Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Wintersemester 2015/2016

Dr. Horn

Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part IV: Transformation of co-ordinates & Gaussian method of solving a system of linear equations)

Stand: 28. Jan. 2016

Teaching & learning contents according to the modular description of LV 200 691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Most of this will be discussed in the standard language of the rather old-fashioned linear algebra or matrix algebra which can be found in most textbooks of business mathematics or mathematical economics.

But in this part we will again adopt a more modern view: The Gaussian method of solving systems of linear will be interpreted geometrically as a transformation of coordinates.

Repetition: Basics of Geometric Algebra

$1 + 3 + 3 + 1 = 2^3 = 8$ different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: $\sigma_x, \sigma_y, \sigma_z$

Three base bivectors: $\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

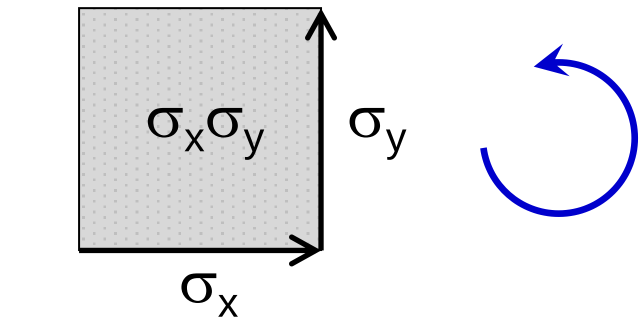
$$1^2 = \sigma_x^{\;2} = \sigma_y^{\;2} = \sigma_z^{\;2} = 1$$

Base bivectors and base trivector square to minus one:

$$(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1$$

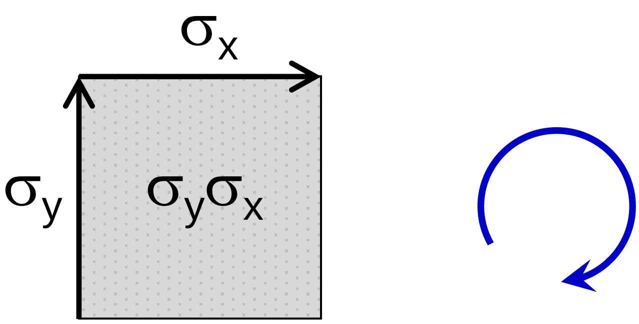
Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the resulting area elements.



right-handed orientation
(anticlockwise orientation)

positive orientation in a right-handed coordinate system



left-handed orientation
(clockwise orientation)

negative orientation in a right-handed coordinate system

Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

$$\sigma_x\sigma_y = -\,\sigma_y\sigma_x$$

$$\sigma_y\sigma_z = -\,\sigma_z\sigma_y$$

$$\sigma_z\sigma_x = -\,\sigma_x\sigma_z$$

Scalars

Scalars are geometric entities without direction. They can be expressed as a multiple of the base scalar:

$$k = k\ 1$$

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

$$r = x\ \sigma_x + y\ \sigma_y + z\ \sigma_z$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

$$A = A_{xy}\ \sigma_x\sigma_y + A_{yz}\ \sigma_y\sigma_z + A_{zx}\ \sigma_z\sigma_x$$

Trivectors

Trivectors are oriented volume elements. They can be expressed as a multiple of the base trivector:

$$V = V_{xyz}\ \sigma_x\sigma_y\sigma_z$$

Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$a\,b = a \bullet b + a \wedge b$$

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

$$a \bullet b = b \bullet a = \frac{1}{2}(a\,b + b\,a)$$

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

$$a \wedge b = -\,b \wedge a = \frac{1}{2}(a\,b - b\,a)$$

Systems of Two Linear Equations

$$\begin{aligned} a_1\, x + b_1\, y &= d_1 \\ a_2\, x + b_2\, y &= d_2 \end{aligned} \quad \Rightarrow \quad a\, x + b\, y = d$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1\,\sigma_x + a_2\,\sigma_y)\, x + (b_1\,\sigma_x + b_2\,\sigma_y)\, y = d_1\,\sigma_x + d_2\,\sigma_y$$

Solutions:

$$x = \frac{1}{a \wedge b}(d \wedge b) = (a \wedge b)^{-1}\,(d \wedge b)$$

$$y = \frac{1}{a \wedge b}(a \wedge d) = (a \wedge b)^{-1}\,(a \wedge d)$$

Systems of Three Linear Equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2 \quad \Rightarrow \quad a x + b y + c z = d$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z)\, x + (b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z)\, y$$
$$+ (c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z)\, z = d_1 \sigma_x + d_2 \sigma_y + d_3 \sigma_z$$

Solutions:

$$x = (a \wedge b \wedge c)^{-1} (d \wedge b \wedge c)$$
$$y = (a \wedge b \wedge c)^{-1} (a \wedge d \wedge c)$$
$$z = (a \wedge b \wedge c)^{-1} (a \wedge b \wedge d)$$

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of former lessons (see part I, II, & III).

Searching for a Graphical Representation of the Gaussian Method

Let's start with an example of a rather simple system of two linear equations:

$$4x + 2y = 14$$
$$x + 4y = 14$$

There are several different strategies to solve this system of two linear equations using the method of Gauss.

First strategy:

x	y	d
4	2	14
1	4	14
1	?	?
0	?	?
1	0	?
0	1	?

Second strategy:

x	y	d
4	2	14
1	4	14
?	0	?
?	1	?
1	0	?
0	1	?

$\Rightarrow$ The left-hand side of the augmented matrix is transformed into an identity matrix.

Third strategy:

x	y	d
4	2	14
1	4	14
0	?	?
1	?	?
0	1	?
1	0	?

Fourth strategy:

x	y	d
4	2	14
1	4	14
?	1	?
?	0	?
0	1	?
1	0	?

$\Rightarrow$ The left-hand side of the augmented matrix is transformed into a permutation matrix, which exchanges the x- and y-coordinates.

This is an example of the most simple permutation symmetry, called S_2 permutation symmetry, as it exchanges only two different entities.

To find out what happens when the Gaussian method is applied, we are looking for diagrams (and equations) which represent these four different strategies graphically (and algebraically).

Searching for a Graphical Representation of the Gaussian Method

In modern linear algebra we transfer all vectors of the system of linear equations

$$4x + 2y = 14$$
$$x + 4y = 14$$

into Pauli vectors:

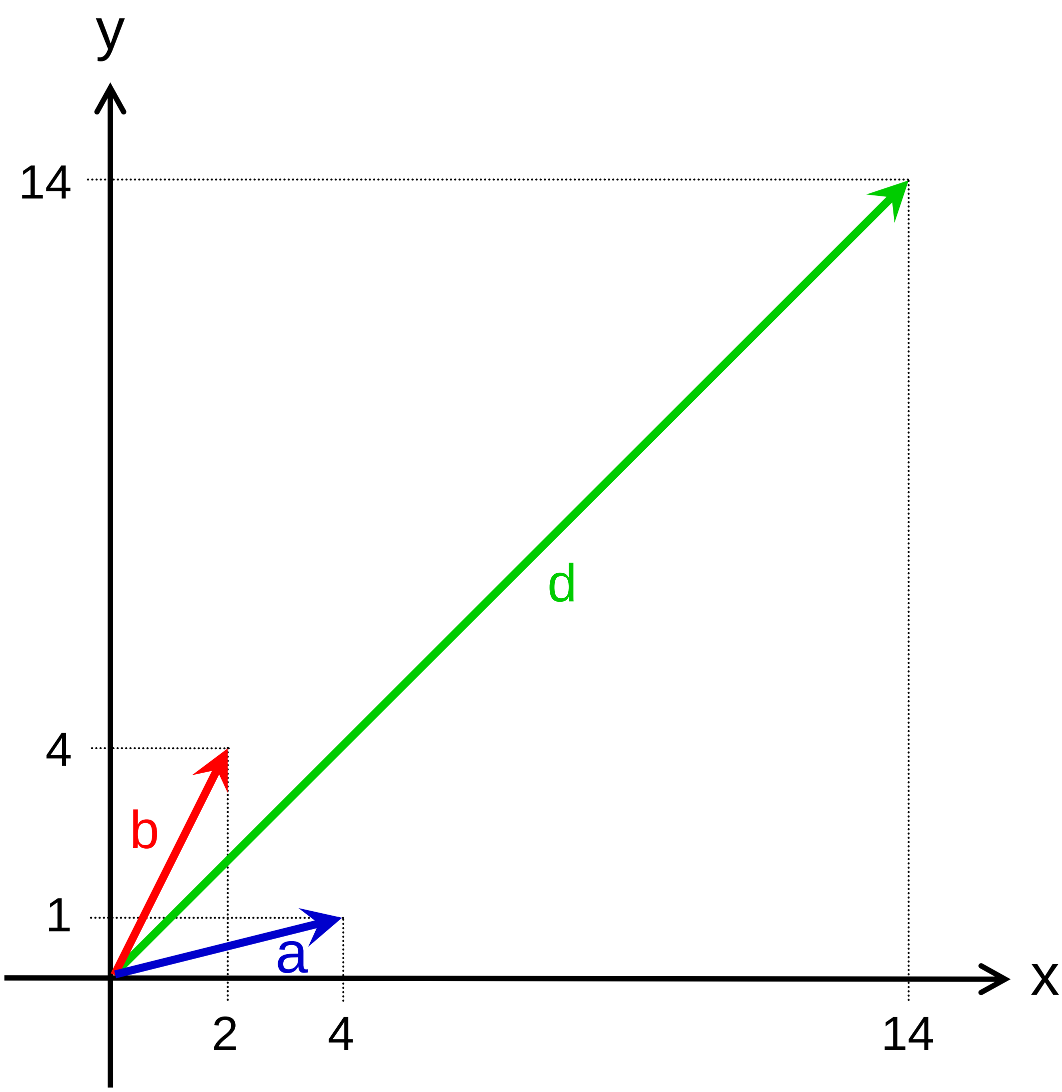
$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \longrightarrow a = 4\sigma_x + \sigma_y$$

$$b = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \longrightarrow b = 2\sigma_x + 4\sigma_y$$

$$d = \begin{bmatrix} 14 \\ 14 \end{bmatrix} \longrightarrow d = 14\sigma_x + 14\sigma_y$$

They can now be represented in a diagram (see following slide).

Diagram of the Starting Point

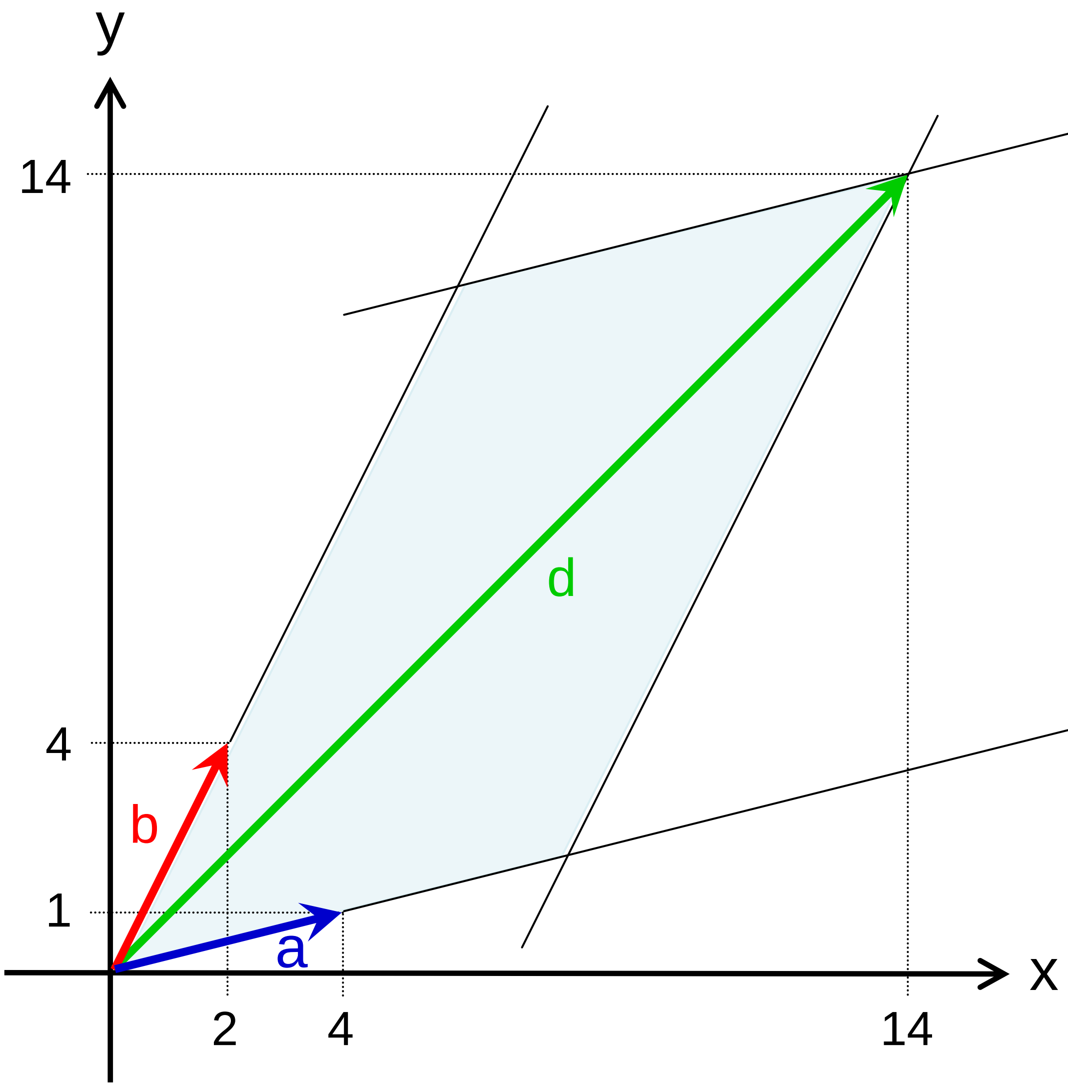


The algebraic starting point is:

$$\underbrace{(4\,\sigma_x + \sigma_y)}_{}^{\mathbf{a}}\ x + \overbrace{(2\,\sigma_x + 4\,\sigma_y)}^{\mathbf{b}}\ y = \overbrace{14\,\sigma_x + 14\,\sigma_y}^{\mathbf{d}}$$

Geometric Analysis of the Starting Point

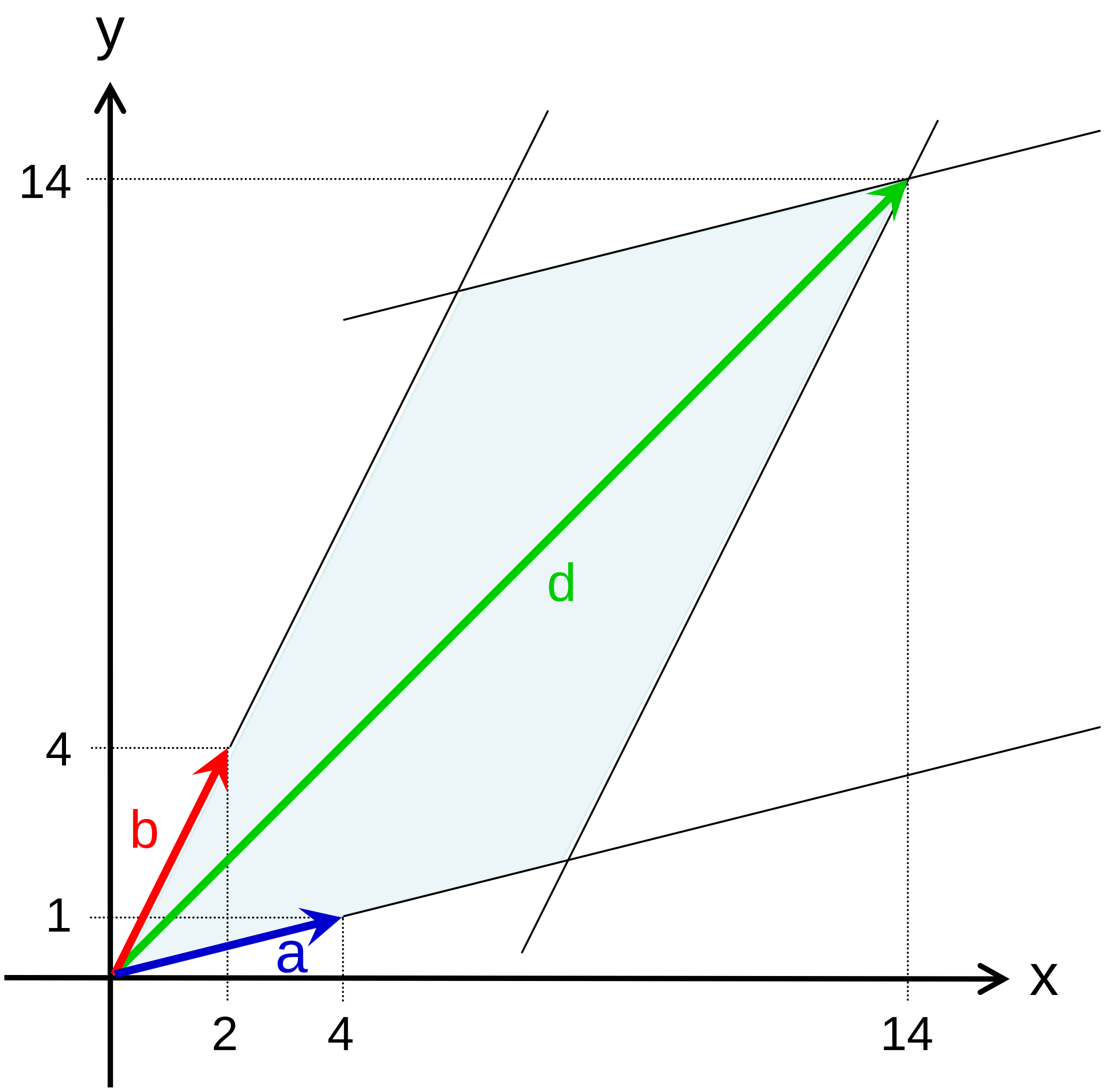
Algebraic starting point: $\mathbf{a}\, x + \mathbf{b}\, y = \mathbf{d}$



Problem: How long are the two sides $\mathbf{a}\,x$ and $\mathbf{b}\,y$ of the parallelogram compared to the lengths of the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$?

