More Examples of Non-Square Matrix Inverses

– Extended version of the paper "Inverse von Rechteck-Matrizen" [5]
(Inverses of Rectangular Matrices), written in German for the annual meeting
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English Abstract

In conventional linear algebra the discussion of matrix inverses is usually limited to inverses of square matrices. But Geometric Algebra opens a didactical path to find and to discuss (left-sided) inverses of non-square matrices as well.
It is shown with examples from business mathematics how such inverses of non-square matrices can be constructed and how they can be used to model economic situations.

German Abstract

Die konventionelle Lineare Algebra bezieht sich bei der Diskussion von Inversen in der Regel nur auf quadratische Matrizen. Die Geometrische Algebra eröffnet einen didaktischen Weg, auch (linksseitige) Inverse von Rechteck-Matrizen zu diskutieren.
Anhand von Beispielen aus der Wirtschaftsmathematik wird erörtert, wie solche Inversen von Rechteck-Matrizen konstruiert und zur Modellierung ökonomischer Sachverhalte eingesetzt werden können.

1. Preliminaries

This paper builds on short talks given at the annual meetings of the Society of Mathematics Education (GDM – Gesellschaft für Didaktik der Mathematik) of last year [2] and of this year [5]. It is intended to show that the left-sided inverse of a non-square m x n matrix $\mathbf{A} = (a_{ij})$ with m rows and n columns (m > n) can be found as n x m matrix

$$\mathbf{A}^{-1} = (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n)^{-1} (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_{i-1} \wedge \sigma_j \wedge \mathbf{a}_{i+1} \wedge \ldots \wedge \mathbf{a}_n) \qquad \{1\}$$

with n rows and m columns in a didactical easily accessible way. This approach relies on two essential didactical foundations:

- While at present the interpretation of matrices is mainly based on the row picture in schools and universities, matrix **A** will be interpreted columnwise in the following. Thus Matrix **A** consists of n coefficient vectors $\mathbf{a}_i$.
- Following Geometric Algebra, vectors will be described as linear combinations of generalized Pauli matrices in the following. Thus vector $\mathbf{a}_j$ is written as

  $$\mathbf{a}_j = a_{1j}\,\sigma_1 + a_{2j}\,\sigma_2 + \ldots + a_{mj}\,\sigma_m \qquad \{2\}$$

  There every Pauli matrix σ_i represents a unit vector pointing into the direction of i.

This didactical approach has been implemented successfully in universities of applied sciences with mathematically interested and highly capable students [3] as well as with students who show a more reluctant attitude towards mathematics [6]. Thus Geometric Algebra has proved its worth as a mathematical language to model Linear Algebra in a modern way.

2. Interlude motivated by physics education

As part-time mathematician with roots in physics the author of this paper will follow the Babylonian tradition which Feynman characterizes as inherent of physics and physics education: The old Babylonians didn't know any methods to write down equations. Instead they produced one example after another – That's all! („Die alten Babylonier kannten keine Methode für das Aufschreiben von Formeln. Stattdessen machten Sie ein Beispiel nach dem anderen – das ist alles" [1, p. 70]).

The mathematical approach described in section 1 thus will be presented and discussed in the following by showing and explaining examples – just in the same way, students are taught mathematical strategies in universities of applied sciences.

3. A linear algebra examination problem

At the written examination of learning module M 22 "Mathematics and Statistics" of students studying for a bachelor's degree "Medical Controlling and Management" the following problem was asked:

> To produce one unit of the first final product E_1 5 units of raw material R_1 and 8 units of raw material R_2 are required.
>
> To produce one unit of the second final product E_2 2 units of raw material R_1 and 4 units of raw material R_2 are required.
>
> Find the quantities of the first and second final products E_1 and E_2 which were produced if at the production process exactly 60 units of raw material R_1 and 100 units of raw material R_2 had been consumed.

This problem can be modeled by a matrix equation of the demand matrix **A**, production vector **p**, and the total demand vector of raw materials **r**:

$$\mathbf{r} = \mathbf{A}\,\mathbf{p} = \begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 60 \\ 100 \end{bmatrix} \qquad \{3\}$$

Usually students decide for a direct solution of this problem by calculating the outer products of the coefficient vectors and the resulting vector. An example of such a solution is shown in figures 1 - 5.

First, the student identified the coefficient vectors **a**, **b** and the resulting vector **r**, using the scheme of Falk as a helpful mathematical tool (see fig. 1).

Then the student calculated the geometric product **a b**. The bivector part of this geometric product can be identified with the outer product $\mathbf{a} \wedge \mathbf{b}$, which represents the determinant of matrix **A** (see fig. 2).

After that the geometric products **r b** and **a r** and thereupon the outer products $\mathbf{r} \wedge \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{r}$ were calculated. By comparing these bivectors with the determinant bivector $\mathbf{a} \wedge \mathbf{b}$, the student was able to identify the solutions of x (see fig. 3) and of y (see fig. 4).

