frac{730}{9}\cdot\sqrt{2}\cdot 100\cdot\frac{1}{\sqrt{2}}\,(\sigma_x + \sigma_y)$$

$$= 7800\,\sigma_x + 8200\,\sigma_y$$

$\Rightarrow$ 7800 units of the first raw material R_1 and 8200 units of the second raw material R_2 had been required.

Solution of Product Engineering Problem II

Check of final result:

Comparison with the result of matrix multi-plication D P = q

D P = q		
		50
		90
30	70	7800
20	80	8200

$\Rightarrow$ The result is correct.

Petrol Station Problem II

There are three petrol stations A, B, and C in a small city in the middle of the Australian Desert.

The eigenvalues und normalized eigenvectors of the transition matrix, which shows the changes of market shares every month, are:

$$\lambda_1 = 1 \qquad v_1 = \frac{3}{\sqrt{38}}\,\sigma_x + \frac{5}{\sqrt{38}}\,\sigma_y + \frac{2}{\sqrt{38}}\,\sigma_z$$

$$\lambda_2 = 0.60 \qquad v_2 = \frac{1}{\sqrt{2}}\,\sigma_x - \frac{1}{\sqrt{2}}\,\sigma_y$$

$$\lambda_3 = 0.50 \qquad v_3 = \frac{1}{\sqrt{2}}\,\sigma_x - \frac{1}{\sqrt{2}}\,\sigma_z$$

In October 2016 the market shares are

Petrol station A: 15 %
Petrol station B: 25 %
Petrol station C: 60 %

Find the market shares in November 2016. Please use only the eigenvalue and eigenvector information to find the result.

Check your result by comparing it with the result of a matrix multiplication (see transition matrix of first petrol station problem).

Solution of Petrol Station Problem II

Vector of market shares in October 2016:

$$x = 0.15\,\sigma_x + 0.25\,\sigma_y + 0.60\,\sigma_z$$

Normalized eigenvectors:

$$v_1 = \frac{1}{\sqrt{38}}\,(3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z)$$

$$v_2 = \frac{1}{\sqrt{2}}\,(\sigma_x - \sigma_y) \qquad\qquad v_3 = \frac{1}{\sqrt{2}}\,(\sigma_x - \sigma_z)$$

Outer products:

$$v_1 \wedge v_2 \wedge v_3 = \frac{1}{2\cdot\sqrt{38}}\,(3\,\sigma_x\sigma_y\sigma_z + 5\,\sigma_y\sigma_z\sigma_x + 2\,\sigma_z\sigma_x\sigma_y)$$

$$= \frac{5}{\sqrt{38}}\,\sigma_x\sigma_y\sigma_z$$

$$x \wedge v_2 \wedge v_3 = \frac{1}{200}\,(15\,\sigma_x\sigma_y\sigma_z + 25\,\sigma_y\sigma_z\sigma_x + 60\,\sigma_z\sigma_x\sigma_y)$$

$$= \frac{1}{2}\,\sigma_x\sigma_y\sigma_z$$

$$v_1 \wedge x \wedge v_3 = \frac{5}{4\cdot\sqrt{19}}\,\sigma_x\sigma_y\sigma_z$$

$$v_1 \wedge v_2 \wedge x = -\frac{2}{\sqrt{19}}\,\sigma_z\sigma_x\sigma_y$$

Solution of Petrol Station Problem II

Eigenvector coefficients of vector of market shares:

$$c_1 = (v_1 \wedge v_2 \wedge v_3)^{-1} (x \wedge v_2 \wedge v_3) = \frac{\sqrt{38}}{5} \cdot \frac{1}{2} = \frac{\sqrt{38}}{10}$$

$$c_2 = (v_1 \wedge v_2 \wedge v_3)^{-1} (v_1 \wedge x \wedge v_3) = \frac{\sqrt{38}}{5} \cdot \frac{5}{4 \cdot \sqrt{19}} = \frac{\sqrt{2}}{4}$$

$$c_3 = (v_1 \wedge v_2 \wedge v_3)^{-1} (v_1 \wedge v_2 \wedge x) = -\frac{\sqrt{38}}{5} \cdot \frac{2}{\sqrt{19}} = -\frac{2}{5}\sqrt{2}$$