Geometric Analysis of the Starting Point

Algebraic starting point: $\mathbf{a}\,x + \mathbf{b}\,y = \mathbf{d}$



Problem: How long are the two sides $\mathbf{a}\,x$ and $\mathbf{b}\,y$ of the parallelogram?

Solution idea: Coefficient vectors **a** and **b** will become unit vectors of a new coordinate system.

Solution Strategy

Algebraic starting point: $\mathbf{a}\,x + \mathbf{b}\,y = \mathbf{d}$

Problem: How long are the two sides $\mathbf{a}\,x$ and $\mathbf{b}\,y$ of the parallelogram compared to the lengths of the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$?

If we measure the lengths of both sides $\mathbf{a}\,x$ and $\mathbf{b}\,y$ of the parallelogram and compare them with the lengths of the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$, the unknown variables x and y will be:

$$\text{unknown variable} = \frac{\text{length of side of parallelogram}}{\text{length of corresponding coefficient vector}}$$

$$\Rightarrow \quad x = \frac{\mathbf{a}\,x}{\mathbf{a}} \qquad y = \frac{\mathbf{b}\,y}{\mathbf{b}}$$

Yes, we are allowed to divide by vectors in Geometric Algebra!

As the division by vectors is a conceptual problem in standard vector algebra, Gauss decided for another idea:

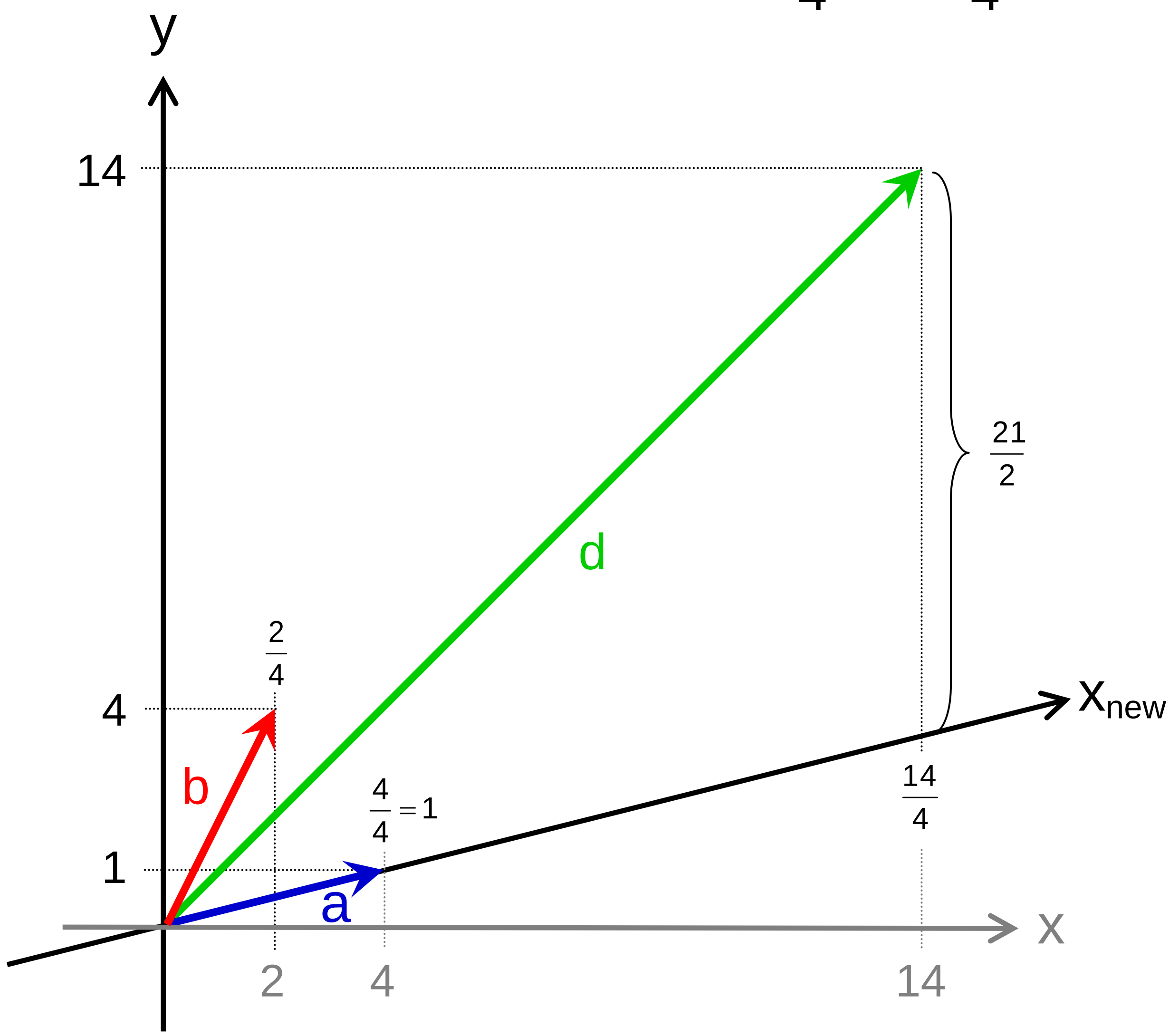
Solution idea: The axes of the coordinate system will be transformed. Coefficient vectors $\mathbf{a}$ and $\mathbf{b}$ will become unit vectors of the new coordinate system.

Intermediate Step (first strategy)

Change of x-coordinates:

$$\underbrace{(4\,\sigma_x + \sigma_y)}_{e_x = 4\,\sigma_x + \sigma_y}\, x + (2\,\sigma_x + 4\,\sigma_y)\, y = 14\,\sigma_x + 14\,\sigma_y$$

$$\Rightarrow \quad \sigma_x = \frac{1}{4}\, e_x - \frac{1}{4}\, \sigma_y$$



New Gaussian Pauli vector equation:

$$e_x\, x + \left(\frac{1}{2}\, e_x + \frac{7}{2}\, \sigma_y\right) y = \frac{7}{2}\, e_x + \frac{21}{2}\, \sigma_y$$

Augmented Matrix of the Inter-mediate Step (first strategy)

The coefficients of this new Gaussian Pauli vector equation

$$(1\, e_x + 0\, \sigma_y)\, x + \left(\frac{1}{2}\, e_x + \frac{7}{2}\, \sigma_y\right) y = \frac{7}{2}\, e_x + \frac{21}{2}\, \sigma_y$$

are the elements of the intermediate step of the augmented matrix:

x	y	d
4	2	14
1	4	14
1	$\frac{1}{2}$	$\frac{7}{2}$
0	$\frac{7}{2}$	$\frac{21}{2}$
1	0	?
0	1	?

In some cases it is helpful to split this inter-mediate step into sub-steps to understand the strategy of Gauss.

Augmented Matrix of the Intermediate Step (first strategy)

x	y	d	
4	2	14	Multiply row one by $\frac{1}{4}$.
1	4	14	
1	$\frac{1}{2}$	$\frac{7}{2}$	
1	4	14	Subtract row one from row two.
1	$\frac{1}{2}$	$\frac{7}{2}$	
0	$\frac{7}{2}$	$\frac{21}{2}$	
1	0	**?**	
0	1	**?**	

As intended, coefficient vector **a** now represents a unit vector which lies parallel to the new x-co-ordinate axis:

$$e_x = \mathbf{a}$$

Allowed Row Operations
to Transform Augmented Matrices

x	y	d
4	2	14
1	4	14
1	$\frac{1}{2}$	$\frac{7}{2}$
1	4	14
1	$\frac{1}{2}$	$\frac{7}{2}$
0	$\frac{7}{2}$	$\frac{21}{2}$
1	0	**?**
0	1	**?**

Multiply row one by $\frac{1}{4}$.

Subtract row one from row two.

The following row operations are allowed to transform the augmented matrices of Gauss:

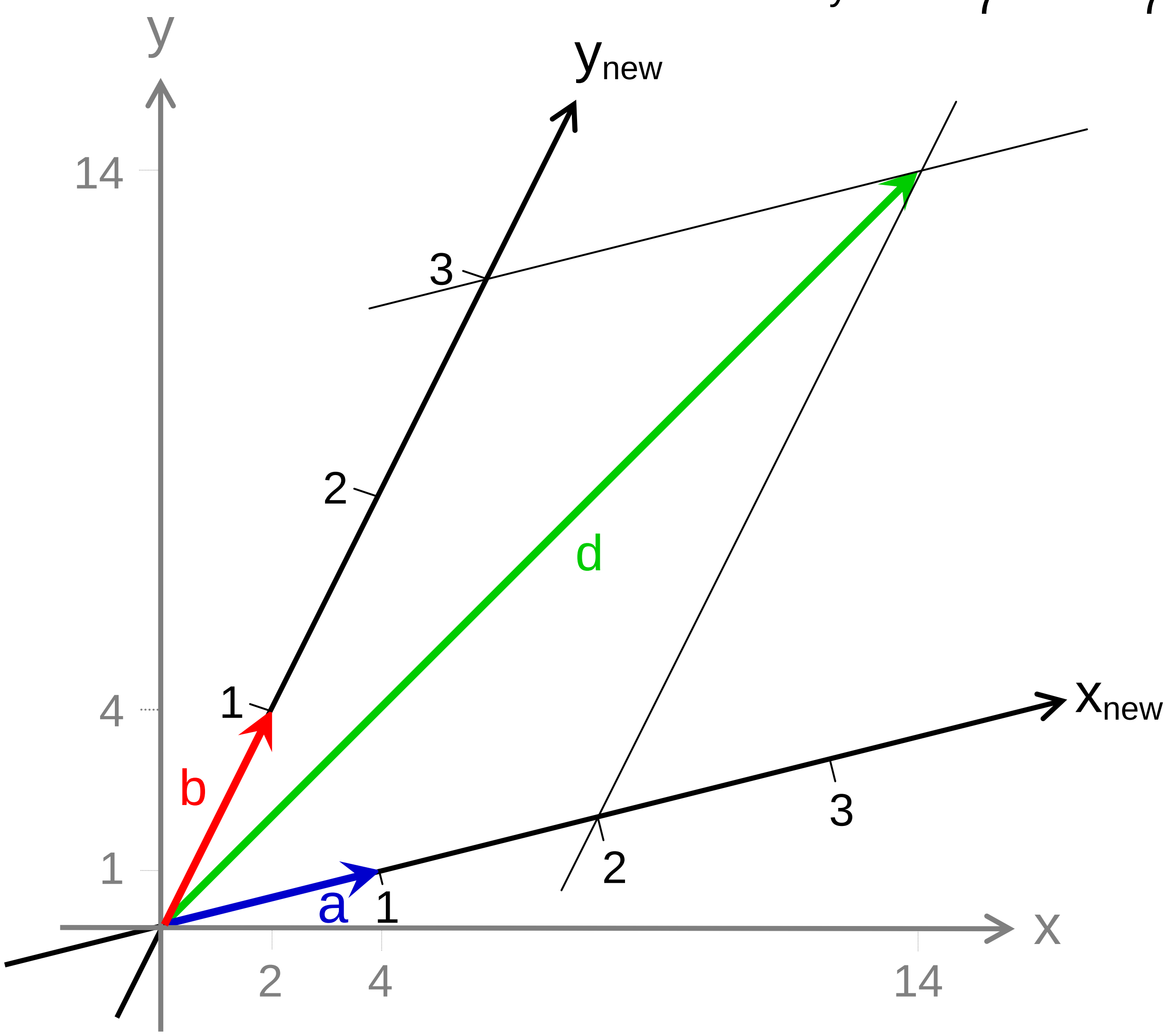
- Any row of the augmented matrix can be multiplied by or divided by a constant (provided the constant does not equal zero).
- Any multiple of a row can be added to or subtracted from any other row.
- Any two rows of the augmented matrix can be interchanged.

Final Step (first strategy)

Change of y-coordinates:

$$e_x\, x + \underbrace{\left(\frac{1}{2} e_x + \frac{7}{2} \sigma_y\right)}_{e_y = \frac{1}{2} e_x + \frac{7}{2} \sigma_y} y = \frac{7}{2} e_x + \frac{21}{2} \sigma_y$$

$$\Rightarrow \quad \sigma_y = -\frac{1}{7} e_x + \frac{2}{7} e_y$$



Final Gaussian Pauli vector equation:

$$e_x\, x + e_y\, y = 2\, e_x + 3\, e_y$$

Augmented Matrix of the Final Step (first strategy)

The coefficients of the final Gaussian Pauli vector equation

$$(1\ e_x + 0\ e_y)\ x + (0\ e_x + 1\ e_y)\ y = 2\ e_x + 3\ e_y$$

are the elements of the final step of the augmented matrix:

x	y	d
4	2	14
1	4	14
1	$\frac{1}{2}$	$\frac{7}{2}$
0	$\frac{7}{2}$	$\frac{21}{2}$
1	0	2
0	1	3

$\Rightarrow$ Solution values: $x = 2$

$y = 3$

In some cases it is helpful to split steps into sub-steps to understand the strategy of Gauss.

Augmented Matrix of the Final Step (first strategy)

x	y	d	
4	2	14	Multiply row one by $\frac{1}{4}$.
1	4	14	
1	$\frac{1}{2}$	$\frac{7}{2}$	
1	4	14	Subtract row one from row two.
1	$\frac{1}{2}$	$\frac{7}{2}$	
0	$\frac{7}{2}$	$\frac{21}{2}$	Multiply row two by $\frac{2}{7}$.
1	$\frac{1}{2}$	$\frac{7}{2}$	Subtract $\frac{1}{2}$(row two) from row one.
0	1	3	
1	0	2	$\Rightarrow$ $x = 2$
0	1	3	$y = 3$

As intended, coefficient vector **b** now represents a unit vector which lies parallel to the new y-co-ordinate axis:

$$e_y = \mathbf{b}$$

Check of Result

System of two linear equations:

$$\begin{aligned} 4x + 2y &= 14 \\ x + 4y &= 14 \end{aligned}$$

Solution: $x = 2$

$y = 3$

Check of result:

$$\begin{aligned} 4 \cdot 2 + 2 \cdot 3 &= 8 + 6 = 14 \\ 2 + 4 \cdot 3 &= 2 + 12 = 14 \end{aligned}$$

$\Rightarrow$ The result is correct.