Finally, the correct results were checked and an answer was given (see fig. 5). While the mathematics of this solution is correct, the interpretation of the student is faulty and confusing. But such interpretative problems concern the basic analysis of the logical structure of the question which the student is expected to answer. This basic logical analysis should be done by students before any mathematical tool is applied and thus does not depend on the mathematical tool chosen later.

Nearly all students solved the mathematical part of this problem in a correct way, showing that it is possible to teach and discuss linear algebra by using Geometric Algebra as a highly efficient mathematical language.

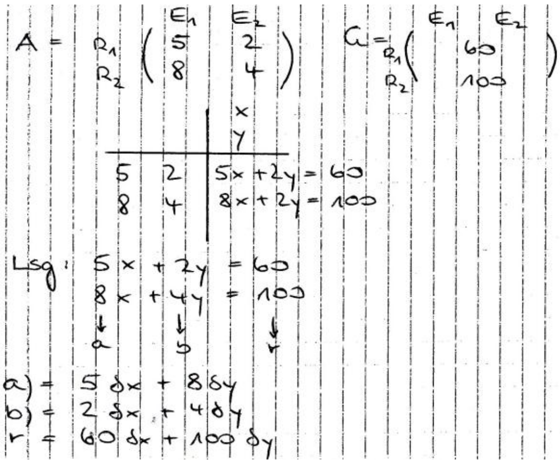

Fig.1: Identification of Pauli vectors **a**, **b**, and **r**.

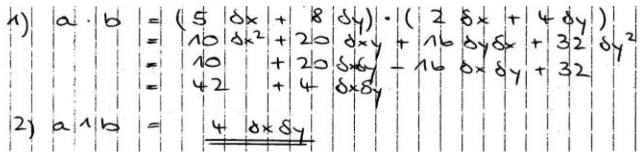

Fig.2: Calculation of the outer product $\mathbf{a} \wedge \mathbf{b}$ which represents the determinant of **A**.

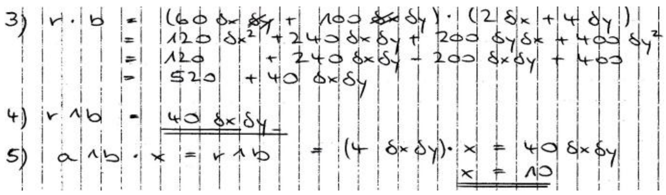

Fig.3: Calculation of outer product $\mathbf{r} \wedge \mathbf{b}$ and comparison of the outer products 2) and 4) to identify the solution of x.

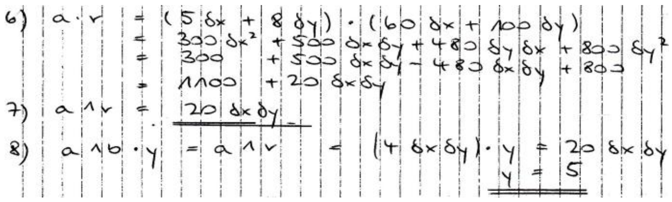

Fig.4: Calculation of outer product $\mathbf{a} \wedge \mathbf{r}$ and comparison of the outer products of 2) and 7) to identify the solution of y.

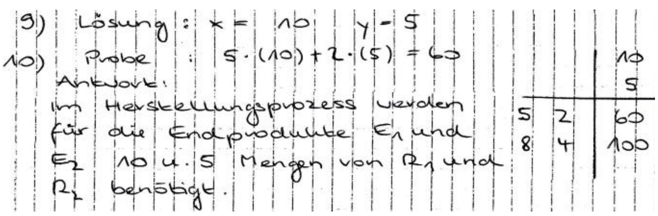

Fig.5: Check of the correct results, while the answer is interpreted in a confusing and faulty way by the student[1].

[1] The arrangement of the student solution has been modified for better clarity.

4. Solving the linear algebra examination problem with the matrix inverse

Besides the direct solution discussed in the previous section, it is possible to solve the given linear algebra examination problem with the inverse of matrix **A**. The definition of the inverse

$$\mathbf{A}\,\mathbf{A}^{-1} = \begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \{4\}$$

shows, that two distinct systems of two linear equation with two different resulting Pauli vectors

$$\begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{r}_1 = \sigma_x \qquad \{5\}$$

$$\begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{r}_2 = \sigma_y \qquad \{6\}$$

emerge. The fictitious (but reasonable) interpretation of the elements of the inverse matrix can be described and explained by the hypothetical questions:

> How many units of the final products E_1 and E_2 would be produced, if at the production process exactly 1 unit of the first raw material R_1 was consumed?
>
> And how many units of the final products E_1 and E_2 would be produced, if at the production process exactly 1 unit of the second raw material R_2 was consumed?