Check of intermediate result:

Reformulation of vector of market shares in October 2016

$$x = c_1\, v_1 + c_2\, v_2 + c_3\, v_3$$

$$= \frac{\sqrt{38}}{10} \cdot \frac{1}{\sqrt{38}} (3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z)$$

$$+ \frac{\sqrt{2}}{4} \cdot \frac{1}{\sqrt{2}} (\sigma_x - \sigma_y) - \frac{2}{5}\sqrt{2} \cdot \frac{1}{\sqrt{2}} (\sigma_x - \sigma_z)$$

$$= 0.15\,\sigma_x + 0.25\,\sigma_y + 0.60\,\sigma_z$$

Solution of Petrol Station Problem II

Final solution:

Vector of market shares in November 2016

$$
\begin{aligned}
x_{Nov} &= c_1\,\lambda_1\,v_1 + c_2\,\lambda_2\,v_2 + c_3\,\lambda_3\,v_3 \\
&= \frac{\sqrt{38}}{10}\cdot 1 \cdot \frac{1}{\sqrt{38}}\,(3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z) \\
&\qquad + \frac{\sqrt{2}}{4}\cdot 0.60 \cdot \frac{1}{\sqrt{2}}\,(\sigma_x - \sigma_y) \\
&\qquad - \frac{2}{5}\sqrt{2}\cdot 0.50 \cdot \frac{1}{\sqrt{2}}\,(\sigma_x - \sigma_z) \\
&= 0.25\,\sigma_x + 0.35\,\sigma_y + 0.40\,\sigma_z
\end{aligned}
$$

$\Rightarrow$ In November 2016 the market shares will be for

Petrol station A: 25 %
Petrol station B: 35 %
Petrol station C: 40 %

Solution of Petrol Station Problem II

Check of final result:

Comparison with the result of matrix multiplication $T\,x = T\,x_{Oct} = x_{Nov}$

$T\,x_{Oct} = x_{Nov}$			0.15
			0.25
			0.60
0.70	0.10	0.20	0.25
0.20	0.80	0.20	0.35
0.10	0.10	0.60	0.40

$\Rightarrow$ The result is correct.

Triangular Matrix Problem II

Matrix A has following eigenvalues und associated eigenvectors:

$$\lambda_1 = 1 \qquad v_1 = \sigma_1$$

$$\lambda_2 = 3 \qquad v_2 = \sigma_1 + \sigma_2$$

$$\lambda_3 = 6 \qquad v_3 = 22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3$$

$$\lambda_4 = 10 \qquad v_4 = 86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4$$

And the following Pauli vector x_2 is given:

$$x_2 = 500\,\sigma_1 + 540\,\sigma_2 + 420\,\sigma_3 + 200\,\sigma_4$$

Please use the given eigenvalue and eigenvector information to find the solution of matrix multiplication

$$A\,x_2 = x_3$$

Check your result by comparing it with the result of a matrix multiplication with matrix A

$$A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

of the first triangular matrix problem.

Solution of Triangular Matrix Problem II

Outer products:

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 = 540\ \sigma_1\sigma_2\sigma_3\sigma_4$$

$$x_2 \wedge v_2 \wedge v_3 \wedge v_4 = 14160\ \sigma_1\sigma_2\sigma_3\sigma_4$$

$$v_1 \wedge x_2 \wedge v_3 \wedge v_4 = 21600\ \sigma_1\sigma_2\sigma_3\sigma_4$$

$$v_1 \wedge v_2 \wedge x_2 \wedge v_4 = -1080\ \sigma_1\sigma_2\sigma_3\sigma_4$$

$$v_1 \wedge v_2 \wedge v_3 \wedge x_2 = 3000\ \sigma_1\sigma_2\sigma_3\sigma_4$$