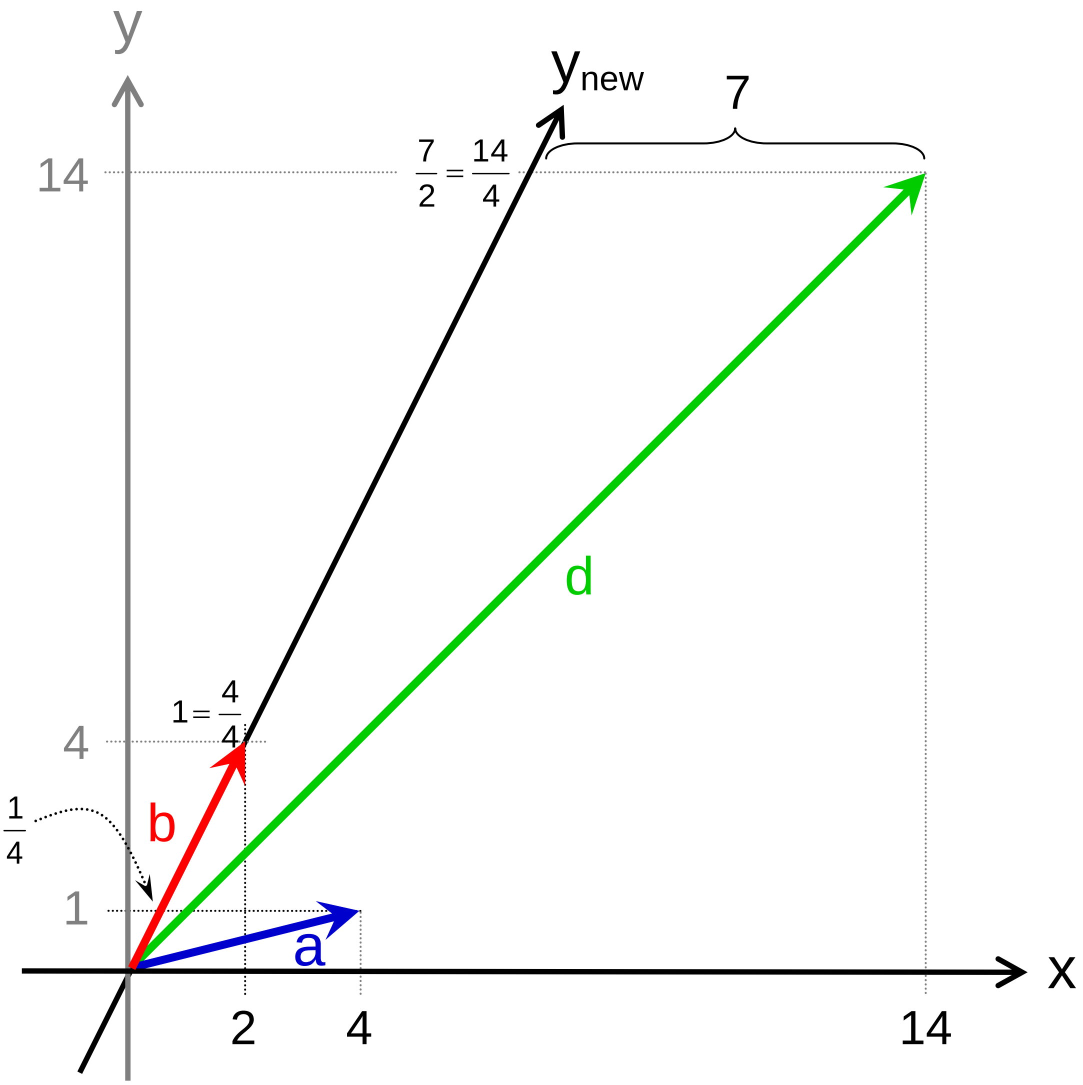
The same results can be found by applying the Gaussian method in a different order (see following slides).

Second strategy

Now the y-coordinates are changed first:

$$(4\,\sigma_x + \sigma_y)\,x + (\underbrace{2\,\sigma_x + 4\,\sigma_y}_{e_y = 2\,\sigma_x + 4\,\sigma_y})\,y = 14\,\sigma_x + 14\,\sigma_y$$

$$\sigma_y = -\frac{1}{2}\sigma_x + \frac{1}{4}e_y \quad \Leftarrow \quad e_y = 2\,\sigma_x + 4\,\sigma_y$$



New Gaussian Pauli vector equation:

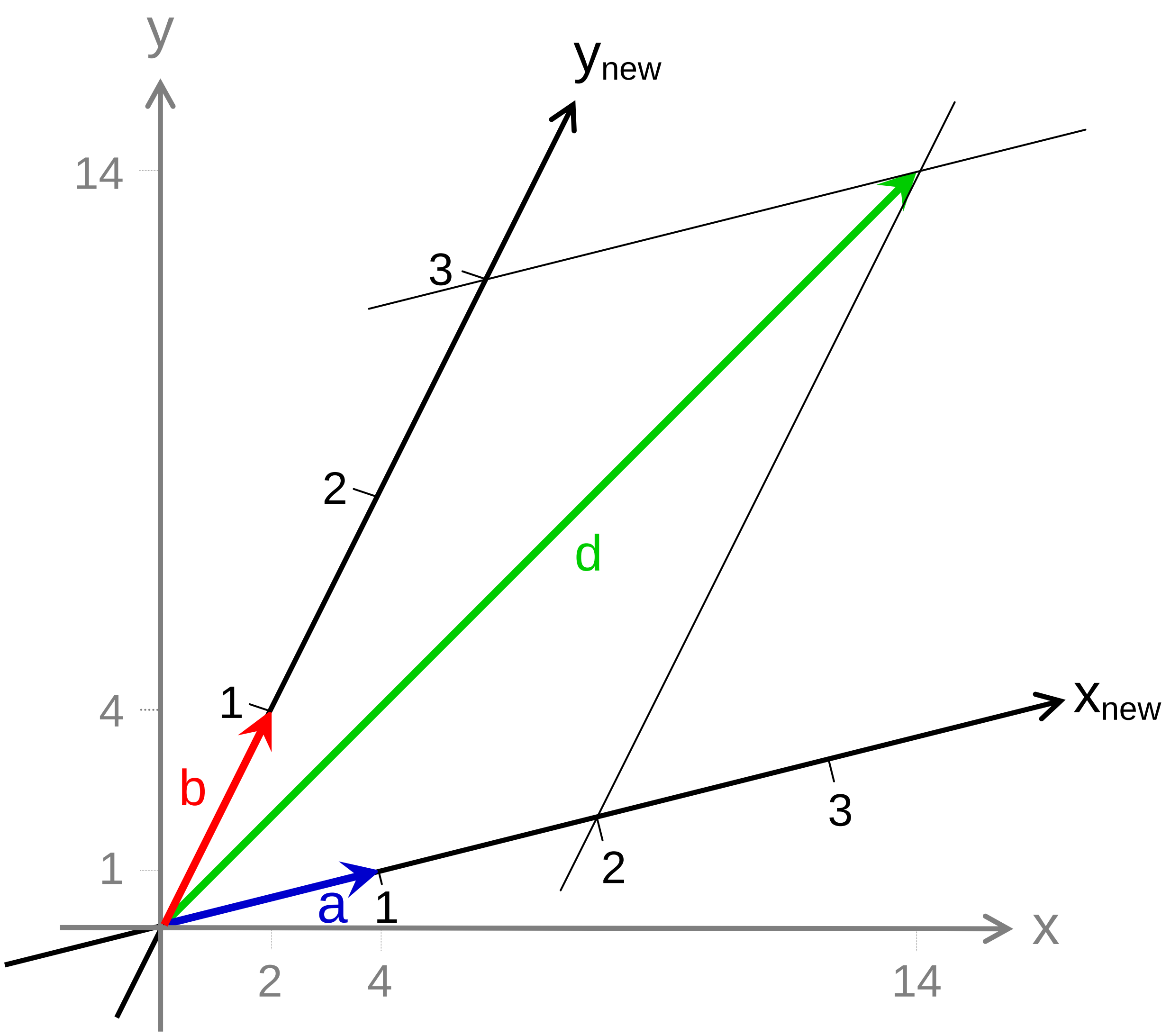
$$\left(\frac{7}{2}\sigma_x + \frac{1}{4}e_y\right)x + e_y\,y = 7\,\sigma_x + \frac{7}{2}e_y$$

Final Step (second strategy)

The x-coordinates are changed in the final step:

$$\underbrace{\left(\frac{7}{2}\,\sigma_x + \frac{1}{4}\,e_y\right)}\,x + e_y\,y = 7\,\sigma_x + \frac{7}{2}\,e_y$$

$$e_x = \frac{7}{2}\sigma_x + \frac{1}{4}e_y \qquad \Rightarrow \qquad \sigma_x = \frac{2}{7}e_x - \frac{1}{14}e_y$$



Final Gaussian Pauli vector equation:

$$e_x\,x + e_y\,y = 2\,e_x + 3\,e_y$$

Augmented Matrices (second strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = 14\,\sigma_x + 14\,\sigma_y$$

$$\left(\frac{7}{2}\,\sigma_x + \frac{1}{4}\,e_y\right)x + (0\,\sigma_x + 1\,e_y)\,y = 7\,\sigma_x + \frac{7}{2}\,e_y$$

$$(1\,e_x + 0\,e_y)\,x + (0\,e_x + 1\,e_y)\,y = 2\,e_x + 3\,e_y$$

are the elements of the augmented matrices:

x	y	d	
4	2	14	Subtract $\frac{1}{2}$(row two) from row one.
1	4	14	Multiply row two by $\frac{1}{4}$.
$\frac{7}{2}$	0	7	Multiply row one by $\frac{2}{7}$.
$\frac{1}{4}$	1	$\frac{7}{2}$	Subtract $\frac{1}{14}$(row one) from row two.
1	0	2	$\Rightarrow$ $x = 2$
0	1	3	$y = 3$

An extended version with more sub-steps can be found at the following slide.

Extended Version of Augmented Matrices (second strategy)

x	y	d	
4	2	14	
1	4	14	Multiply row two by $\frac{1}{4}$.
4	2	14	Subtract 2 times row two from row one.
$\frac{1}{4}$	1	$\frac{7}{2}$	
$\frac{7}{2}$	0	7	Multiply row one by $\frac{2}{7}$.
$\frac{1}{4}$	1	$\frac{7}{2}$	
1	0	2	
$\frac{1}{4}$	1	$\frac{7}{2}$	Subtract $\frac{1}{4}$(row one) from row two.
1	0	2	$\Rightarrow$ $x = 2$
0	1	3	$y = 3$

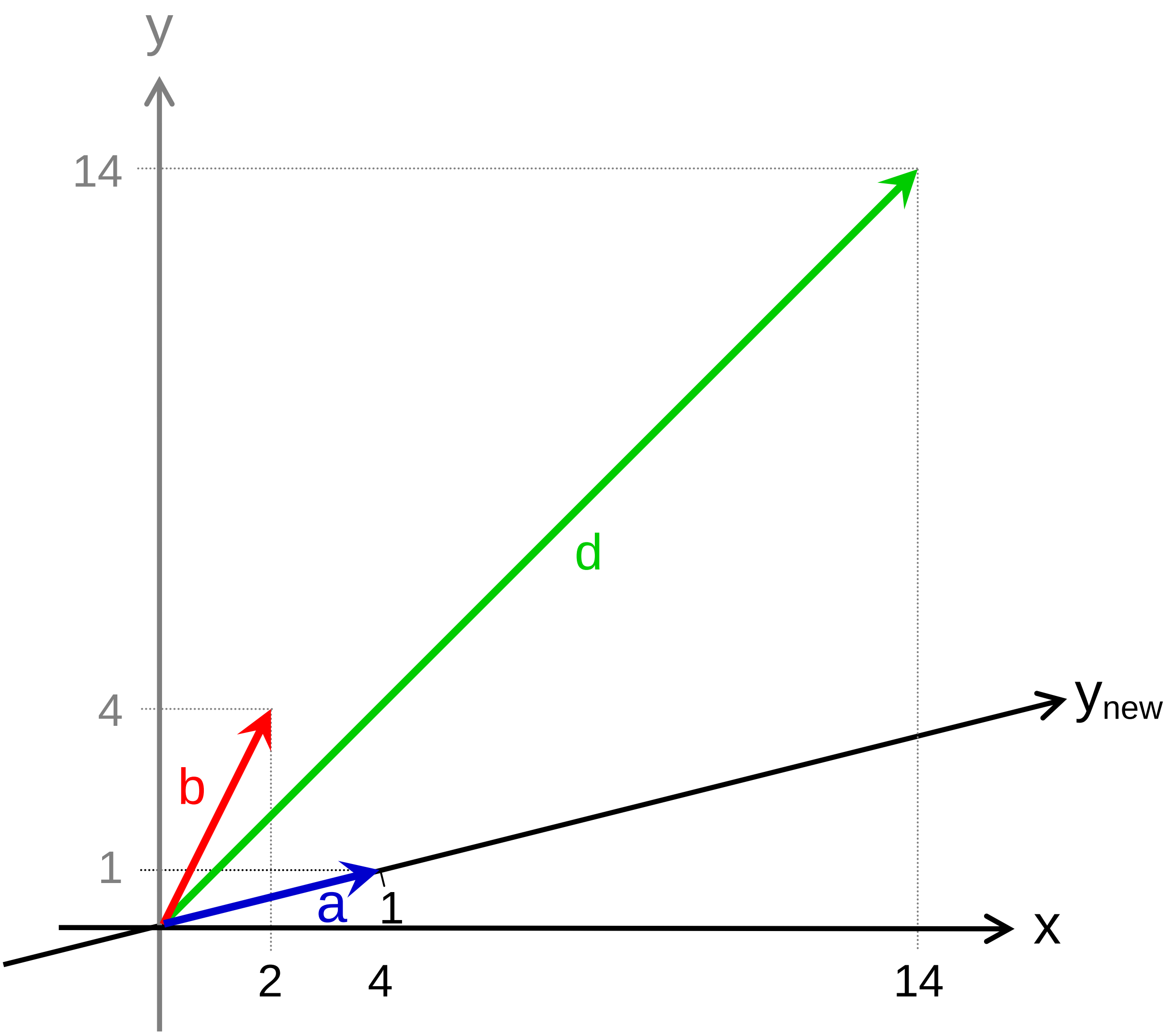
Of course the result is identical to the result of the first strategy.

Third Strategy

Now coefficient vector **a** will be chosen as unit vector which points into the direction of the new y-axis:

$$\underbrace{(4\,\sigma_x + \sigma_y)}_{e_y = 4\,\sigma_x + \sigma_y}\, x + (2\,\sigma_x + 4\,\sigma_y)\, y = 14\,\sigma_x + 14\,\sigma_y$$

$$e_y = 4\,\sigma_x + \sigma_y \quad \Rightarrow \quad \sigma_y = -4\,\sigma_x + e_y$$

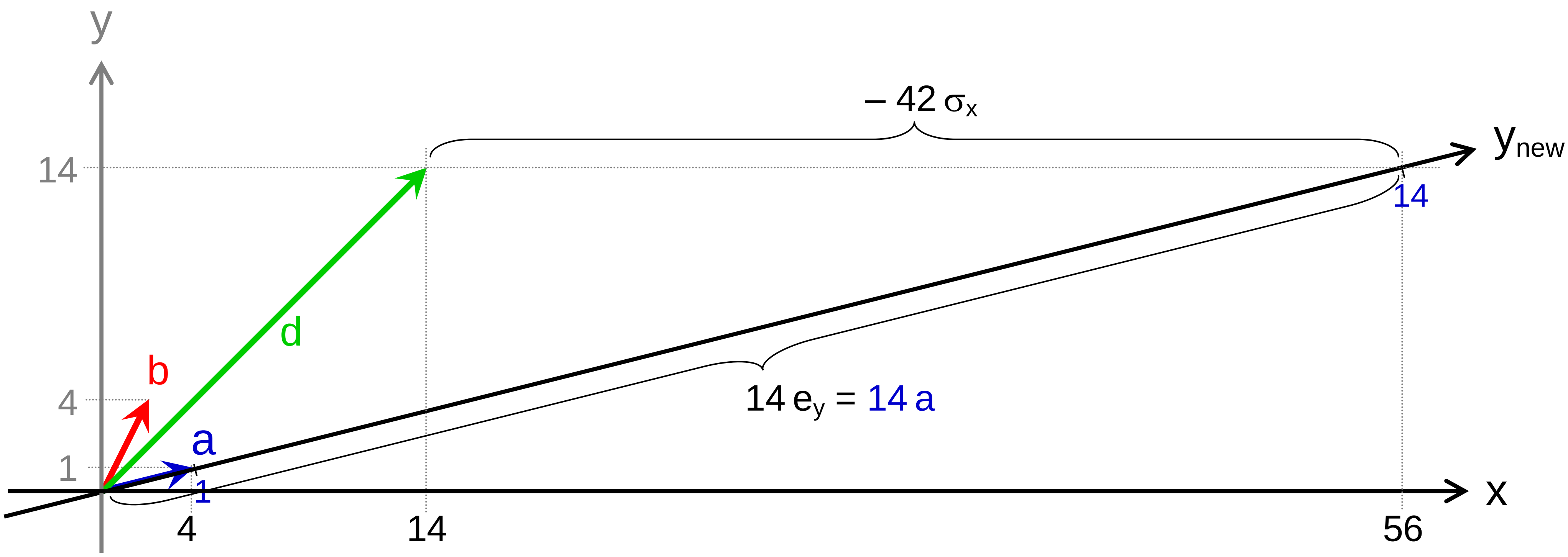


New Gaussian Pauli vector equation:

$$e_y\, x + (-14\,\sigma_x + 4\,e_y)\, y = -42\,\sigma_x + 14\,e_y$$

Complete Diagram of Intermediate Step (third strategy)

After scaling the coordinate axes down, the complete diagram of the new equation $e_y\, x + (-14\, \sigma_x + 4\, e_y)\, y = -42\, \sigma_x + 14\, e_y$ can be shown.