These elements of the inverse matrix $\mathbf{A}^{-1}$ can be found by the same mathematical idea discussed in section 3: Outer products (or bivectors or oriented area elements) are compared. The results then are:

$$x_1 = (\mathbf{a}_1 \wedge \mathbf{a}_2)^{-1}\,(\sigma_x \wedge \mathbf{a}_2) = 1 \qquad \{7\}$$

$$y_1 = (\mathbf{a}_1 \wedge \mathbf{a}_2)^{-1}\,(\mathbf{a}_1 \wedge \sigma_x) = -2 \qquad \{8\}$$

and

$$x_2 = (\mathbf{a}_1 \wedge \mathbf{a}_2)^{-1}\,(\sigma_y \wedge \mathbf{a}_2) = -0.5 = -1/2 \qquad \{9\}$$

$$y_2 = (\mathbf{a}_1 \wedge \mathbf{a}_2)^{-1}\,(\mathbf{a}_1 \wedge \sigma_y) = 1.25 = 5/4 \qquad \{10\}$$

These results are a simple application of eq. {1}. The negative values indicate, that this hypothetical consumption is not realized in real economic situations. Nevertheless it makes sense mathematically, because the expected results

$$\mathbf{p} = \mathbf{A}^{-1}\,\mathbf{r} = \begin{bmatrix} 1 & -1/2 \\ -2 & 5/4 \end{bmatrix} \begin{bmatrix} 60 \\ 100 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \{11\}$$

can be found using these values of the inverse matrix. Of course they are identical to the results (see fig. 5, #9) found by direct calculation.

A more detailed explanation of this solution strategy can be found in the literature [3], [6]. The OHP slides, which had been used in several business mathematics courses of universities of applied sciences [4], [7] can be downloaded.

5. Modified linear algebra examination problem

In real product engineering situations it will be a rare coincident if the number of required raw materials and the number of final products, which are produced, are equal. Generally more raw materials are required to produce some final products, resulting in a non-square demand matrix with more rows than columns (m > n).

Therefore the linear algebra examination problem is extended, and a third raw material is taken into account to model a more realistic situation:

> To produce one unit of the first final product E_1 5 units of raw material R_1, 8 units of raw material R_2, and **one unit of raw material R_3** are required.
>
> To produce one unit of the second final product E_2 2 units of raw material R_1, 4 units of raw material R_2, and **6 units of raw material R_3** are required.
>
> Find the quantities of the first and second final products E_1 and E_2 which were produced if at the production process exactly 60 units of raw material R_1, 100 units of raw material R_2, and **40 units of raw material R_3** had been consumed.

Again this problem might be solved directly, transforming the non-square demand matrix **B** and the total demand vector of raw materials **r**

$$\mathbf{r} = \mathbf{B}\,\mathbf{p} = \begin{bmatrix} 5 & 2 \\ 8 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 60 \\ 100 \\ 40 \end{bmatrix} \qquad \{12\}$$

into Pauli vectors

$$\mathbf{b}_1 = 5\,\sigma_1 + 8\,\sigma_2 + \sigma_3 \qquad \{13\}$$

$$\mathbf{b}_2 = 2\,\sigma_1 + 4\,\sigma_2 + 6\,\sigma_3 \qquad \{14\}$$

$$\mathbf{r} = 60\,\sigma_1 + 100\,\sigma_2 + 40\,\sigma_3 \qquad \{15\}$$

and calculating the outer products

$$\mathbf{b}_1 \wedge \mathbf{b}_2 = 4\,\sigma_1\sigma_2 + 44\,\sigma_2\sigma_3 - 28\,\sigma_3\sigma_1 \qquad \{16\}$$

$$\mathbf{r} \wedge \mathbf{b}_2 = 40\,\sigma_1\sigma_2 + 440\,\sigma_2\sigma_3 - 280\,\sigma_3\sigma_1 \qquad \{17\}$$

$$\mathbf{b}_1 \wedge \mathbf{r} = 20\,\sigma_1\sigma_2 + 220\,\sigma_2\sigma_3 - 140\,\sigma_3\sigma_1 \qquad \{18\}$$

As every Pauli vector is parallel to the same plane represented by unit bivector

$$\hat{\mathbf{B}} = \frac{\mathbf{b}_1 \wedge \mathbf{b}_2}{|\mathbf{b}_1 \wedge \mathbf{b}_2|} = \frac{\mathbf{b}_1 \wedge \mathbf{b}_2}{|\mathbf{b}_1 \wedge \mathbf{b}_2|} = \frac{\mathbf{b}_1 \wedge \mathbf{b}_2}{|\mathbf{b}_1 \wedge \mathbf{b}_2|}$$

$$= \frac{1}{3\sqrt{19}}\,(\sigma_1\sigma_2 + 11\,\sigma_2\sigma_3 - 7\,\sigma_3\sigma_1) \qquad \{19\}$$

this system of three linear equations is consistent and a unique solution exists. This solution can again be found by comparing the outer products {17} & {18} with determinant bivector {16}, resulting once again in

$$x = 10 \qquad \text{and} \qquad y = 5 \qquad \{20\}$$

6. Solving the modified linear algebra examination problem with a matrix inverse

As seen, the overconstrained, but consistent system of three linear equations and only two variables {12} can be solved directly by comparing outer products. We might be happy by having found the solution.