Eigenvector coefficients of original vector x_2:

$$c_1 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1}\,(x_2 \wedge v_2 \wedge v_3 \wedge v_4)$$

$$= \frac{1}{540} \cdot 14160 = \frac{236}{9}$$

$$c_2 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1}\,(v_1 \wedge x_2 \wedge v_3 \wedge v_4)$$

$$= \frac{1}{540} \cdot 21600 = 40$$

$$c_3 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1}\,(v_1 \wedge v_2 \wedge x_2 \wedge v_4)$$

$$= \frac{1}{540} \cdot (-1080) = -2$$

$$c_4 = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{-1}\,(v_1 \wedge v_2 \wedge v_3 \wedge x_2)$$

$$= \frac{1}{540} \cdot 3000 = \frac{50}{9}$$

Solution of Triangular Matrix Problem II

Check of intermediate result:

Reformulation of original vector x_2

$$x_2 = c_1\, v_1 + c_2\, v_2 + c_3\, v_3$$
$$= \frac{236}{9}\,\sigma_1 + 40\,(\sigma_1 + \sigma_2) - 2\,(22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3)$$
$$+ \frac{50}{9}\,(86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4)$$
$$= 500\,\sigma_1 + 540\,\sigma_2 + 420\,\sigma_3 + 200\,\sigma_4$$

Final solution: New vector x_3

$$x_3 = c_1\,\lambda_1\, v_1 + c_2\,\lambda_2\, v_2 + c_3\,\lambda_3\, v_3 + c_4\,\lambda_4\, v_4$$
$$= \frac{236}{9}\cdot 1\,\sigma_1 + 40\cdot 3\,(\sigma_1 + \sigma_2)$$
$$- 2\cdot 6\,(22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3)$$
$$+ \frac{50}{9}\cdot 10\,(86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4)$$
$$= 4660\,\sigma_1 + 5320\,\sigma_2 + 4320\,\sigma_3 + 2000\,\sigma_4$$

$\Rightarrow$ Result in conventional column vector notation:

$$x_3 = \begin{bmatrix} 4660 \\ 5320 \\ 4320 \\ 2000 \end{bmatrix}$$

Solution of Triangular Matrix Problem II

Original vector in conventional column vector notation:

$$x_2 = \begin{bmatrix} 500 \\ 540 \\ 420 \\ 200 \end{bmatrix}$$

Check of result:

$A\,x_2 = x_3$

				500
				540
				420
				200
1	2	4	7	4660
0	3	5	8	5320
0	0	6	9	4320
0	0	0	10	2000

$\Rightarrow$ The result is correct.

Solving Linear Equations

An important objective of this short Geometric Algebra crash course is to discuss different strategies of solving systems of linear equations.

If all eigenvalues and eigenvectors of a system of linear equations are known, the solution of it can be found in a simple way:

given matrix

$$A\,x = y$$

unknown vector x

vector y given as linear combination of eigenvectors:

$$y = c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n$$

The unknown vector x can now be written in a straight-forward way as:

$$x = \frac{c_1}{\lambda_1} v_1 + \frac{c_2}{\lambda_2} v_2 + \frac{c_3}{\lambda_3} v_3 + \ldots + \frac{c_n}{\lambda_n} v_n$$

Product Engineering Problem III

A firm manufactures two different types of final products P_1 and P_2. To produce these products two different raw materials R_1 and R_2 are required.

The eigenvalues und associated eigenvectors of the demand matrix, which shows the demand of raw materials to produce one unit of the final products, are:

$$\lambda_1 = 10 \qquad v_1 = 7\,\sigma_x - 2\,\sigma_y$$

$$\lambda_2 = 100 \qquad v_2 = \sigma_x + \sigma_y$$

The day after Halloween the demand of raw materials is given by the following linear combination of eigenvectors:

$$q = 150\,v_1 + 6000\,v_2$$

Find the quantities of raw materials R_1 and R_2 which had been consumed at the day after Halloween. And find the quantities of final products P_1 and P_2 which had been produced at this day.

Please use only the eigenvalue and eigenvector information to find the result (and and compare it with the result of a standard matrix multiplication).