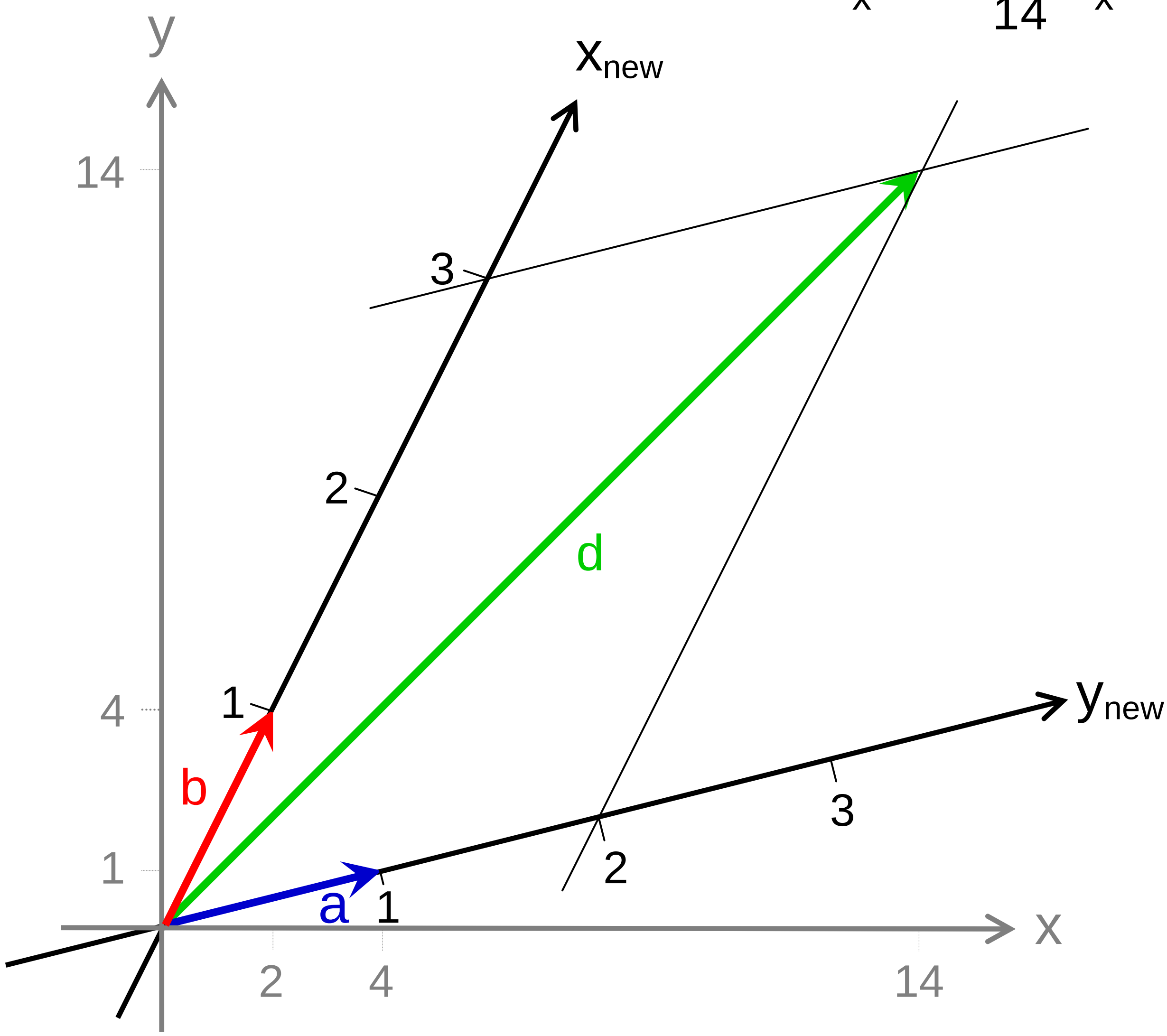


Final Step (third strategy)

And the new x-axis will point into the direction of coefficient vector **b**:

$$e_y\, x + (\underbrace{-14\,\sigma_x + 4\,e_y}_{e_x \,=\, -14\,\sigma_x + 4\,e_y})\, y = -42\,\sigma_x + 14\,e_y$$

$$\Rightarrow \quad \sigma_x = -\frac{1}{14} e_x + \frac{2}{7} e_y$$



Final Gaussian Pauli vector equation:

$$e_y\, x + e_x\, y = 3\, e_x + 2\, e_y$$

Augmented Matrices (third strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \quad 14\,\sigma_x + 14\,\sigma_y$$

$$(0\,\sigma_x + 1\,e_y)\,x + (-14\,\sigma_x + 4\,e_y)\,y = -42\,\sigma_x + 14\,e_y$$

$$(0\,e_x + 1\,e_y)\,x + (1\,e_x + 0\,e_y)\,y = \quad 3\,e_x + 2\,e_y$$

are the elements of the augmented matrices:

x	y	d	
4	2	14	Subtract 4 times row two from row one.
1	4	14	
0	–14	–42	Divide row one by (–14).
1	4	14	Add $\frac{2}{7}$(row one) to row two.
0	1	3	$\Rightarrow$ $y = 3$
1	0	2	$x = 2$

An extended version with more sub-steps can be found at the following slide.

Extended Version of Augmented Matrices (third strategy)

x	y	d	
4	2	14	Subtract 4 times row two from row one.
1	4	14	
0	– 14	– 42	Divide row one by (– 14).
1	4	14	
0	1	3	
1	4	14	Subtract 4 times row one from row two.
0	1	3	$\Rightarrow$ y = 3
1	0	2	x = 2

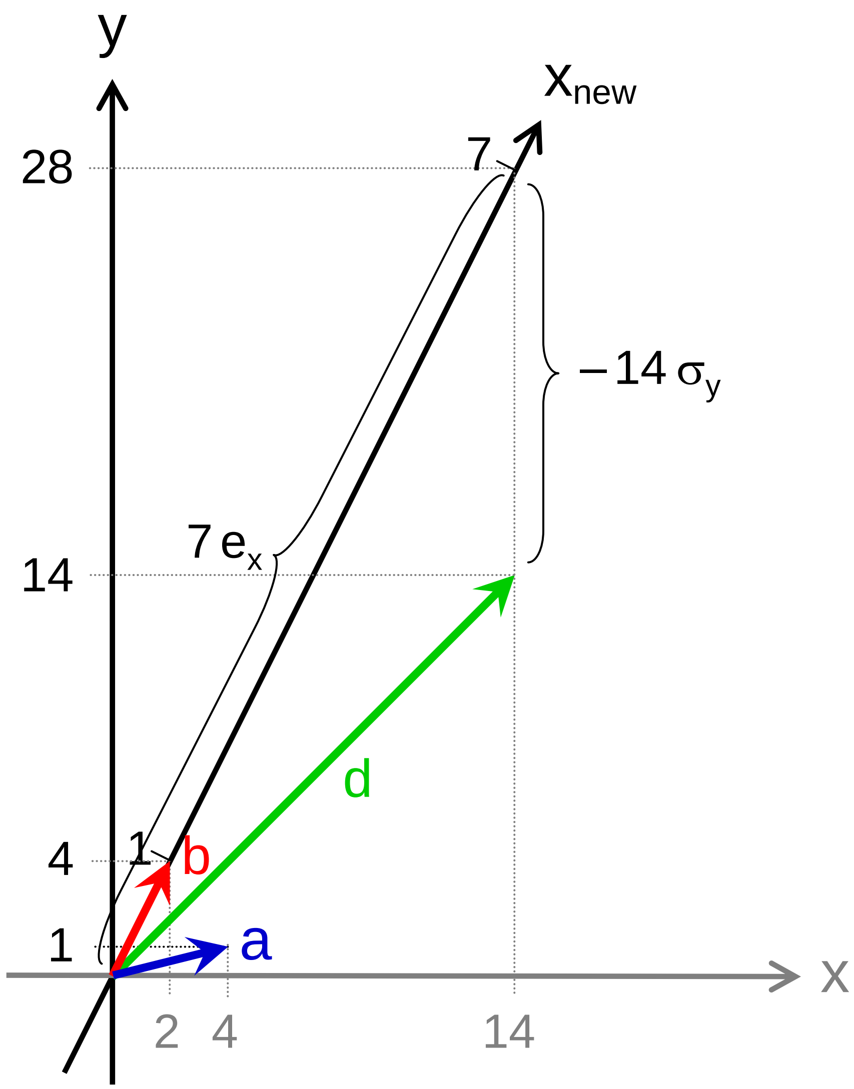
Again the result is identical to the results of previous strategies.

Fourth Strategy

Now coefficient vector **b** is chosen as unit vector which points into the direction of the new x-axis:

$$(4\,\sigma_x + \sigma_y)\,x + (\underbrace{2\,\sigma_x + 4\,\sigma_y}_{})\,y = 14\,\sigma_x + 14\,\sigma_y$$

$$\sigma_x = \frac{1}{2}e_x - 2\,\sigma_y \quad \Leftarrow \quad e_x = 2\,\sigma_x + 4\,\sigma_y$$



New Gaussian Pauli vector equation:

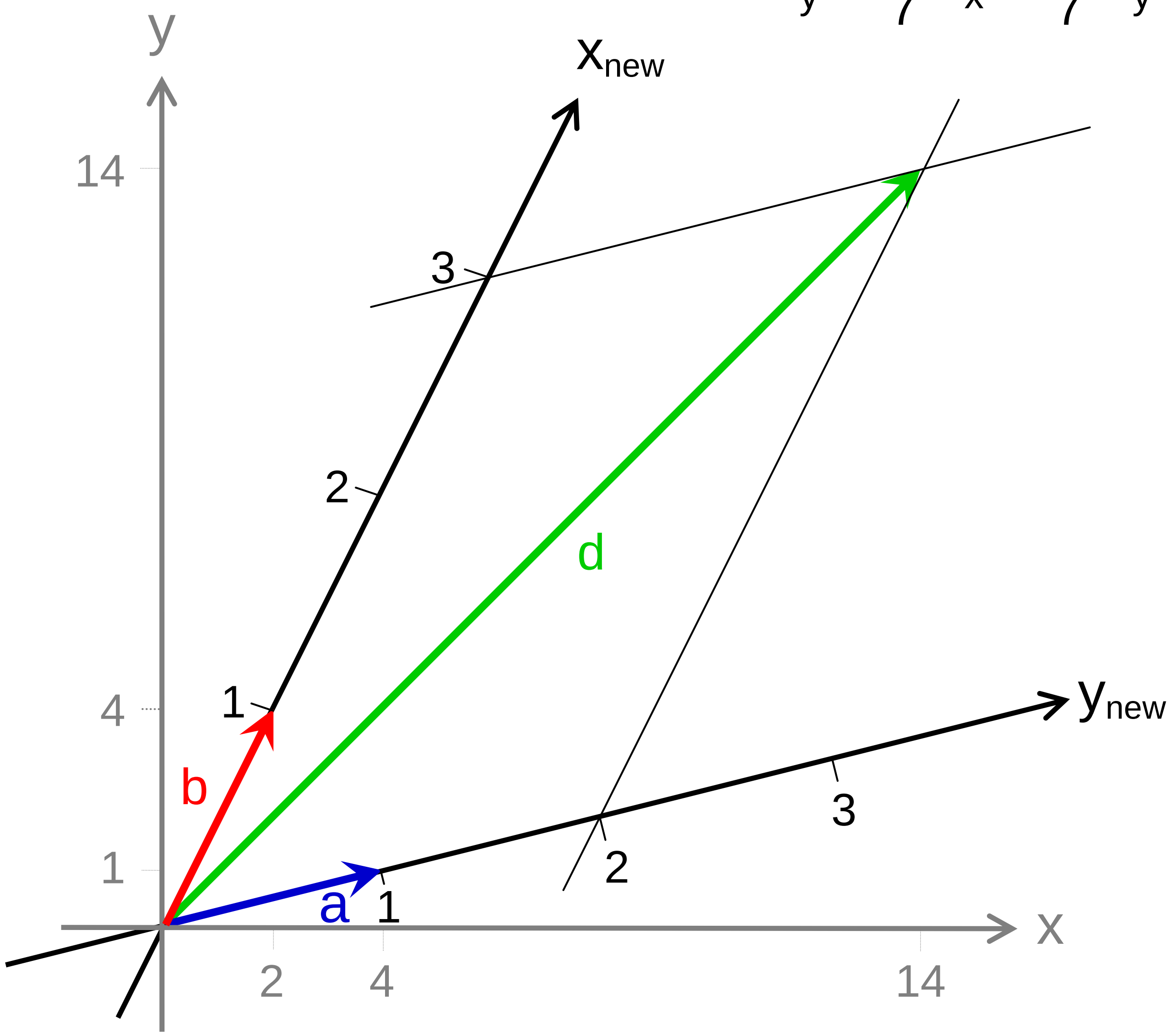
$$(2\,e_x - 7\,\sigma_y)\,x + e_x\,y = 7\,e_x - 14\,\sigma_y$$

Final Step (fourth strategy)

And the new y-axis will point into the direction of coefficient vector **a**:

$$\underbrace{(2\,e_x - 7\,\sigma_y)}_{e_y \,=\, 2\,e_x - 7\,\sigma_y}\,x + e_x\,y = 7\,e_x - 14\,\sigma_y$$

$$\Rightarrow \quad \sigma_y = \frac{2}{7}\,e_x - \frac{1}{7}\,e_y$$



Final Gaussian Pauli vector equation:

$$e_y\,x + e_x\,y = 3\,e_x + 2\,e_y$$

Augmented Matrices (fourth strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \ \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = 14\,\sigma_x + 14\,\sigma_y$$

$$(2\,e_x - 7\,\sigma_y)\,x + (1\,e_x + 0\,\sigma_y)\,y = 7\,e_x - 14\,\sigma_y$$

$$(0\,e_x + 1\,e_y)\,x + (1\,e_x + 0\,e_y)\,y = 3\,e_x + 2\,e_y$$

are the elements of the augmented matrices:

x	y	d	
4	2	14	Divide row one by 2.
1	4	14	Subtract 2 times row one from row two.
2	1	7	Add $\frac{2}{7}$(row two) to row one.
–7	0	–14	Divide row two by (–7).
0	1	3	$\Rightarrow$ y = 3
1	0	2	x = 2

An extended version with more sub-steps can be found at the following slide.

Extended Version of Augmented Matrices (fourth strategy)

x	y	d	
4	2	14	Divide row one by 2.
1	4	14	
2	1	7	
1	4	14	Subtract 4 times row one from row two.
2	1	7	
–7	0	–14	Divide row two by (–7).
2	1	7	Subtract 2 times row two from row one.
1	0	2	
0	1	3	$\Rightarrow$ y = 3
1	0	2	x = 2

Again the result is identical to the results of previous strategies.

Inverse Matrices

As already discussed in previous lessons the solution of a system of linear equations can also be found with the help of the inverse matrix.

$$\begin{aligned} 4x + 2y &= 14 \\ x + 4y &= 14 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

Repetition: Definition of Inverse Matrices

Inverse matrices $\mathbf{A}^{-1}$ satisfy the relationship

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \boldsymbol{I}$$

The product of a matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$ equals the identity matrix $\boldsymbol{I}$.

As already discussed this relationship can be used to solve systems of linear equations.

Repetition: Inverse Matrices

As already discussed in previous lessons the solution of a system of linear equations can also be found with the help of the inverse matrix.

$$\begin{aligned} 4x + 2y &= 14 \\ x + 4y &= 14 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

Multiplying $\mathbf{A}^{-1}$ from the left will result in:

$$\mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

solution vector of the
system of linear equations

The Gaussian method can be used to find the inverse of a matrix.

Finding Inverse Matrices with the Gaussian Method

If matrix **A** is a (n x n)-matrix, the definition of an inverse matrix $\mathbf{A}\,\mathbf{A}^{-1} = \boldsymbol{I}$ can be split into n different systems of n different linear equations.

Thus in our (2 x 2)-matrix example, we are able to split the defining equation

$$
\begin{array}{cc|cc}
 & & x_1 & x_2 \\
 & & y_1 & y_2 \\
\hline
4 & 2 & 1 & 0 \\
1 & 4 & 0 & 1
\end{array}
$$

into the following two systems of two linear equations:

$$
\begin{array}{cc|c}
 & & x_1 \\
 & & y_1 \\
\hline
4 & 2 & 1 \\
1 & 4 & 0
\end{array}
\qquad\qquad
\begin{array}{cc|c}
 & & x_2 \\
 & & y_2 \\
\hline
4 & 2 & 0 \\
1 & 4 & 1
\end{array}
$$

Finding Inverse Matrices with the Gaussian Method

First (blue) system of linear equations:

$$4x + 2y = 1$$
$$x + 4y = 0$$

Second (green) system of linear equations:

$$4x + 2y = 0$$
$$x + 4y = 1$$

These two systems of two linear equations can be written as Pauli vector equations.

First (blue) system of linear equations:

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x$$

Second (green) system of linear equations:

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_y$$

All row operations which will be carried out in the first (blue) system of linear equations are identical to the row operations which will be carried out in the second (green) system of linear equations.