But from a mathematical perspective, it is interesting to look for another solution strategy which is based on the idea to first find an inverse matrix $\mathbf{B}^{-1}$. Using this matrix inverse, the solution can then be calculated on the analogy of section 4.

Arguing by analogy one could claim that the definition of the right-sided inverse

$$\mathbf{B}\,\mathbf{B}^{-1} = \begin{bmatrix} 5 & 2 \\ 8 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{21\}$$

shows, that the three distinct systems of three linear equation with three different resulting Pauli vectors

$$\begin{bmatrix} 5 & 2 \\ 8 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{r}_1 = \sigma_1 \qquad \{22\}$$

$$\begin{bmatrix} 5 & 2 \\ 8 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{r}_2 = \sigma_2 \qquad \{23\}$$

$$\begin{bmatrix} 5 & 2 \\ 8 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{r}_3 = \sigma_3 \qquad \{24\}$$

will give the following elements of the right-sided matrix inverse:

$$x_1 = (\mathbf{b}_1 \wedge \mathbf{b}_2)^{-1} (\sigma_1 \wedge \mathbf{b}_2) \qquad \{25\}$$
$$= \frac{1}{684} (46 - 66\, \sigma_1\sigma_2 - 22\, \sigma_2\sigma_3 - 44\, \sigma_3\sigma_1)$$

$$y_1 = (\mathbf{b}_1 \wedge \mathbf{b}_2)^{-1} (\mathbf{b}_1 \wedge \sigma_1) \qquad \{26\}$$
$$= \frac{1}{684} (-15 + 11\, \sigma_1\sigma_2 + 55\, \sigma_2\sigma_3 + 88\, \sigma_3\sigma_1)$$

and

$$x_2 = (\mathbf{b}_1 \wedge \mathbf{b}_2)^{-1} (\sigma_2 \wedge \mathbf{b}_2) \qquad \{27\}$$
$$= \frac{1}{684} (64 + 42\, \sigma_1\sigma_2 + 14\, \sigma_2\sigma_3 + 28\, \sigma_3\sigma_1)$$

$$y_2 = (\mathbf{b}_1 \wedge \mathbf{b}_2)^{-1} (\mathbf{b}_1 \wedge \sigma_2) \qquad \{28\}$$
$$= \frac{1}{684} (-6 - 7\, \sigma_1\sigma_2 - 35\, \sigma_2\sigma_3 - 56\, \sigma_3\sigma_1)$$

and

$$x_3 = (\mathbf{b}_1 \wedge \mathbf{b}_2)^{-1} (\sigma_3 \wedge \mathbf{b}_2) \qquad \{29\}$$
$$= \frac{1}{684} (-58 - 6\, \sigma_1\sigma_2 - 2\, \sigma_2\sigma_3 - 4\, \sigma_3\sigma_1)$$

$$y_3 = (\mathbf{b}_1 \wedge \mathbf{b}_2)^{-1} (\mathbf{b}_1 \wedge \sigma_3) \qquad \{30\}$$
$$= \frac{1}{684} (123 + \sigma_1\sigma_2 + 5\, \sigma_2\sigma_3 + 8\, \sigma_3\sigma_1)$$

But as Rota perhaps said, “many great teachers are a bit like con men” [8, p. 9], it was withhold till now, that the coefficient vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and the resulting vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are no longer parallel to a joint plane. Therefore a right-sided matrix inverse has indeed not been found. But a second look at it is promising [9].

Instead the results {25} to {30} make up perfect elements of a left-sided matrix inverse $\mathbf{B}^{-1}$.

$$\mathbf{B}^{-1}\mathbf{B} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 8 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \{31\}$$

Thus the result of the modified linear algebra examination problem as expected is

$$\mathbf{p} = \mathbf{B}^{-1}\,\mathbf{r} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 60 \\ 100 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \{32\}$$

7. Quaternionic interpretation

Inverses of non-square matrices can be constructed and used to solve consistent systems of linear equations in the way shown above. They allow an alternative mathematical approach to systems of linear equations.

In addition, they not only allow an alternative approach, but also an alternative view of systems of linear equations. An essential property of the inverse calculated above is interesting: The elements of the left-sided inverse $\mathbf{B}^{-1}$ {25} to {30} are not real numbers, but linear combinations of real numbers and bivectors. Thus they show a quaternionic structure.