Solution of Product Engineering Problem III

Finding the quantities of raw materials R_1 and R_2 which had been required:

$$q = 150\, v_1 + 6000\, v_2$$

$$= 150\,(7\,\sigma_x - 2\,\sigma_y) + 6000\,(\sigma_x + \sigma_y)$$

$$= 7050\,\sigma_x + 5700\,\sigma_y$$

$\Rightarrow$ 7050 units of the first raw material R_1 and 5700 units of the second raw material R_2 had been consumed in the production process.

Solution of Product Engineering Problem III

$c_1 = 150 \qquad \lambda_1 = 10 \qquad \Rightarrow \qquad \frac{c_1}{\lambda_1} = 15$

$c_2 = 6000 \qquad \lambda_2 = 100 \qquad \Rightarrow \qquad \frac{c_2}{\lambda_2} = 60$

Production vector at the day after Halloween:

$$P = \frac{c_1}{\lambda_1} v_1 + \frac{c_2}{\lambda_2} v_2$$

$$= 15\,(7\,\sigma_x - 2\,\sigma_y) + 60\,(\sigma_x + \sigma_y)$$

$$= 165\,\sigma_x + 30\,\sigma_y$$

$\Rightarrow$ 165 units of the first final product P_1 and 30 units of the second final product P_2 had been produced.

Check of final result:

As the given eigenvectors are eigenvectors of the demand matrix D of the first Halloween product engineering problem, this demand matrix has to be used for the check.

D P = q		
		165
		30
30	70	7050
20	80	5700

$\Rightarrow$ The result is correct.

Petrol Station Problem III

There are three petrol stations A, B, and C in a small city in the middle of the Australian Desert.

The eigenvalues und associated eigenvectors of the transition matrix, which shows the changes of market shares every month, are:

$\lambda_1 = 1 \qquad v_1 = 3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z$

$\lambda_2 = 0.60 \qquad v_2 = \sigma_x - \sigma_y$

$\lambda_3 = 0.50 \qquad v_3 = \sigma_x - \sigma_z$

In the fourth month after an advertising campaign of petrol station A the market shares are given by the following linear combination of eigenvectors:

$$x_4 = 0.1000\, v_1 + 0.0648\, v_2 + 0.0125\, v_3$$

Find the market shares … in the fourth month
… in the third month
… in the second month
… in the month directly
after the advertising campaign of petrol station A.

Please use only the eigenvalue and eigenvector information to find the results (and compare them with the results of standard matrix multiplications).

Solution of Petrol Station Problem III

Given market share vector four months after the advertising campaign of petrol station A:

$$x_4 = 0.1000\, v_1 + 0.0648\, v_2 + 0.0125\, v_3$$

$$= 0.1000\,(3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z) + 0.0648\,(\sigma_x - \sigma_y) + 0.0125\,(\sigma_x - \sigma_z)$$

$$= (0.3000 + 0.0648 + 0.0125)\,\sigma_x + (0.5000 - 0.0648)\,\sigma_y + (0.2000 - 0.0125)\,\sigma_z$$

$$= 0.3773\,\sigma_x + 0.4352\,\sigma_y + 0.1875\,\sigma_z$$

$\Rightarrow$ The market shares in the fourth month after the advertising campaign of petrol station A are

Petrol station A: 37.73 %
Petrol station B: 43.52 %
Petrol station C: 18.75 %

Solution of Petrol Station Problem III

$c_1 = 0.1000 \qquad \lambda_1 = 1 \qquad \Rightarrow \quad \dfrac{c_1}{\lambda_1} = 0.1000$

$c_2 = 0.0648 \qquad \lambda_2 = 0.60 \qquad \Rightarrow \quad \dfrac{c_2}{\lambda_2} = 0.1080$

$c_3 = 0.0125 \qquad \lambda_3 = 0.50 \qquad \Rightarrow \quad \dfrac{c_3}{\lambda_3} = 0.0250$