Condensed Way of Writing Both Systems of Linear Equations

As all row operations which will be carried out in the first (blue) system of linear equations are identical to the row operations which will be carried out in the second (green) system of linear equations, we are able to write both equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x$$

or $$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_y$$

in a condensed way in one line only:

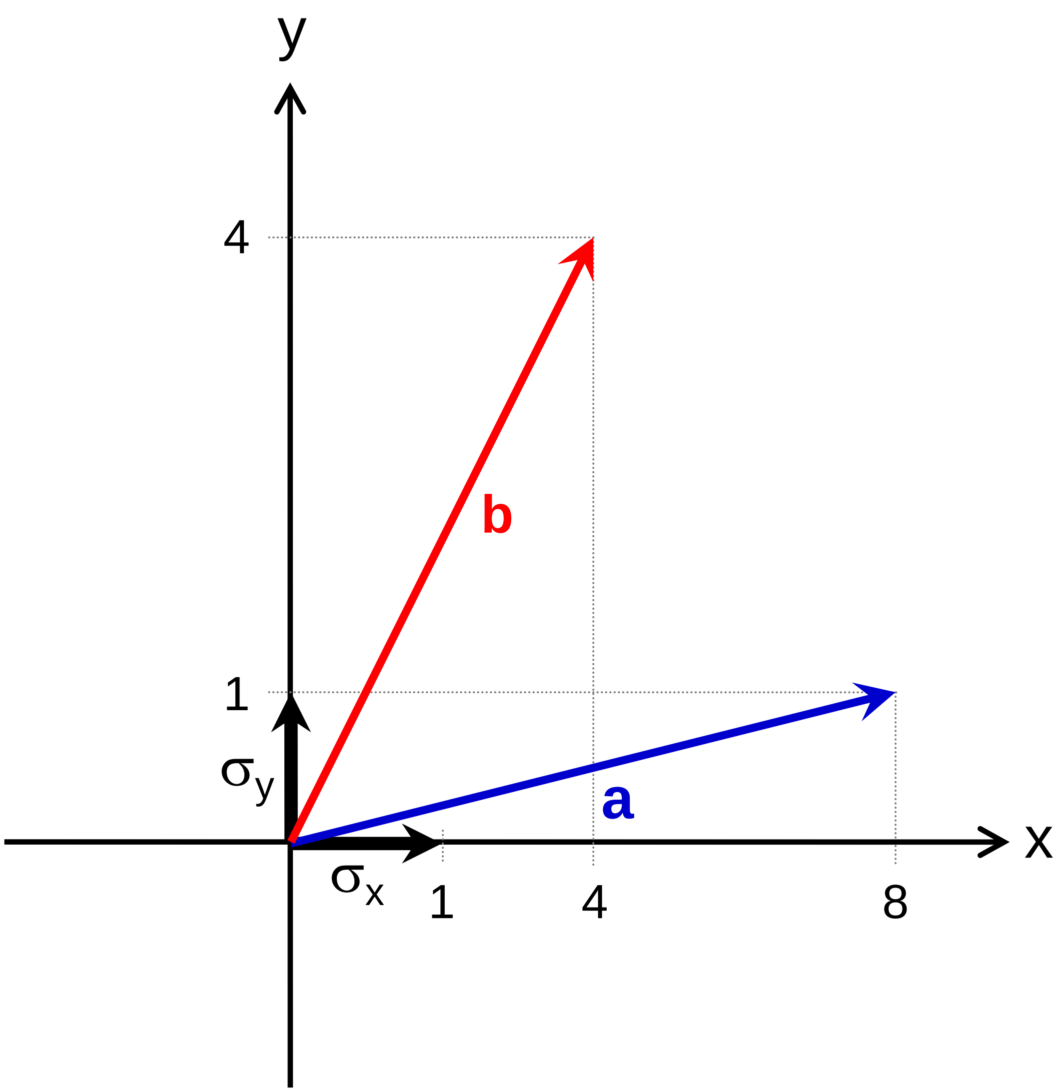
$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

Of course, $\sigma_x \neq \sigma_y$.

These are still two different equations.

These two equations can again be visualized in a diagram (see following slide).

Diagram of the Starting Point



The algebraic starting point is:

$$\overbrace{(4\,\sigma_x + \sigma_y)}^{\mathbf{a}}\, x + \overbrace{(2\,\sigma_x + 4\,\sigma_y)}^{\mathbf{b}}\, y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

Transformation of Coordinates (first strategy)

Change of x-coordinates:

$$\underbrace{(4\,\sigma_x + \sigma_y)}\; x + (2\,\sigma_x + 4\,\sigma_y)\; y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

$$e_x = 4\,\sigma_x + \sigma_y \quad \Rightarrow \quad \sigma_x = \underbrace{\frac{1}{4} e_x - \frac{1}{4}\,\sigma_y}$$

$$1\,e_x \quad x + \left(\frac{1}{2} e_x + \frac{7}{2}\sigma_y\right)\; y = \frac{1}{4} e_x - \frac{1}{4}\,\sigma_y \underset{\text{or}}{\mp} \sigma_y$$

Change of y-coordinates:

$$1\,e_x \quad x + \left(\underbrace{\frac{1}{2} e_x + \frac{7}{2}\sigma_y}\right)\; y = \frac{1}{4} e_x - \frac{1}{4}\,\sigma_y \underset{\text{or}}{\mp} \sigma_y$$

$$e_y = \frac{1}{2} e_x + \frac{7}{2}\sigma_y \quad \Rightarrow \quad \sigma_y = \underbrace{-\frac{1}{7} e_x + \frac{2}{7} e_y}$$

$$1\,e_x \quad x + \quad 1\,e_y \quad y = \frac{2}{7} e_x - \frac{1}{14} e_y \underset{\text{or}}{\mp} -\frac{1}{7} e_x + \frac{2}{7} e_y$$

Transformation of Coordinates (second strategy)

Change of y-coordinates:

$$(4\,\sigma_x + \sigma_y)\,x + (\underbrace{2\,\sigma_x + 4\,\sigma_y})\,y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

$$e_y = 2\,\sigma_x + 4\,\sigma_y \;\Rightarrow\; \sigma_y = \underbrace{-\frac{1}{2}\sigma_x + \frac{1}{4}e_y}$$

$$\left(\frac{7}{2}\sigma_x + \frac{1}{4}e_y\right)x + \quad 1\,e_y \qquad y = \sigma_x \underset{\text{or}}{\mp} -\frac{1}{2}\sigma_x + \frac{1}{4}e_y$$

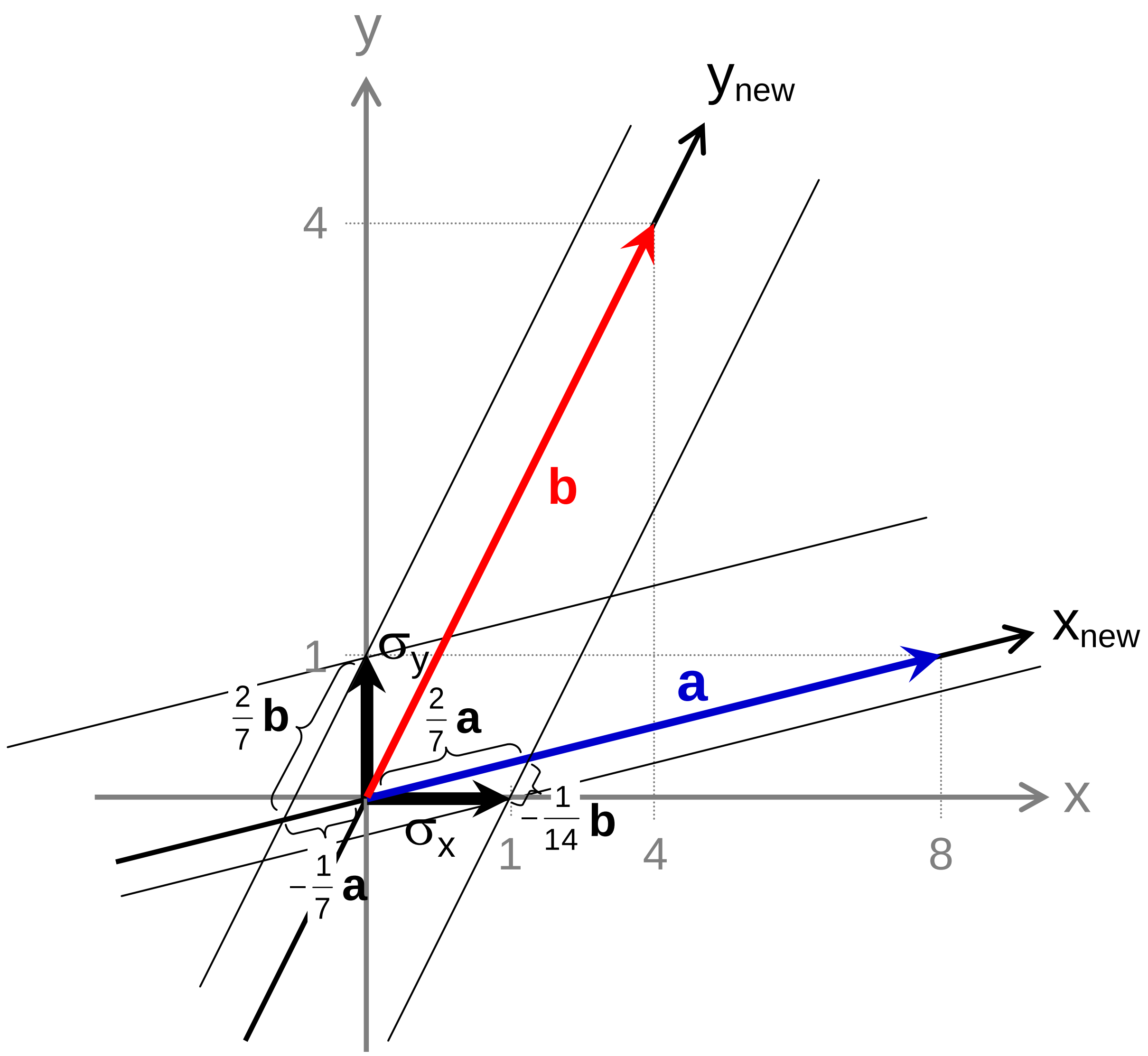
Change of x-coordinates:

$$\left(\underbrace{\frac{7}{2}\sigma_x + \frac{1}{4}e_y}\right)x + \quad 1\,e_y \qquad y = \sigma_x \underset{\text{or}}{\mp} -\frac{1}{2}\sigma_x + \frac{1}{4}e_y$$

$$e_x = \frac{7}{2}\sigma_x + \frac{1}{4}e_y \;\Rightarrow\; \sigma_x = \underbrace{\frac{2}{7}e_x - \frac{1}{14}e_y}$$

$$1\,e_x \qquad x + \qquad 1\,e_y \qquad y = \frac{2}{7}e_x - \frac{1}{14}e_y \underset{\text{or}}{\mp} -\frac{1}{7}e_x + \frac{2}{7}e_y$$

Diagram of the Final Situation
(first & second strategy)



The final algebraic equation equals:

$$e_x\, x + e_y\, y = \frac{2}{7}\, e_x - \frac{1}{14}\, e_y \underset{\text{or}}{=} -\frac{1}{7}\, e_x + \frac{2}{7}\, e_y$$

Augmented Matrices (first strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

$$1\,e_x \quad x + \left(\frac{1}{2}e_x + \frac{7}{2}\sigma_y\right)\,y = \frac{1}{4}e_x - \frac{1}{4}\,\sigma_y \underset{\text{or}}{\mp} \sigma_y$$

$$1\,e_x \quad x + \quad 1\,e_y \quad y = \frac{2}{7}e_x - \frac{1}{14}e_y \underset{\text{or}}{\mp} -\frac{1}{7}e_x + \frac{2}{7}e_y$$

are the elements of the augmented matrices:

x	y	e_1	e_2	
4	2	1	0	Multiply row one by $\frac{1}{4}$.
1	4	0	1	Subtract $\frac{1}{4}$(row one) from row two.
1	$\frac{1}{2}$	$\frac{1}{4}$	0	Subtract $\frac{1}{7}$(row two) from row one.
0	$\frac{7}{2}$	$-\frac{1}{4}$	1	Multiply row two by $\frac{2}{7}$.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Extended Version of Augmented Matrices (first strategy)

x	y	e_1	e_2	
4	2	1	0	Multiply row one by $\frac{1}{4}$.
1	4	0	1	
1	$\frac{1}{2}$	$\frac{1}{4}$	0	
1	4	0	1	Subtract row one from row two.
1	$\frac{1}{2}$	$\frac{1}{4}$	0	
0	$\frac{7}{2}$	$-\frac{1}{4}$	1	Multiply row two by $\frac{2}{7}$.
1	$\frac{1}{2}$	$\frac{1}{4}$	0	Subtract $\frac{1}{2}$(row two) from row one.
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Augmented Matrices (second strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

$$\left(\frac{7}{2}\sigma_x + \frac{1}{4}e_y\right)x + \quad 1\,e_y \quad y = \sigma_x \underset{\text{or}}{\mp} -\frac{1}{2}\sigma_x + \frac{1}{4}e_y$$

$$1\,e_x \quad x + \quad 1\,e_y \quad y = \frac{2}{7}e_x - \frac{1}{14}e_y \underset{\text{or}}{\mp} -\frac{1}{7}e_x + \frac{2}{7}e_y$$

are the elements of the augmented matrices:

x	y	e_1	e_2	
4	2	1	0	Subtract $\frac{1}{2}$(row two) from row one.
1	4	0	1	Multiply row two by $\frac{1}{4}$.
$\frac{7}{2}$	0	1	$-\frac{1}{2}$	Multiply row one by $\frac{2}{7}$.
$\frac{1}{4}$	1	0	$\frac{1}{4}$	Subtract $\frac{1}{14}$(row one) from row two.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

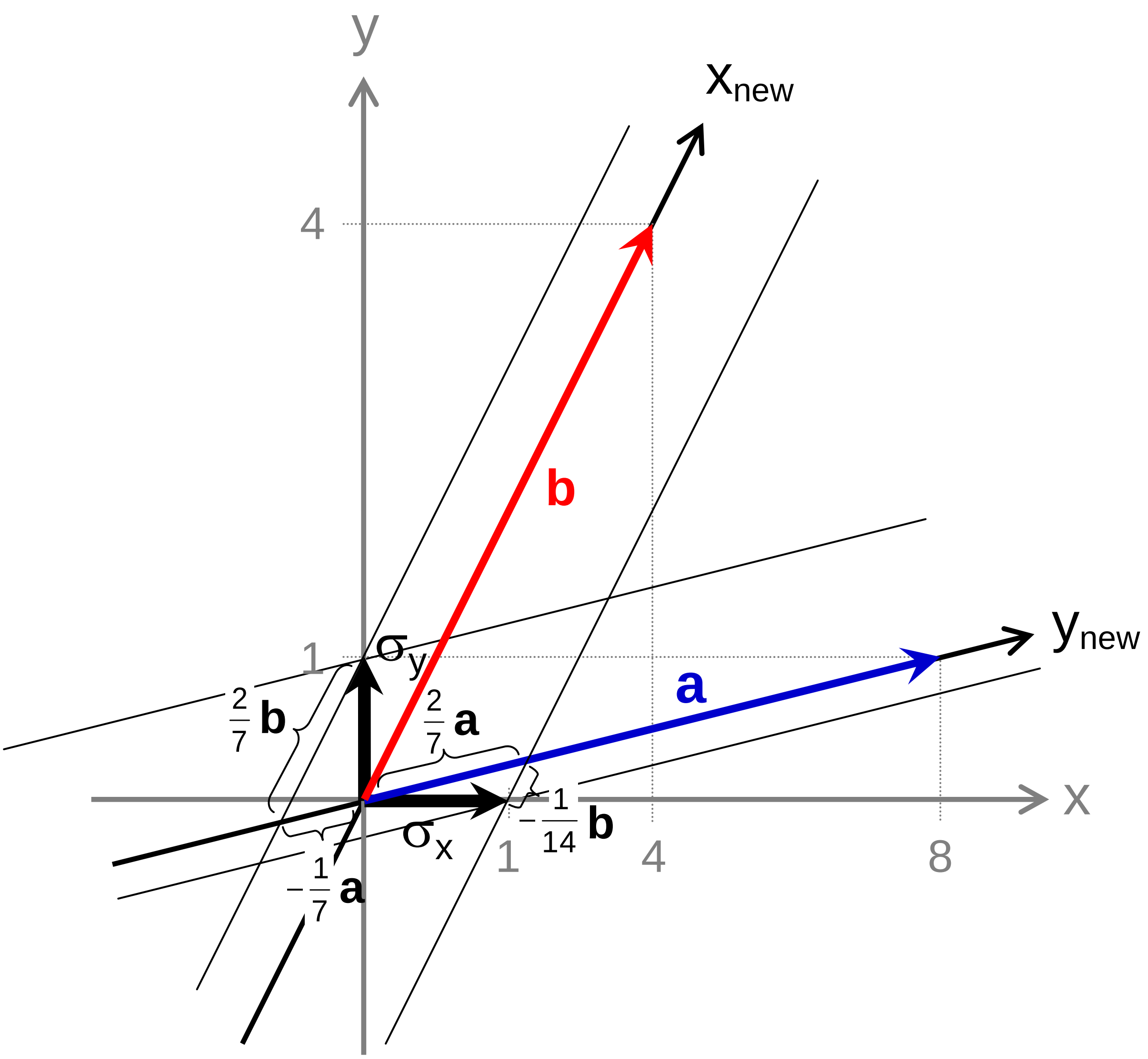
$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Extended Version of Augmented Matrices (second strategy)

x	y	e_1	e_2	
4	2	1	0	
1	4	0	1	Multiply row two by $\frac{1}{4}$.
4	2	1	0	Subtract 2 times row two from row one.
$\frac{1}{4}$	1	0	$\frac{1}{4}$	
$\frac{7}{2}$	0	1	$-\frac{1}{2}$	Multiply row one by $\frac{2}{7}$.
$\frac{1}{4}$	1	0	$\frac{1}{4}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
$\frac{1}{4}$	1	0	$\frac{1}{4}$	Subtract $\frac{1}{4}$(row one) from row two.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Diagram of the Final Situation (third & forth strategy)



The final algebraic equation equals:

$$e_y\, x + e_x\, y = -\frac{1}{14} e_x + \frac{2}{7} e_y \underset{\text{or}}{=} \frac{2}{7} e_x - \frac{1}{7} e_y$$

But the interpretation of this equation is not so easily found.