If the unit bivectors are identified with unit quaternions

$$i = \sigma_2\sigma_3 \qquad j = \sigma_3\sigma_1 \qquad k = -\sigma_1\sigma_2 \qquad \{33\}$$

the left-sided matrix inverse $\mathbf{B}^{-1}$ can be written as

$$\mathbf{B}^{-1} = \frac{1}{684} \begin{bmatrix} 46 - 22\,i - 44\,j + 66\,k & 64 + 14\,i + 28\,j - 42\,k & -58 - 2\,i - 4\,j + 6\,k \\ -15 + 55\,i + 88\,j - 11\,k & -6 - 35\,i - 56\,j + 7\,k & 123 + 5\,i + 8\,j - k \end{bmatrix} \qquad \{34\}$$

This approach thus leads to a convincing motivation for quaternions: Quaternionic numbers are necessary to construct non-square matrix inverses.

And it shows, how quaternions can be generalized in an easy and elegant way: We simply have to increase the number of raw materials and of final products to get higher grade elements of inverse matrices. These higher grade elements might then be identified with generalized quaternions.

8. Rotating systems of linear equations

We now have two different systems of linear equations {3} and {12} which possess identical solutions {11} and {32}.

If solutions are identical, why shouldn't be systems of linear equations be identical, too? As a physicist with a strong emotional commitment to relativity, it makes sense to ask whether two systems of linear equations might be nothing else than one and only one system which we look at from two different perspectives.

Thus it might be possible to rotate the system of linear equations with Pauli vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{r}_2${13}, {14}, {15} which are parallel to a plane represented by unit bivector $\hat{\mathbf{B}}$ {19} into a system of linear equations with Pauli vectors which are parallel to a plane represented by unit bivector

$$\hat{\mathbf{A}} = \sigma_1\sigma_2 \qquad \{35\}$$

The axis of rotation it then represented by unit vector $\hat{\mathbf{n}}$ which points into the direction of the dual of the commutator product of bivectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, while the plane, in which the rotation takes place, is represented by unit bivector $\hat{\mathbf{N}}$ which is parallel to the commutator product:

$$\hat{\mathbf{A}} \times \hat{\mathbf{B}} = -\frac{1}{\sqrt{170}}\,(7\,\sigma_2\sigma_3 + 11\,\sigma_3\sigma_1) \qquad \{36\}$$

$$\hat{\mathbf{n}} = \frac{\mathbf{I}\,(\hat{\mathbf{A}} \times \hat{\mathbf{B}})}{\left|\hat{\mathbf{A}} \times \hat{\mathbf{B}}\right|} = \frac{1}{\sqrt{170}}\,(7\,\sigma_1 + 11\,\sigma_2) \qquad \{37\}$$

$$\hat{\mathbf{N}} = \mathbf{I}\,\hat{\mathbf{n}} = \frac{1}{\sqrt{170}}\,(7\,\sigma_2\sigma_3 + 11\,\sigma_3\sigma_1) \qquad \{38\}$$

The angle of rotation can be found by comparing the unit vectors of both planes

$$\hat{\mathbf{n}}_a = \hat{\mathbf{A}}\,\hat{\mathbf{n}} = \frac{1}{\sqrt{170}}\,(11\,\sigma_1 - 7\,\sigma_2) \qquad \{39\}$$

$$\hat{\mathbf{n}}_b = \hat{\mathbf{B}}\,\hat{\mathbf{n}} = \frac{1}{\sqrt{170 \cdot 171}}\,(11\,\sigma_1 - 7\,\sigma_2 - 170\,\sigma_3) \qquad \{40\}$$

perpendicular to the axis of rotation:

$$\cos\alpha = \mathbf{n}_a \bullet \mathbf{n}_b = \frac{1}{\sqrt{171}} = \frac{1}{3\sqrt{19}} \qquad \{41a\}$$

$$\Rightarrow \quad \alpha \approx 85.6142° \qquad \{41b\}$$

Thus the rotor to permit this rotation is then given by

$$R = \hat{\mathbf{n}}_a \frac{\hat{\mathbf{n}}_a + \hat{\mathbf{n}}_b}{\left|\hat{\mathbf{n}}_a + \hat{\mathbf{n}}_b\right|} = \frac{\hat{\mathbf{n}}_a + \hat{\mathbf{n}}_b}{\left|\hat{\mathbf{n}}_a + \hat{\mathbf{n}}_b\right|}\hat{\mathbf{n}}_b$$

$$= \frac{1}{\sqrt{2 + \frac{2}{\sqrt{171}}}}\,(1 + \hat{\mathbf{n}}_a\hat{\mathbf{n}}_b) \qquad \{42a\}$$

$$= \frac{1 + \sqrt{171} + 7\,\sigma_2\sigma_3 + 11\,\sigma_3\sigma_1}{\sqrt{2}\cdot\sqrt{171 + \sqrt{171}}} \qquad \{42b\}$$