Market share vector three months after the advertising campaign of petrol station A:

$$
\begin{aligned}
x_3 &= \frac{c_1}{\lambda_1} v_1 + \frac{c_2}{\lambda_2} v_2 + \frac{c_3}{\lambda_3} v_3 \\
&= 0.1000\,(3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z) \\
&\qquad + 0.1080\,(\sigma_x - \sigma_y) + 0.0250\,(\sigma_x - \sigma_z) \\
&= (0.3000 + 0.1080 + 0.0250)\,\sigma_x \\
&\qquad + (0.5000 - 0.1080)\,\sigma_y \\
&\qquad + (0.2000 - 0.0250)\,\sigma_z \\
&= 0.4330\,\sigma_x + 0.3920\,\sigma_y + 0.1750\,\sigma_z
\end{aligned}
$$

$\Rightarrow$ The market shares in the third month after the advertising campaign of petrol station A are

Petrol station A: 43.30 %
Petrol station B: 39.20 %
Petrol station C: 17.50 %

Solution of Petrol Station Problem III

$c_1 = 0.1000 \qquad \lambda_1 = 1 \qquad \Rightarrow \qquad \dfrac{c_1}{\lambda_1^{\,2}} = 0.1000$

$c_2 = 0.0648 \qquad \lambda_2 = 0.60 \qquad \Rightarrow \qquad \dfrac{c_2}{\lambda_2^{\,2}} = 0.1800$

$c_3 = 0.0125 \qquad \lambda_3 = 0.50 \qquad \Rightarrow \qquad \dfrac{c_3}{\lambda_3^{\,2}} = 0.0500$

Market share vector two months after the advertising campaign of petrol station A:

$$x_2 = \frac{c_1}{\lambda_1^{\,2}}\, v_1 + \frac{c_2}{\lambda_2^{\,2}}\, v_2 + \frac{c_3}{\lambda_3^{\,2}}\, v_3$$

$$= 0.1000\,(3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z) + 0.1800\,(\sigma_x - \sigma_y) + 0.0500\,(\sigma_x - \sigma_z)$$

$$= (0.3000 + 0.1800 + 0.0500)\,\sigma_x + (0.5000 - 0.1800)\,\sigma_y + (0.2000 - 0.0500)\,\sigma_z$$

$$= 0.5300\,\sigma_x + 0.3200\,\sigma_y + 0.1500\,\sigma_z$$

$\Rightarrow$ The market shares in the second month after the advertising campaign of petrol station A are

Petrol station A: 53.00 %
Petrol station B: 32.00 %
Petrol station C: 15.00 %

Solution of Petrol Station Problem III

$c_1 = 0.1000 \qquad \lambda_1 = 1 \qquad \Rightarrow \qquad \dfrac{c_1}{\lambda_1^{\,3}} = 0.1000$

$c_2 = 0.0648 \qquad \lambda_2 = 0.60 \qquad \Rightarrow \qquad \dfrac{c_2}{\lambda_2^{\,3}} = 0.3000$

$c_3 = 0.0125 \qquad \lambda_3 = 0.50 \qquad \Rightarrow \qquad \dfrac{c_3}{\lambda_3^{\,3}} = 0.1000$

Market share vector one month after the advertising campaign of petrol station A:

$$\begin{aligned}
x_1 &= \frac{c_1}{\lambda_1^{\,3}}\, v_1 + \frac{c_2}{\lambda_2^{\,3}}\, v_2 + \frac{c_3}{\lambda_3^{\,3}}\, v_3 \\
&= 0.1000\,(3\,\sigma_x + 5\,\sigma_y + 2\,\sigma_z) \\
&\qquad + 0.3000\,(\sigma_x - \sigma_y) + 0.1000\,(\sigma_x - \sigma_z) \\
&= (0.3000 + 0.3000 + 0.1000)\,\sigma_x \\
&\qquad + (0.5000 - 0.3000)\,\sigma_y \\
&\qquad + (0.2000 - 0.1000)\,\sigma_z \\
&= 0.7000\,\sigma_x + 0.2000\,\sigma_y + 0.1000\,\sigma_z
\end{aligned}$$