Augmented Matrices (third strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

$$1\,e_y\,x + (-14\,\sigma_x + 4\,e_y)\,y = \sigma_x \underset{\text{or}}{\mp} -4\,\sigma_x + e_y$$

$$1\,e_y\,x + \qquad 1\,e_x\,y \qquad = -\frac{1}{14}e_x + \frac{2}{7}e_y \underset{\text{or}}{\mp} \frac{2}{7}e_x - \frac{1}{7}e_y$$

are the elements of the augmented matrices:

x	y	e_1	e_2	
4	2	1	0	Subtract 4 times row two from row one.
1	4	0	1	
0	–14	1	–4	Divide row one by (–14).
1	4	0	1	Add $\frac{2}{7}$(row one) to row two.
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	

$$\Rightarrow \quad \mathbf{A}{\updownarrow}^{-1} = \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$$

Matrix $\mathbf{A}{\updownarrow}^{-1}$ isn't the inverse of matrix **A** yet:

$$\mathbf{A}{\updownarrow}^{-1} \neq \mathbf{A}^{-1}$$

A last step still is missing.

Augmented Matrices (third strategy)

Compared to the inverse $\mathbf{A}^{-1}$, first and second row of matrix $\mathbf{A}\updownarrow^{-1}$ are interchanged. To change the x-direction into a new y-direction and the y-direction into a new x-direction at the same time, matrix $\mathbf{A}^{-1}$ has to be multiplied with the permutation matrix from the left side:

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{A}\updownarrow^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

The complete table of augmented matrices then is:

x	y	e_1	e_2	
4	2	1	0	Subtract 4 times row two from row one.
1	4	0	1	
0	-14	1	-4	Divide row one by (-14).
1	4	0	1	Add $\frac{2}{7}$(row one) to row two.
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	Interchange row one and row two.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	$\Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$

Extended Version of Augmented Matrices (third strategy)

x	y	e_1	e_2	
4	2	1	0	Subtract 4 times row two from row one.
1	4	0	1	
0	–14	1	–4	Divide row one by (–14).
1	4	0	1	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	
1	4	0	1	Subtract 4 times row one from row two.
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	Interchange row one and row two.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Augmented Matrices (fourth strategy)

The coefficients of the Pauli vector equations

$$(4\,\sigma_x + \sigma_y)\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = \sigma_x \underset{\text{or}}{\mp} \sigma_y$$

$$(2\,e_x - 7\,\sigma_y)\,x + \quad 1\,e_x\,y \quad = \frac{1}{2}\,e_x - 2\,\sigma_y \underset{\text{or}}{\mp} \sigma_y$$

$$1\,e_y\,x \quad + \quad 1\,e_x\,y \quad = -\frac{1}{14}\,e_x + \frac{2}{7}\,e_y \underset{\text{or}}{\mp} \frac{2}{7}\,e_x - \frac{1}{7}\,e_y$$

are the elements of the augmented matrices:

x	y	e_1	e_2	
4	2	1	0	Divide row one by 2.
1	4	0	1	Subtract 2 times row one from row two.
2	1	$\frac{1}{2}$	0	Add $\frac{2}{7}$(row two) to row one.
-7	0	-2	1	Divide row two by (-7).
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	Interchange row one and row two.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Extended Version of Augmented Matrices (fourth strategy)

x	y	e_1	e_2	
4	2	1	0	Divide row one by 2.
1	4	0	1	
2	1	$\frac{1}{2}$	0	
1	4	0	1	Subtract 4 times row one from row two.
2	1	$\frac{1}{2}$	0	
-7	0	-2	1	Divide row two by (-7).
2	1	$\frac{1}{2}$	0	Subtract 2 times row two from row one.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	Interchange row one and row two.
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
1	0	$\frac{2}{7}$	$-\frac{1}{7}$	
0	1	$-\frac{1}{14}$	$\frac{2}{7}$	

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

Result

Original matrix: $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$

Inverse matrix: $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$

Check of the Inverse Matrix

To check the result, the original matrix **A** should be multiplied by the inverse matrix $\mathbf{A}^{-1}$ (or the inverse matrix $\mathbf{A}^{-1}$ should be multiplied by the original matrix **A**).

$$\begin{array}{cc|cc} & & \frac{2}{7} & -\frac{1}{7} \\ & & -\frac{1}{14} & \frac{2}{7} \\ \hline 4 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array} \qquad \begin{array}{cc|cc} & & 4 & 2 \\ & & 1 & 4 \\ \hline \frac{2}{7} & -\frac{1}{7} & 1 & 0 \\ -\frac{1}{14} & \frac{2}{7} & 0 & 1 \end{array}$$

$\Rightarrow$ The inverse matrix is correct.

Solution of the System of Linear Equations

Now the solution of the system of two linear equations we took as an example can be found:

$$\begin{aligned} 4\,x + 2\,y &= 14 \\ x + 4\,y &= 14 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

solution vector of the system of linear equations

$$\begin{array}{cc|c} & & 14 \\ & & 14 \\ \hline \frac{2}{7} & -\frac{1}{7} & 2 \\ -\frac{1}{14} & \frac{2}{7} & 3 \end{array}$$

$\Rightarrow$ solution values:

$x = 2$

$y = 3$

Interpretation of the Inverse Matrix

Original matrix: $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$

The original matrix can be interpreted as a set of two coefficient vectors:

$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \longrightarrow a = 4\,\sigma_x + \sigma_y$$

$$b = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \longrightarrow b = 2\,\sigma_x + 4\,\sigma_y$$

Inverse matrix: $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$

The inverse matrix can be interpreted as a set of two other coefficient vectors:

$$\sigma_x = \begin{bmatrix} 4/14 \\ -1/14 \end{bmatrix} \longrightarrow \sigma_x = \frac{4}{14}a - \frac{1}{14}b$$

$$\sigma_y = \begin{bmatrix} -2/14 \\ 4/14 \end{bmatrix} \longrightarrow \sigma_y = -\frac{2}{14}a + \frac{4}{14}b$$

Interpretation of the Inverse Matrix

If we view the Gaussian method as a transformation of coordinates, the original matrix **A** is equal to the set of new frame vectors*

$$a = e_x \qquad\qquad b = e_y$$

The elements of **A** then are the coefficients of these new frame vectors* expressed as linear combinations of the base vectors σ_x, σ_y of the old coordinate system.

And the inverse matrix $\mathbf{A}^{-1}$ is equal to the set of old base vectors

$$\sigma_x \qquad\qquad \sigma_y$$

The elements of $\mathbf{A}^{-1}$ then are the coefficients of the old base vectors expressed as linear combinations of the frame vectors* a, b of the new coordinate system.

* Base vectors usually are supposed to be orthonormal. As vectors $a = e_x$ and $b = e_y$ are not orthogonal, they are called frame vectors of the new coordinate system with oblique coordinate axes.

Finding the Lag Matrix

The Gaussian method of finding the inverse matrix can be generalized into a method of finding the lag matrix **B** of a matrix product **A B** = **D**.

Example Problem

A firm manufactures two different types of final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

4 units of R_1 and 1 unit of R_2 are required to produce one unit of the first final product P_1.
2 units of R_1 and 4 units of R_2 are required to produce one unit of the second final product P_2.

In the first quarter 140 units of the first raw material R_1 and 140 units of the second raw material R_2 had been used up.
In the second quarter 210 units of the first raw material R_1 and 350 units of the second raw material R_2 had been used up.
In the third quarter 280 units of the first raw material R_1 and 70 units of the second raw material R_2 had been used up.

Find the production matrix which shows how many units of each final product were produced in the first, second, and third quarter.

Example Problem

Demand matrices $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$ and

$$\mathbf{D} = \begin{array}{c} \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{1st quarter} & \text{2nd quarter} & \text{3rd quarter} \end{array} \end{array}$$

are given.

The production matrix $\mathbf{B} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ is unknown.

The matrix equation (see scheme of Falk)

$$\begin{array}{cc|ccc} & & x_1 & x_2 & x_3 \\ & & y_1 & y_2 & y_3 \\ \hline 4 & 2 & 140 & 210 & 280 \\ 1 & 4 & 140 & 350 & 70 \end{array}$$

can now be split into three separate systems of linear equations:

$$\begin{array}{cc|c} & & x_1 \\ & & y_1 \\ \hline 4 & 2 & 140 \\ 1 & 4 & 140 \end{array} \qquad \begin{array}{cc|c} & & x_2 \\ & & y_2 \\ \hline 4 & 2 & 210 \\ 1 & 4 & 350 \end{array} \qquad \begin{array}{cc|c} & & x_3 \\ & & y_3 \\ \hline 4 & 2 & 280 \\ 1 & 4 & 70 \end{array}$$

Diagram of the Starting Point

We are now looking for a representation of the three vectors of constants

$$d_1 = 140\,\sigma_x + 140\,\sigma_y$$
$$d_2 = 210\,\sigma_x + 350\,\sigma_y$$
$$d_3 = 280\,\sigma_x + 70\,\sigma_y$$

as linear combinations of the coefficient vectors

$$a = 4\,\sigma_x + \sigma_y$$

and $b = 2\,\sigma_x + 4\,\sigma_y$

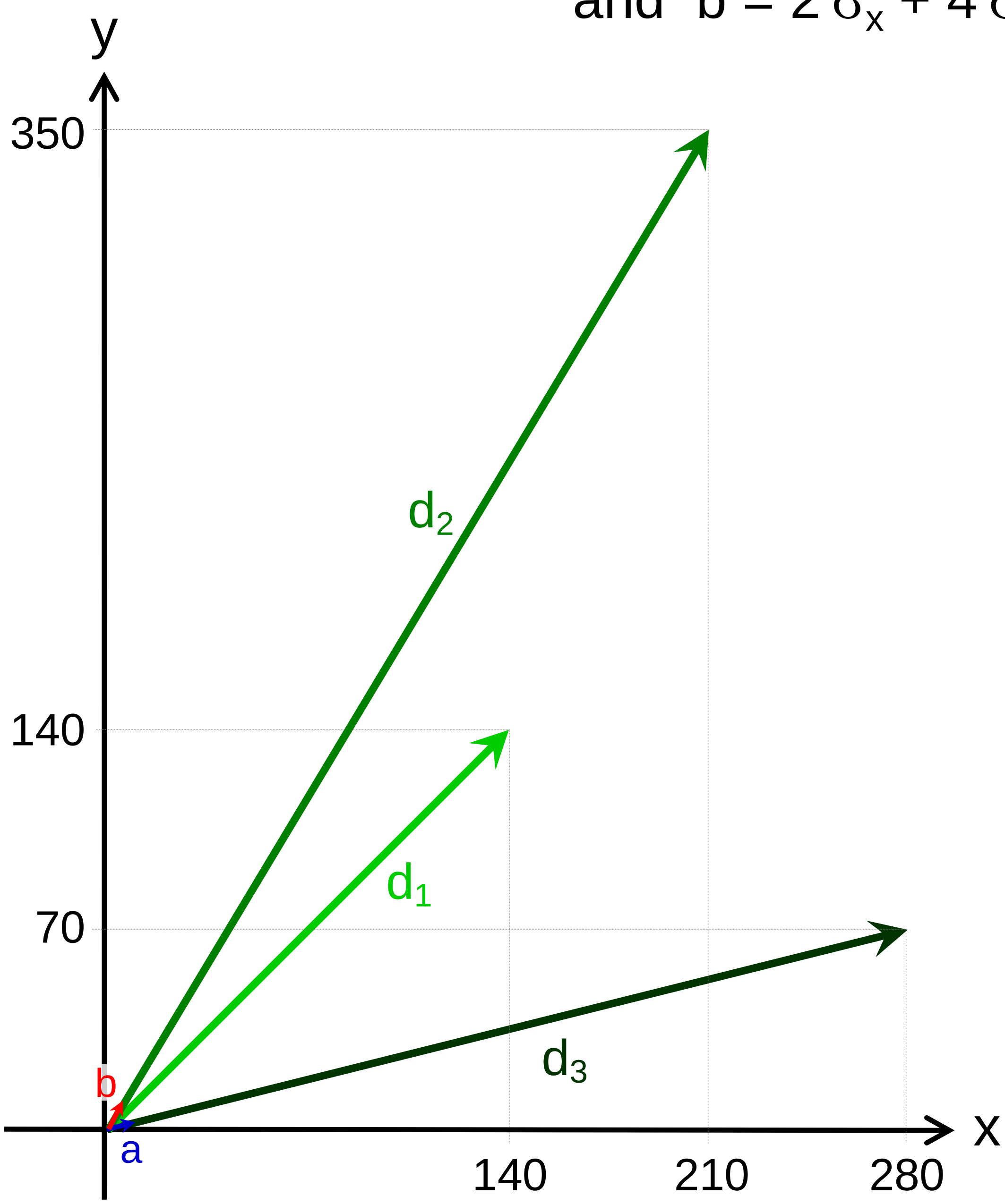
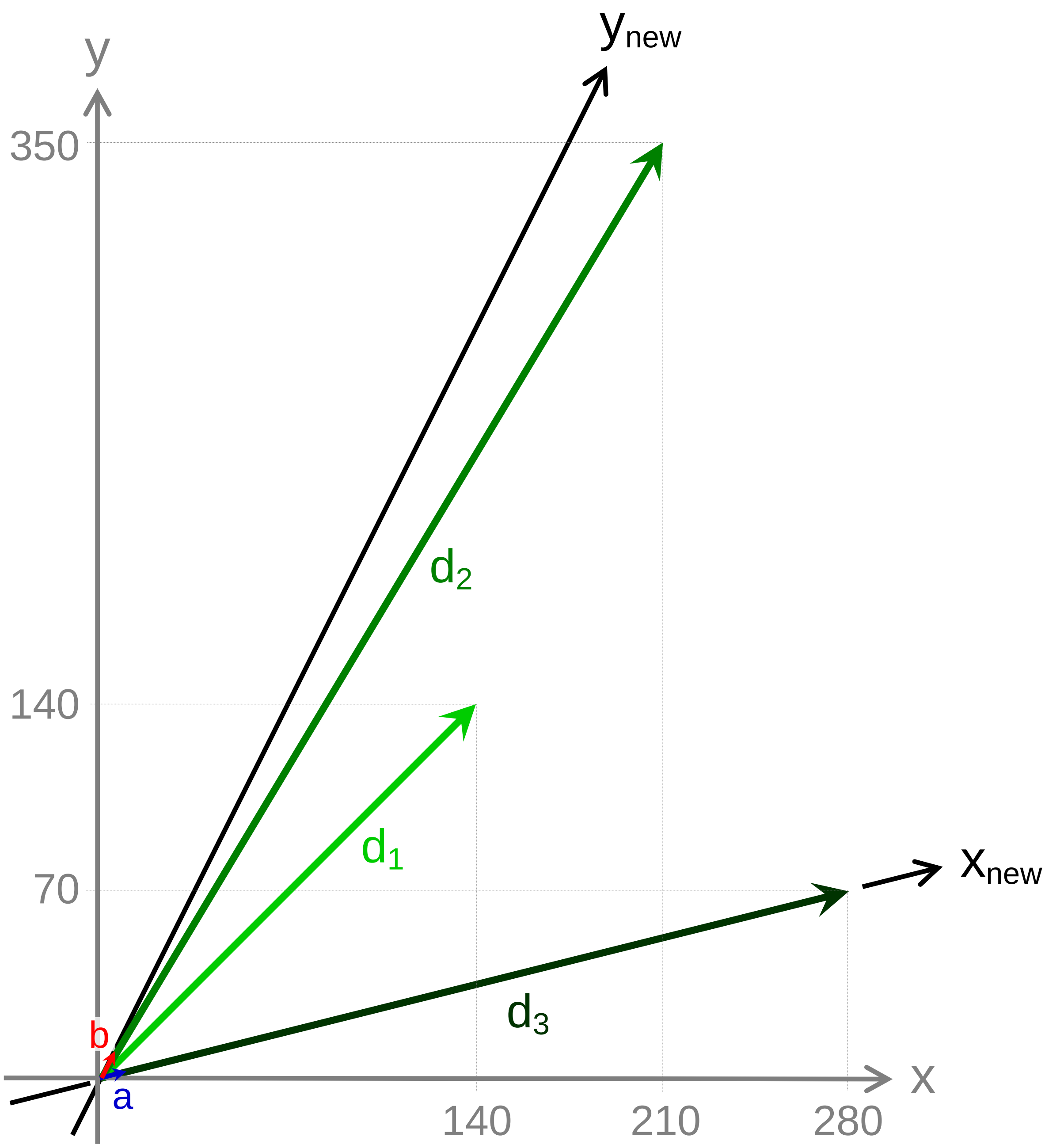


Diagram of the Final Situation



As the third vector of constants $d_3 = 280\,\sigma_x + 70\,\sigma_y$ is parallel to coefficient vector $a = e_x = 4\,\sigma_x + \sigma_y$, we immediately know that $y_3 = 0$.