with

$$\cos\frac{\alpha}{2} = \frac{1 + \sqrt{171}}{\sqrt{2}\cdot\sqrt{171 + \sqrt{171}}} = \frac{1}{\sqrt{2}}\cdot\sqrt{1 + \frac{1}{\sqrt{171}}} \qquad \{43a\}$$

$$\Rightarrow \quad \alpha/2 \approx 42.8071° \qquad \{43b\}$$

Now the consistent system of three linear equations of plane $\mathbf{B}$ {12} can be rotated into a system of only two linear equations lying in the xy-plane with Pauli vectors

$$\mathbf{c}_1 = R\,\mathbf{b}_1\,\tilde{R} \qquad \{44\}$$
$$= \frac{1}{1 + \sqrt{171}}\left((-6 + 5\sqrt{171})\,\sigma_1 + (15 + 8\sqrt{171})\,\sigma_2\right)$$

$$\mathbf{c}_2 = R\,\mathbf{b}_2\,\tilde{R} \qquad \{45\}$$
$$= \frac{1}{1 + \sqrt{171}}\left((-64 + 2\sqrt{171})\,\sigma_1 + (46 + 4\sqrt{171})\,\sigma_2\right)$$

$$\mathbf{r}' = R\,\mathbf{r}\,\tilde{R} \qquad \{47\}$$
$$= \frac{1}{1 + \sqrt{171}}\left((-380 + 60\sqrt{171})\,\sigma_1 + (380 + 100\sqrt{171})\,\sigma_2\right)$$

From a mathematical point of view this rotated system {48} of two linear equations

$$\mathbf{c}_1\,x + \mathbf{c}_2\,y = \mathbf{r}' \qquad \{48\}$$

is completely equivalent to the system {12}, {49} of three linear equations

$$\mathbf{b}_1\,x + \mathbf{b}_2\,y = \mathbf{r} \qquad \{49\}$$

as the geometric properties of all vectors are identical. They include identical angles, possess identical lengths and will result in an identical solution {20}, {53}.

But from an economical point of view the rotated system of linear equations of course describes a completely different product engineering situation, as the 'rotated' problem includes negative consumption rates for the second raw material. Thus the second raw material was not consumed, but produced when producing the second final product.

The corresponding 'rotated' problem can then be stated as:

> To produce one unit of the first final product E_1 4.2186 units of raw material R_1 and 8.4973 units of raw material R_2 are required.
>
> To produce not only one unit of the second final product E_2 but also 2.6886 units of the first raw material R_1, 6.9837 units of raw material R_2 are required.
>
> Find the quantities of the first and second final products E_1 and E_2 which were produced if at the production process 28.7427 units of the first raw material R_1 had been produced and 119.8910 units of the second raw material R_2 had been consumed.

The solution of this problem can be found by rotating the outer products {16}, {17}, {18} into the xy-plane

$$\mathbf{c}_1 \wedge \mathbf{c}_2 = \mathrm{R}\,(\mathbf{b}_1 \wedge \mathbf{b}_2)\,\tilde{\mathrm{R}} = 12\sqrt{19}\,\sigma_1\sigma_2 \qquad \{50\}$$

$$\mathbf{r}' \wedge \mathbf{c}_2 = \mathrm{R}\,(\mathbf{r} \wedge \mathbf{b}_2)\,\tilde{\mathrm{R}} = 120\sqrt{19}\,\sigma_1\sigma_2 \qquad \{51\}$$

$$\mathbf{c}_1 \wedge \mathbf{r}' = \mathrm{R}\,(\mathbf{r} \wedge \mathbf{b}_2)\,\tilde{\mathrm{R}} = 60\sqrt{19}\,\sigma_1\sigma_2 \qquad \{52\}$$

resulting as expected in

$$x = 10 \qquad \text{and} \qquad y = 5 \qquad \{53\}$$

As these results are scalars, it is even possible to declare these results as the outcome of a rotation of scalars

$$x = \mathrm{R}\,x\,\tilde{\mathrm{R}} = x\,\mathrm{R}\,\tilde{\mathrm{R}} = 10 \qquad \{54a\}$$

$$y = \mathrm{R}\,y\,\tilde{\mathrm{R}} = y\,\mathrm{R}\,\tilde{\mathrm{R}} = 5 \qquad \{54b\}$$

which do not change when rotated.

Unfortunately the values of $\mathbf{r}'$ {47} and $\mathbf{r}$ {3} and the angles between their coefficient vectors are different. Therefore the two equivalent systems of linear equations {48}, {49} are not equivalent to the first simple system of linear equations {3}.

9. Right-sided matrix inverses

If a matrix consists of more columns than rows the left-sided inverse equals to zero. The coefficient vectors will be not linear independent and the outer product will disappear.

As the matrix equation then produce less equations than existing unknowns no unique solution can be found.

But in this case a right-sided matrix inverse can be defined. This right-sided matrix inverse can simply be constructed by transposing the matrix equation, thus resulting again in a transposed matrix which possess more rows than columns.