$\Rightarrow$ The market shares in the month directly after the advertising campaign of petrol station A are

Petrol station A: 70.00 %
Petrol station B: 20.00 %
Petrol station C: 10.00 %

Solution of Petrol Station Problem III

Check of results:

Comparison with the result of matrix multiplications

$T x_i = x_{i+1}$			
			0.7000
			0.2000
			0.1000
0.70	0.10	0.20	0.5300
0.20	0.80	0.20	0.3200
0.10	0.10	0.60	0.1500
0.70	0.10	0.20	0.4330
0.20	0.80	0.20	0.3920
0.10	0.10	0.60	0.1750
0.70	0.10	0.20	0.3773
0.20	0.80	0.20	0.4352
0.10	0.10	0.60	0.1875

As the given eigenvectors are eigenvectors of the transition matrix T of the first petrol station problem, this transition matrix has to be used for the check.

$\Rightarrow$ The results are correct.

Triangular Matrix Problem III

Matrix A has following eigenvalues und associated eigenvectors:

$$\begin{array}{lll} \lambda_1 = 1 & \qquad & v_1 = \sigma_1 \\ \lambda_2 = 3 & & v_2 = \sigma_1 + \sigma_2 \\ \lambda_3 = 6 & & v_3 = 22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3 \\ \lambda_4 = 10 & & v_4 = 86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4 \end{array}$$

The following matrix multiplication

$$A\,x_1 = x_2$$

results in vector x_2, which was already given in triangular matrix problem II:

$$\begin{aligned} x_2 &= 500\,\sigma_1 + 540\,\sigma_2 + 420\,\sigma_3 + 200\,\sigma_4 \\ &= \frac{236}{9}\,v_1 + 40\,v_2 - 2\,v_3 + \frac{50}{9}\,v_4 \end{aligned}$$

Use the given eigenvalue and eigenvector information to find vector x_1.

Check your result by comparing it with the result of a matrix multiplication with matrix A.

Solution of Triangular Matrix Problem III

$$c_1 = \frac{236}{9} \qquad \lambda_1 = 1 \quad \Rightarrow \quad \frac{c_1}{\lambda_1} = \frac{236}{9}$$

$$c_2 = 40 \qquad \lambda_2 = 3 \quad \Rightarrow \quad \frac{c_2}{\lambda_2} = \frac{40}{3}$$

$$c_3 = -2 \qquad \lambda_3 = 6 \quad \Rightarrow \quad \frac{c_3}{\lambda_3} = -\frac{1}{3}$$

$$c_4 = \frac{50}{9} \qquad \lambda_4 = 10 \quad \Rightarrow \quad \frac{c_4}{\lambda_4} = \frac{5}{9}$$

Pauli vector x_1:
(see triangular matrix problem II)

$$\begin{aligned} x_1 &= \frac{c_1}{\lambda_1} v_1 + \frac{c_2}{\lambda_2} v_2 - \frac{c_3}{\lambda_3} v_3 + \frac{c_4}{\lambda_4} v_4 \\ &= \frac{236}{9} \sigma_1 + \frac{40}{3} (\sigma_1 + \sigma_2) \\ &\qquad - \frac{1}{3} (22\,\sigma_1 + 25\,\sigma_2 + 15\,\sigma_3) \\ &\qquad + \frac{5}{9} (86\,\sigma_1 + 99\,\sigma_2 + 81\,\sigma_3 + 36\,\sigma_4) \\ &= 80\,\sigma_1 + 60\,\sigma_2 + 40\,\sigma_3 + 20\,\sigma_4 \end{aligned}$$

Solution of Triangular Matrix Problem III

Result in conventional column vector notation:

$$x_1 = \begin{bmatrix} 80 \\ 60 \\ 40 \\ 20 \end{bmatrix}$$

Check of result:

$A\,x_1 = x_2$				
				80
				60
				40
				20
1	2	4	7	500
0	3	5	8	540
0	0	6	9	420
0	0	0	10	200

$\Rightarrow$ The result is correct.