Transformation of Coordinates (first strategy)

The three systems of linear equations result in the following Pauli vector equations:

$$\underbrace{(4\,\sigma_x + \sigma_y)}\,x + (2\,\sigma_x + 4\,\sigma_y)\,y = 140\,\sigma_x + 140\,\sigma_y \underset{\text{or}}{\mp} 210\,\sigma_x + 350\,\sigma_y \underset{\text{or}}{\mp} 280\,\sigma_x + 70\,\sigma_y$$

$$e_x = 4\,\sigma_x + \sigma_y \quad\Rightarrow\quad \sigma_x = 0.25\,e_x - 0.25\,\sigma_y \quad\Leftrightarrow\quad \text{change of x-coordinates}$$

$$\downarrow$$

$$e_x\,x + \underbrace{(0.5\,e_x + 3.5\,\sigma_y)}\,y = 35\,e_x + 105\,\sigma_y \underset{\text{or}}{\mp} 52.5\,e_x + 297.5\,\sigma_y \underset{\text{or}}{\mp} 70\,e_x$$

$$e_y = 0.5\,e_x + 3.5\,\sigma_y \quad\Rightarrow\quad \sigma_y = -\,1/7\,e_x + 2/7\,e_y \quad\Leftrightarrow\quad \text{change of y-coordinates}$$

$$\downarrow$$

$$e_x\,x + e_y\,y = 20\,e_x + 30\,e_y \underset{\text{or}}{\mp} 10\,e_x + 85\,e_y \underset{\text{or}}{\mp} 70\,e_x + 0\,e_y$$

$\Rightarrow$ Results: $x_1 = 20 \quad y_1 = 30 \quad x_2 = 10 \quad y_2 = 85 \quad x_3 = 70 \quad y_3 = 0$

$\Rightarrow$ 20 units of the first final product P_1 and 30 units of the second final product P_2 were produced in the first quarter. 10 units of the first final product P_1 and 85 units of the second final product P_2 were produced in the second quarter. 70 units of the first final product P_1 were produced in the third quarter.

Transformation of Coordinates (second strategy)

The three systems of linear equations result in the following Pauli vector equations:

$$(4\,\sigma_x + \sigma_y)\,x + \underbrace{(2\,\sigma_x + 4\,\sigma_y)}\,y = 140\,\sigma_x + 140\,\sigma_y \underset{\text{or}}{\mp} 210\,\sigma_x + 350\,\sigma_y \underset{\text{or}}{\mp} 280\,\sigma_x + 70\,\sigma_y$$

$$e_y = 2\,\sigma_x + 4\,\sigma_y \quad \Rightarrow \quad \sigma_y = -\,0.5\,\sigma_x + 0.25\,e_y \quad \Leftrightarrow$$ change of y-coordinates

$$\underbrace{(3.5\,\sigma_x + 0.25\,e_y)}\,x + e_y\,y = 70\,\sigma_x + 35\,e_y \underset{\text{or}}{\mp} 35\,\sigma_x + 87.5\,e_y \underset{\text{or}}{\mp} 245\,\sigma_x + 17.5\,e_x$$

$$e_x = 3.5\,\sigma_x + 0.25\,e_y \quad \Rightarrow \quad \sigma_x = 2/7\,e_x - 1/14\,e_y \quad \Leftrightarrow$$ change of y-coordinates

$$e_x\,x + e_y\,y = 20\,e_x + 30\,e_y \underset{\text{or}}{\mp} 10\,e_x + 85\,e_y \underset{\text{or}}{\mp} 70\,e_x + 0\,e_y$$

$\Rightarrow$ Results: $x_1 = 20 \quad y_1 = 30 \quad x_2 = 10 \quad y_2 = 85 \quad x_3 = 70 \quad y_3 = 0$

$\Rightarrow$ 20 units of the first final product P_1 and 30 units of the second final product P_2 were produced in the first quarter. 10 units of the first final product P_1 and 85 units of the second final product P_2 were produced in the second quarter. 70 units of the first final product P_1 were produced in the third quarter.

Augmented Matrices (first strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

x	y	d_1	d_2	d_3	
4	2	140	210	280	Multiply row one by $\frac{1}{4}$.
1	4	140	350	70	Subtract $\frac{1}{4}$(row one) from row two.
1	0.5	35	52.5	70	Subtract $\frac{1}{7}$(row two) from row one.
0	3.5	105	297.5	0	Multiply row two by $\frac{2}{7}$.
1	0	20	10	70	
0	1	30	85	0	

$$\Rightarrow \quad B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$$

Augmented Matrices (second strategy)

x	y	d_1	d_2	d_3	
4	2	140	210	280	Subtract $\frac{1}{2}$(row two) from row one.
1	4	140	350	70	Multiply row two by $\frac{1}{4}$.
3.5	0	70	35	245	Multiply row one by $\frac{2}{7}$.
0.25	1	35	87.5	17.5	Subtract $\frac{1}{14}$(row one) from row two.
1	0	20	10	70	
0	1	30	85	0	

$$\Rightarrow \quad B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$$

Augmented Matrices (third strategy)

x	y	d_1	d_2	d_3	
4	2	140	210	280	Subtract 4 times row two from row one.
1	4	140	350	70	
0	–14	–420	–1190	0	Divide row one by (–14).
1	4	140	350	70	Add $\frac{2}{7}$(row one) to row two.
0	1	30	85	0	Interchange row one and row two.
1	0	20	10	70	
1	0	20	10	70	
0	1	30	85	0	

$$\Rightarrow \quad B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$$

Augmented Matrices (fourth strategy)

x	y	d_1	d_2	d_3	
4	2	140	210	280	Divide row one by 2.
1	4	140	350	70	Subtract 2 times row one from row two.
2	1	70	105	140	Add $\frac{2}{7}$(row two) to row one.
–7	0	–140	–70	–490	Divide row two by (–7).
0	1	30	85	0	Interchange row one and row two.
1	0	20	10	70	
1	0	20	10	70	
0	1	30	85	0	

$$\Rightarrow \quad B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$$

Result

If $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix}$

are part of matrix equation $\mathbf{A}\,\mathbf{B} = \mathbf{D}$, then the unknown lag matrix will be $\mathbf{B} = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$.

Check of Result

$$
\begin{array}{cc|ccc}
 & & 20 & 10 & 70 \\
 & & 30 & 85 & 0 \\
\hline
4 & 2 & 140 & 210 & 280 \\
1 & 4 & 140 & 350 & 70
\end{array}
$$

$\Rightarrow$ The lag matrix is correct.

Finding the Lead Matrix

In the following problem the same values will be used to show how the lead matrix can be found.

Finding the Lead Matrix

Now matrix equation **A B** = **D** is given and matrices **B** and **D** are known, while matrix **A** has to be found.

Example Problem

A firm manufactures two different types of final products P_1 and P_2. To produce these products two different raw materials R_1 and R_2 are required.

In the first quarter 20 units of the first final product P_1 and 30 units of the second final product P_2 were produced.
In the second quarter 10 units of the first final product P_1 and 85 units of the second final product P_2 were produced.
In the third quarter 70 units of the first final product P_1 and no unit of the second final product P_2 were produced.

To produce these quantities of final products, the following quantities of raw materials had been used up:

In the first quarter 140 units of the first raw material R_1 and 140 units of the second raw material R_2 had been used up.
In the second quarter 210 units of the first raw material R_1 and 350 units of the second raw material R_2 had been used up.
In the third quarter 280 units of the first raw material R_1 and 70 units of the second raw material R_2 had been used up.

Find the demand matrix which shows how many units of each raw material R_1 and R_2 are required to produce one unit of each final product.

Finding the Lead Matrix

Now demand matrix $\mathbf{A} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ is unknown,

while the production matrix $\mathbf{B} = \begin{array}{ccc} \left[\begin{array}{ccc} 20 & 10 & 70 \\ 30 & 85 & 0 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{1st quarter} & \text{2nd quarter} & \text{3rd quarter} \end{array} \end{array}$

and the matrix which shows the total demand of raw materials are given.

$$\mathbf{D} = \begin{array}{c} \left[\begin{array}{ccc} 140 & 210 & 280 \\ 140 & 350 & 70 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{1st quarter} & \text{2nd quarter} & \text{3rd quarter} \end{array} \end{array}$$

The unknown matrix A is the lead matrix of matrix product

$$\mathbf{A}\,\mathbf{B} = \mathbf{D}$$

But the solution strategies discussed in previous slides only show how a lag matrix can be found. Therefore the unknown lead matrix should be transformed into a lag matrix by transposing the matrix product.

$$\mathbf{B}^{T}\,\mathbf{A}^{T} = \mathbf{D}^{T}$$

Transposing a matrix product changes the of the matrices.

Transposition of Matrices

Transpose matrices can be found by converting columns of the original matrices into rows and rows of the original matrices into columns:

$$\mathbf{B} = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix} \longrightarrow \mathbf{B}^T = \begin{bmatrix} 20 & 30 \\ 10 & 85 \\ 70 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix} \longrightarrow \mathbf{D}^T = \begin{bmatrix} 140 & 140 \\ 210 & 350 \\ 280 & 70 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \longrightarrow \mathbf{A}^T = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$\uparrow \quad \uparrow$$

Now the subscripts of the elements are in accordance with the column vector notation again.

Finding the Lead Matrix

The transposed matrix equation $\mathbf{B}^T \mathbf{A}^T = \mathbf{D}^T$ (see scheme of Falk)

			x_1	x_2
			y_1	y_2
20	30		140	140
10	85		210	350
70	0		280	70

can now be split into two separate systems of three linear equations:

			x_1
			y_1
20	30		140
10	85		210
70	0		280

			x_2
			y_2
20	30		140
10	85		350
70	0		70

Pauli Vectors in Three-dimensional Space

Coefficient vectors and the two vectors of constants will be transferred again into Pauli vectors. But this time each system of linear equations consists of three linear equations.

Therefore the Pauli vectors will have three components, and three different base vectors σ_x, σ_y, and σ_z are required.

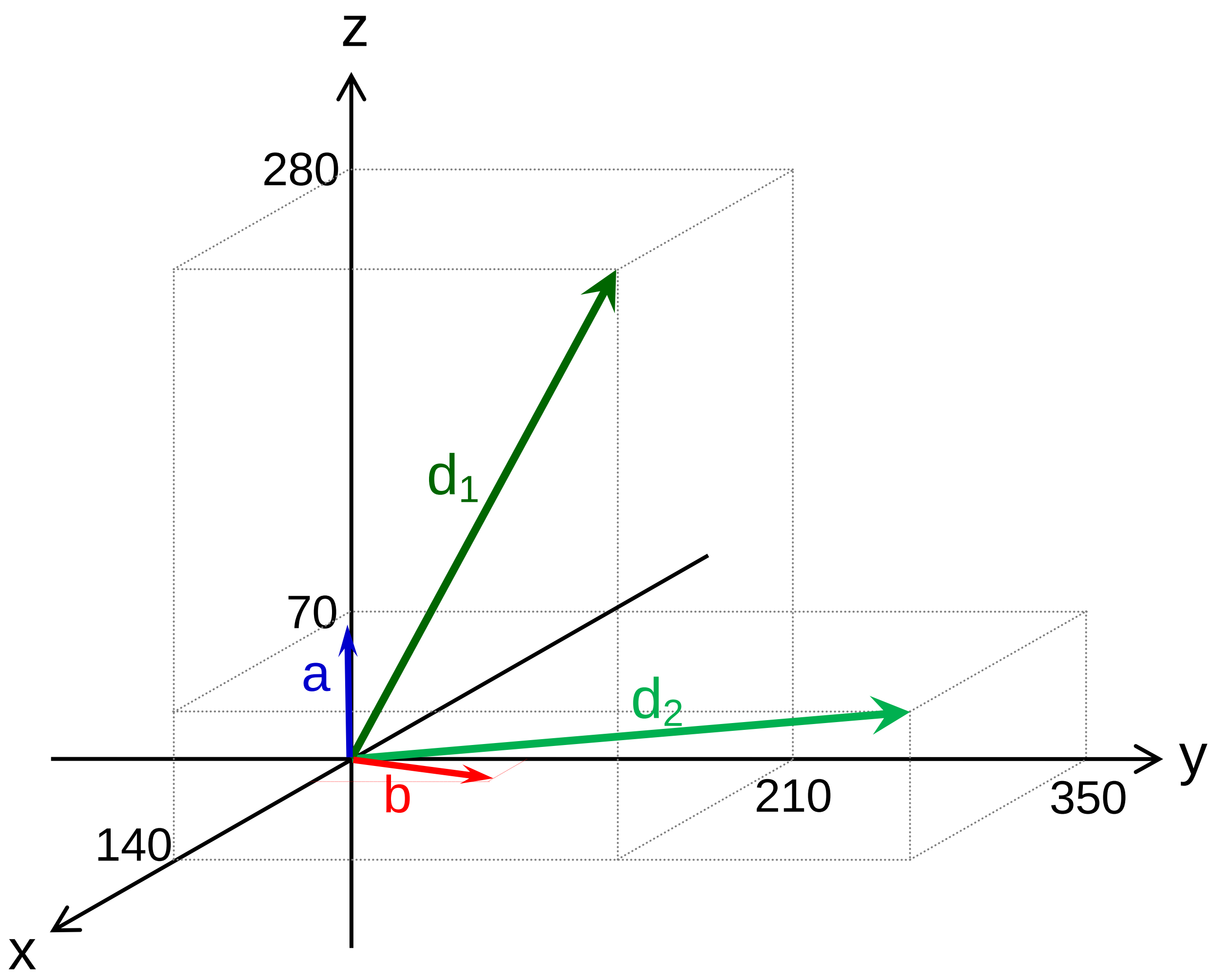
$$a = \begin{bmatrix} 20 \\ 10 \\ 70 \end{bmatrix} \longrightarrow a = 20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z$$

$$b = \begin{bmatrix} 30 \\ 85 \\ 0 \end{bmatrix} \longrightarrow b = 30\,\sigma_x + 85\,\sigma_y + 0\,\sigma_z$$

$$d_1 = \begin{bmatrix} 140 \\ 210 \\ 280 \end{bmatrix} \longrightarrow d_1 = 140\,\sigma_x + 210\,\sigma_y + 280\,\sigma_z$$

$$d_2 = \begin{bmatrix} 140 \\ 350 \\ 70 \end{bmatrix} \longrightarrow d_2 = 140\,\sigma_x + 350\,\sigma_y + 70\,\sigma_z$$

Diagram of the Starting Point



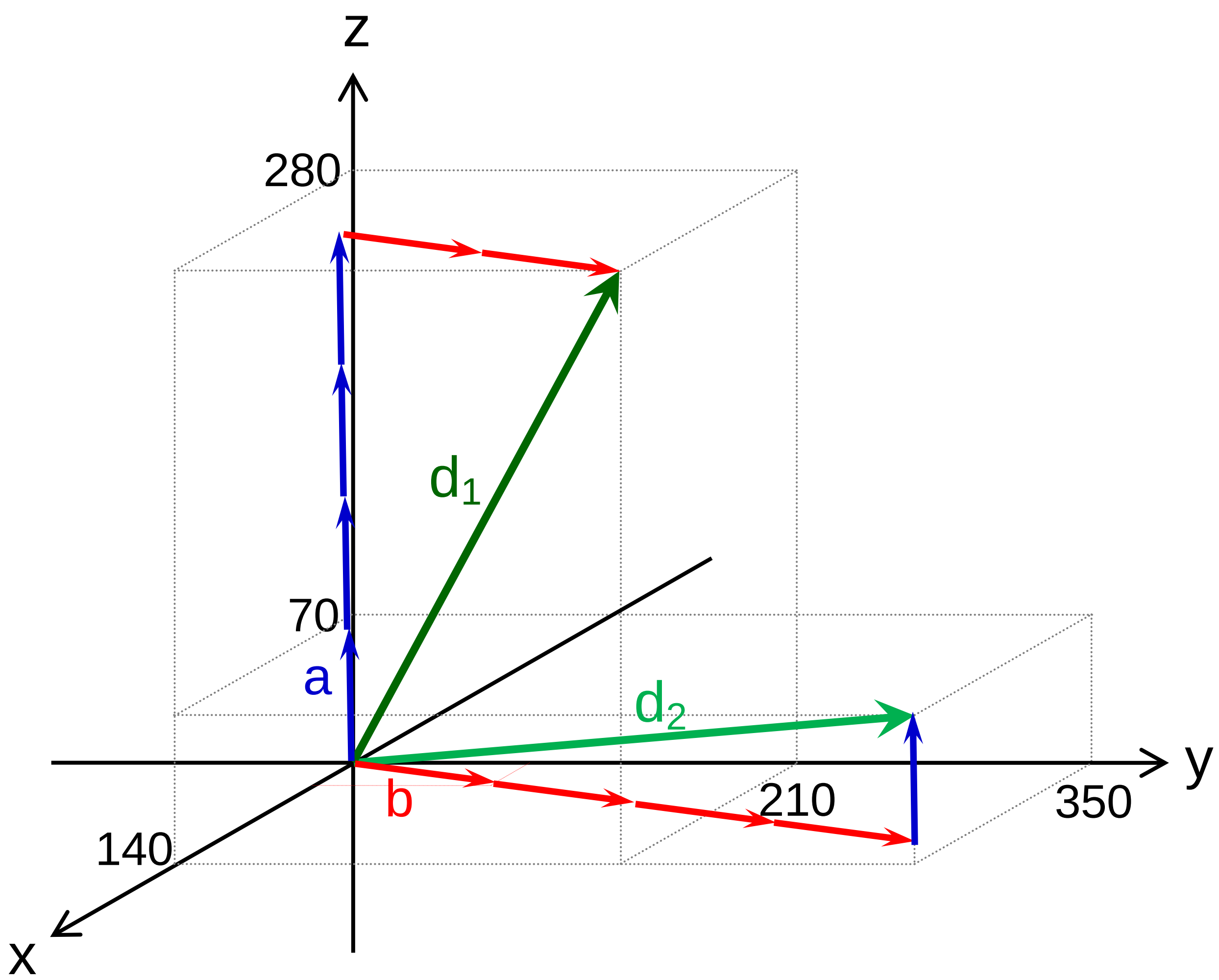
The algebraic starting point is:

$$\underbrace{(20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z)}_{\textbf{a}}\ x + \underbrace{(30\,\sigma_x + 85\,\sigma_y)}_{\textbf{b}}\ y = d_1 \underset{\text{or}}{\mp} d_2$$

$$= 140\,\sigma_x + 210\,\sigma_y + 280\,\sigma_z$$

$$\underset{\text{or}}{\mp}\ 140\,\sigma_x + 350\,\sigma_y + 70\,\sigma_z$$

Diagram of the Solution



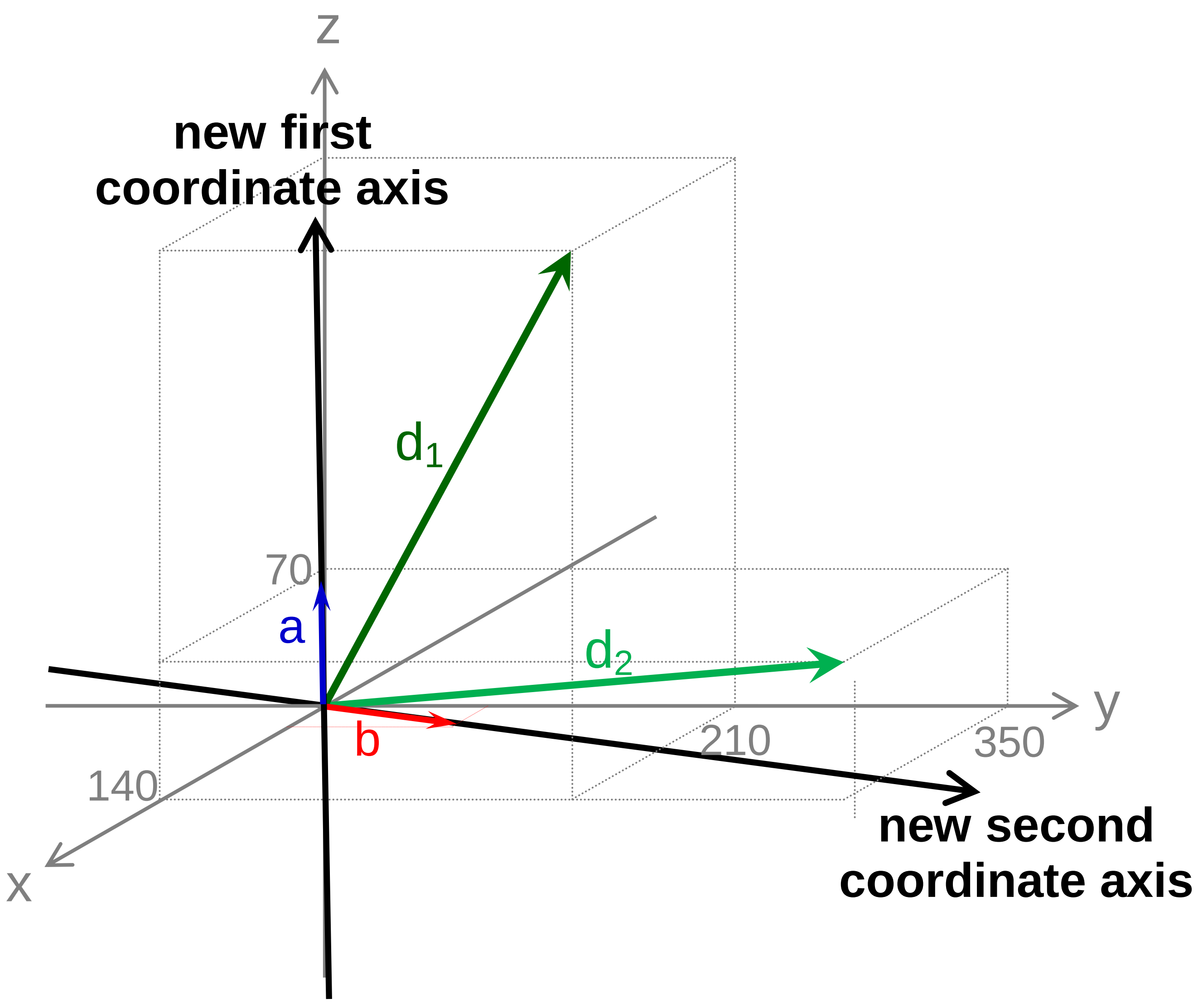
As we already know the solution of this system of linear equations is given by

$$4\,\mathbf{a} + 2\,\mathbf{b} = d_1$$

and $$1\,\mathbf{a} + 4\,\mathbf{b} = d_2$$

The solution can be found with appropriate transformations of the coordinates.

Diagram of the Transformation of Coordinates



Again there are several different strategies to implement the transformation of coordinates:

> The first new coordinate axis (and equally the second new axis) can be considered as a new x-axis or as a new y-axis or as a new z-axis.

Transformation of Coordinates (first strategy)

The two systems of linear equations result in the following Pauli vector equations:

$$\underbrace{(20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z)}_{e_x}\,x + (30\,\sigma_x + 85\,\sigma_y)\,y$$
$$= 140\,\sigma_x + 210\,\sigma_y + 280\,\sigma_z$$
$$\underset{\text{or}}{\mp}\ 140\,\sigma_x + 350\,\sigma_y + 70\,\sigma_z$$

Change of x-coordinates:

$$e_x = 20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z$$
$$\Rightarrow \quad \sigma_x = 0.05\,e_x - 0.5\,\sigma_y - 3.5\,\sigma_z$$

$$e_x\,x + \underbrace{(1.5\,e_x + 70\,\sigma_y - 105\,\sigma_z)}_{e_y}\,y = 7\,e_x + 140\,\sigma_y - 210\,\sigma_z$$
$$\underset{\text{or}}{\mp}\ 7\,e_x + 280\,\sigma_y - 420\,\sigma_z$$

Change of y-coordinates:

$$e_y = 1.5\,e_x + 70\,\sigma_y - 105\,\sigma_z$$
$$\Rightarrow \quad \sigma_y = -3/140\,e_x + 1/70\,e_y + 3/2\,\sigma_y$$

$$e_x\,x + e_y\,y = 4\,e_x + 2\,e_y + 0\,\sigma_z \mp 1\,e_x + 4\,e_y + 0\,\sigma_z$$

$\Rightarrow$ Results: $x_1 = 4 \quad y_1 = 2 \qquad x_2 = 1 \quad y_2 = 4$

$\Rightarrow$ 4 units of the first raw material R_1 and 1 unit of the second raw material R_2 are required to produce one unit of the first final product P_1. 1 unit of the first raw material R_1 and 4 units of the second raw material R_2 are required to produce one unit of the second final product P_2.

Transformation of Coordinates (second strategy)

The two systems of linear equations result in the following Pauli vector equations:

$$\underbrace{(20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z)}_{e_y}\,x + (30\,\sigma_x + 85\,\sigma_y)\,y$$

$$= 140\,\sigma_x + 210\,\sigma_y + 280\,\sigma_z$$

$$\underset{\text{or}}{=} 140\,\sigma_x + 350\,\sigma_y + 70\,\sigma_z$$

Change of y-coordinates:

$$e_y = 20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z$$

$$\Rightarrow \quad \sigma_y = -2\,\sigma_x + 0.1\,e_y - 7\,\sigma_z$$

$$e_y\,x + \underbrace{(-140\,\sigma_x + 8.5\,e_y - 595\,\sigma_z)}_{e_z}\,y = -280\,\sigma_x + 21\,e_y - 1190\,\sigma_z$$

$$\underset{\text{or}}{=} -560\,\sigma_x + 35\,e_y - 2380\,\sigma_z$$

Change of z-coordinates:

$$e_z = -140\,\sigma_x + 8.5\,e_y - 595\,\sigma_z$$

$$\Rightarrow \quad \sigma_z = -4/17\,\sigma_x + 1/70\,e_y - 1/595\,e_z$$

$$e_y\,x + e_z\,y = 0\,\sigma_x + 4\,e_y + 2\,e_z \underset{\text{or}}{=} 0\,\sigma_x + 1\,e_y + 4\,e_z$$

$\Rightarrow$ Results: $x_1 = 4 \quad y_1 = 2 \qquad x_2 = 1 \quad y_2 = 4$

$\Rightarrow$ 4 units of the first raw material R_1 and 1 unit of the second raw material R_2 are required to produce one unit of the first final product P_1. 1 unit of the first raw material R_1 and 4 units of the second raw material R_2 are required to produce one unit of the second final product P_2.

Transformation of Coordinates (third strategy)

The two systems of linear equations result in the following Pauli vector equations:

$$\underbrace{(20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z)}_{e_z}\,x + (30\,\sigma_x + 85\,\sigma_y + 0\,\sigma_z)\,y$$
$$= 140\,\sigma_x + 210\,\sigma_y + 280\,\sigma_z$$
$$\underset{\text{or}}{=}\ 140\,\sigma_x + 350\,\sigma_y + 70\,\sigma_z$$

Change of z-coordinates:

$$e_z = 20\,\sigma_x + 10\,\sigma_y + 70\,\sigma_z \qquad \Rightarrow \quad \sigma_z = -\,2/7\,\sigma_x - 1/7\,\sigma_y + 1/70\,e_z$$

$$e_z\,x + \underbrace{(30\,\sigma_x + 85\,\sigma_y)}_{e_x}\,y = 60\,\sigma_x + 170\,\sigma_y + 4\,e_z$$
$$\underset{\text{or}}{=}\ 120\,\sigma_x + 340\,\sigma_y + 1\,e_z$$

Change of x-coordinates:

$$e_x = 30\,\sigma_x + 85\,\sigma_y \qquad \Rightarrow \quad \sigma_x = 1/30\,e_x - 17/6\,\sigma_y$$

$$e_z\,x + e_x\,y = 2\,e_x + 0\,\sigma_y + 4\,e_z \ \underset{\text{or}}{=}\ 4\,e_x + 0\,\sigma_y + 1\,e_z$$

$\Rightarrow$ Results: $\quad y_1 = 2 \qquad x_1 = 4 \quad y_2 = 4 \qquad x_2 = 1$

$\Rightarrow$ 4 units of the first raw material R_1 and 1 unit of the second raw material R_2 are required to produce one unit of the first final product P_1. 1 unit of the first raw material R_1 and 4 units of the second raw material R_2 are required to produce one unit of the second final product P_2.

Augmented Matrices (first strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

x	y	d_1	d_2	
20	30	140	140	Multiply row one by $\frac{1}{20}$.
10	85	210	350	Subtract $\frac{1}{2}$ (row one) from row two.
70	0	280	70	Subtract $\frac{7}{2}$ (row one) from row three.
1	1.5	7	7	Subtract $\frac{3}{140}$ (row two) from row one.
0	70	140	280	Multiply row two by $\frac{1}{70}$.
0	–105	–210	–420	Add $\frac{3}{2}$ (row two) to row three.
1	0	4	1	
0	1	2	4	
0	0	0	0	

$$\mathbf{A}^{T} = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$$

$\Rightarrow$ As expected the unknown lead matrix equals:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$$

Augmented Matrices (second strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

x	y	d_1	d_2	
20	30	140	140	Subtract 2 times row two from row one.
10	85	210	350	Divide row two by 10.
70	0	280	70	Subtract 7 times row two from row three.
0	–140	–280	–560	Subtract $\frac{4}{17}$(row three) from row one.
1	8.5	21	35	Add $\frac{1}{70}$(row three) to row two.
0	–595	–1190	–2380	Divide row one by (–595).
0	0	0	0	
1	0	4	1	
0	1	2	4	

$$\mathbf{A}^T = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$$

$\Rightarrow$ As expected the unknown lead matrix equals:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$$

Augmented Matrices (third strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

x	y	d_1	d_2	
20	30	140	140	Subtract $\frac{2}{7}$(row three) from row one.
10	85	210	350	Subtract $\frac{1}{7}$(row three) from row two.
70	0	280	70	Divide row three by 70.
0	30	60	120	Divide row one by 30.
0	85	170	340	Subtract $\frac{17}{6}$(row one) from row two.
1	0	4	1	
0	1	2	4	$\Rightarrow$ Second row of $\mathbf{A}^T$
0	0	0	0	
1	0	4	1	$\Rightarrow$ First row of $\mathbf{A}^T$

$$\mathbf{A}^T = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$$

$\Rightarrow$ As expected the unknown lead matrix equals:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$$

Overconstrained Systems of Linear Equations

The example above showed how to solve a system of three linear equations with two variables. These systems, which have more equations than variables, are called overconstrained systems of linear equations.

The final augmented matrices of the examples possess a row which has only zeros as elements. This indicates that the mathematical information stored in these rows has not been required to find the solution.

We have been able to find the solution using the first two rows only (first strategy), using the last two rows only (second strategy), or using only the first and last rows (third strategy).

Inconsistent Systems of Linear Equations

Sometimes the equations of a system of linear equations are contradictory. Then no unique solution can be found. These systems of linear equations are called inconsistent.

As an example, the given equations of the problem discussed above are changed into:

		x_1 y_1	x_2 y_2
20	30	140	140
10	85	210	350
70	0	280	70

consistent systems of linear equations

		x_1 y_1	x_2 y_2
20	30	140	140
10	85	210	350
70	0	285	68

inconsistent systems of linear equations

$\Downarrow$

The first two rows still result in the solution of the consistent systems, while the last row cannot be solved with these values.

Inconsistent Systems of Linear Equations

The inconsistency of linear equations is indicated by rows of the augmented matrices which have coefficient values of zero, but values of constants which differ from zero.

x	y	d_1	d_2	
20	30	140	140	Multiply row one by $\frac{1}{20}$.
10	85	210	350	Subtract $\frac{1}{2}$ (row one) from row two.
70	0	285	68	Subtract $\frac{7}{2}$ (row one) from row three.
1	1.5	7	7	Subtract $\frac{3}{140}$ (row two) from row one.
0	70	140	280	Multiply row two by $\frac{1}{70}$.
0	–105	–205	–422	Add $\frac{3}{2}$ (row two) to row three.
1	0	4	1	
0	1	2	4	
0	0	5	–2	

$$\begin{cases} 0\,x + 0\,y = 5 \\ 0\,x + 0\,y = -2 \end{cases}$$

No real numbers x, y can solve these equations.

Multiples of zero can never be different from zero.