10. Outlook

If a system of n linear equations can be described by a matrix $\mathbf{A}$ which consists of n linear independent coefficient vectors, we are able to find a matrix inverse as described in the previous sections (and shown again in the attachment).

This system of n linear equations

$$\mathbf{r} = \mathbf{A}\,\mathbf{p} \qquad \{55\}$$

is called consistent, if the resulting vector $\mathbf{r}$ and the coefficient vectors are not linear independent. Then a unique solution vector $\mathbf{p}$ of the system of linear equations can be found by pre-multiplication of the matrix inverse $\mathbf{A}^{-1}$

$$\mathbf{p} = \mathbf{A}^{-1}\,\mathbf{r} \qquad \{56\}$$

The system of n linear equations {55} is called inconsistent, if the resulting vector $\mathbf{r}$ and the coefficient vectors are linear independent. Then a unique solution with real elements does not exist. But solution vector $\mathbf{p}$ {56}, which will have generalized quaternionic elements, will exist.

Therefore it makes sense to call solution vector $\mathbf{p}$ {56} a solution of the inconsistent system of linear equations, which will be discussed in [9].

11. Literature

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12. Attachment: More example problems

The following problems can be solved directly (see section 5). Solutions which use non-square matrix inverses are given below.

- Additional example problem 1:

  To produce one unit of the first final product E_1 5 units of raw material R_1 and 6 units of raw material R_2 are required.

  To produce one unit of the second final product E_2 3 units of raw material R_1 and 4 units of raw material R_2 are required.

  To produce one unit of the third final product E_3 1 unit of raw material R_1 and 2 units of raw material R_2 are required.

  The total costs of raw materials to produce one unit of the first final product E_1 are € 170.
  The total costs of raw materials to produce one unit of the second final product E_2 are € 110.
  The total costs of raw materials to produce one unit of the third final product E_3 are € 50.

  Find the prices of the raw materials R_1 and R_2.

- Solution of additional example problem 1:

Demand matrix: $\mathbf{D} = \begin{bmatrix} 5 & 3 & 1 \\ 6 & 4 & 2 \end{bmatrix}$

Price vector: $\mathbf{p}^T = \begin{bmatrix} x & y \end{bmatrix} = ?$

Total cost vector: $\mathbf{c}^T = \begin{bmatrix} 170 & 110 & 50 \end{bmatrix}$

System of linear equations: $\mathbf{p}^T \mathbf{D} = \mathbf{c}^T$

Transposed system of linear equations:

$$\mathbf{D}^T \mathbf{p} = \begin{bmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 170 \\ 110 \\ 50 \end{bmatrix} = \mathbf{c}$$

Coefficient vectors:
$\mathbf{a} = 5\,\sigma_1 + 3\,\sigma_2 + \sigma_3$
$\mathbf{b} = 6\,\sigma_1 + 4\,\sigma_2 + 2\,\sigma_3$

Resulting vector: $\mathbf{c} = 170\,\sigma_1 + 110\,\sigma_2 + 50\,\sigma_3$

Outer products (determinant):

$$\mathbf{a} \wedge \mathbf{b} = 2\,\sigma_1\sigma_2 + 2\,\sigma_2\sigma_3 - 4\,\sigma_3\sigma_1$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{12}\,(-\,\sigma_1\sigma_2 - \sigma_2\sigma_3 + 2\,\sigma_3\sigma_1)$$

Outer products (sub-determinants):

$$\sigma_1 \wedge \mathbf{b} = 4\,\sigma_1\sigma_2 - 2\,\sigma_3\sigma_1$$
$$\sigma_2 \wedge \mathbf{b} = -\,6\,\sigma_1\sigma_2 + 2\,\sigma_2\sigma_3$$
$$\sigma_3 \wedge \mathbf{b} = -\,4\,\sigma_2\sigma_3 + 6\,\sigma_3\sigma_1$$
$$\mathbf{a} \wedge \sigma_1 = -\,3\,\sigma_1\sigma_2 + \sigma_3\sigma_1$$
$$\mathbf{a} \wedge \sigma_2 = 5\,\sigma_1\sigma_2 - \sigma_2\sigma_3$$
$$\mathbf{a} \wedge \sigma_3 = 3\,\sigma_2\sigma_3 - 5\,\sigma_3\sigma_1$$

Elements of inverse matrix:

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_1 \wedge \mathbf{b}) = \frac{1}{12}\,(8 - 2\,\sigma_1\sigma_2 - 6\,\sigma_2\sigma_3 - 4\,\sigma_3\sigma_1)$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_2 \wedge \mathbf{b}) = \frac{1}{12}\,(-\,4 + 4\,\sigma_1\sigma_2 + 12\,\sigma_2\sigma_3 + 8\,\sigma_3\sigma_1)$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_3 \wedge \mathbf{b}) = \frac{1}{12}\,(-\,16 - 2\,\sigma_1\sigma_2 - 6\,\sigma_2\sigma_3 - 4\,\sigma_3\sigma_1)$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_1) = \frac{1}{12}\,(-\,5 + \sigma_1\sigma_2 + 5\,\sigma_2\sigma_3 + 3\,\sigma_3\sigma_1)$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_2) = \frac{1}{12}\,(4 - 2\,\sigma_1\sigma_2 - 10\,\sigma_2\sigma_3 - 6\,\sigma_3\sigma_1)$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_3) = \frac{1}{12}\,(13 + \sigma_1\sigma_2 + 5\,\sigma_2\sigma_3 + 3\,\sigma_3\sigma_1)$$

Inverse matrix:

$$(\mathbf{D}^T)^{-1} = \frac{1}{12}\begin{pmatrix} 8 - 2\sigma_1\sigma_2 - 6\sigma_2\sigma_3 - 4\sigma_3\sigma_1 & -4 + 4\sigma_1\sigma_2 + 12\sigma_2\sigma_3 + 8\sigma_3\sigma_1 & -16 - 2\sigma_1\sigma_2 - 6\sigma_2\sigma_3 - 4\sigma_3\sigma_1 \\ -5 + \sigma_1\sigma_2 + 5\sigma_2\sigma_3 + 3\sigma_3\sigma_1 & 4 - 2\sigma_1\sigma_2 - 10\sigma_2\sigma_3 - 6\sigma_3\sigma_1 & 13 + \sigma_1\sigma_2 + 5\sigma_2\sigma_3 + 3\sigma_3\sigma_1 \end{pmatrix}$$

Solution of price vector:

$$\mathbf{p} = (\mathbf{D}^T)^{-1}\,\mathbf{c} = (\mathbf{D}^T)^{-1}\begin{bmatrix} 170 \\ 110 \\ 50 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$\Rightarrow$ One unit of the first raw material R_1 costs € 10.
One unit of the second raw material R_2 costs € 20.

- Additional example problem 2:

  To produce one unit of the first final product E_1 7 units of raw material R_1, 5 units of raw material R_2, 3 units of raw material R_3, and one unit of raw material R_4 are required.

  To produce one unit of the second final product E_2 8 units of raw material R_1, 6 units of raw material R_2, 4 units of raw material R_3, and 2 units of raw material R_4 are required.

  Find the quantities of the first and second final products E_1 and E_2 which were produced if at the production process exactly 2070 units of raw material R_1, 1530 units of raw material R_2, 990 units of raw material R_3, and 450 units of raw material R_4 had been consumed.

- Solution of additional example problem 2:

Demand matrix: $\mathbf{D} = \begin{bmatrix} 7 & 8 \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}$

Production vector: $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} = ?$

Total demand of raw materials: $\mathbf{r} = \begin{bmatrix} 2070 \\ 1530 \\ 990 \\ 450 \end{bmatrix}$

System of linear equations:

$$\mathbf{D\,p} = \begin{bmatrix} 7 & 8 \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2070 \\ 1530 \\ 990 \\ 450 \end{bmatrix} = \mathbf{r}$$

Coefficient vectors: $\mathbf{a} = 7\,\sigma_1 + 5\,\sigma_2 + 3\,\sigma_3 + \sigma_4$

$\mathbf{b} = 8\,\sigma_1 + 6\,\sigma_2 + 4\,\sigma_3 + 2\,\sigma_4$

Resulting vector:

$$\mathbf{r} = 2070\,\sigma_1 + 1530\,\sigma_2 + 990\,\sigma_3 + 450\,\sigma_4$$

Outer products (determinant):

$$\mathbf{a} \wedge \mathbf{b} = 2\,\sigma_1\sigma_2 + 4\,\sigma_1\sigma_3 + 6\,\sigma_1\sigma_4 + 2\,\sigma_2\sigma_3 + 4\,\sigma_2\sigma_4 + 2\,\sigma_3\sigma_4$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{40}\,(\sigma_1\sigma_2 + 2\,\sigma_1\sigma_3 + 3\,\sigma_1\sigma_4 + \sigma_2\sigma_3 + 2\,\sigma_2\sigma_4 + \sigma_3\sigma_4)$$

Outer products (sub-determinants):

$$\sigma_1 \wedge \mathbf{b} = 6\,\sigma_1\sigma_2 + 4\,\sigma_1\sigma_3 + 2\,\sigma_1\sigma_4$$
$$\sigma_2 \wedge \mathbf{b} = -8\,\sigma_1\sigma_2 + 4\,\sigma_2\sigma_3 + 2\,\sigma_2\sigma_4$$
$$\sigma_3 \wedge \mathbf{b} = -8\,\sigma_1\sigma_3 - 6\,\sigma_2\sigma_3 + 2\,\sigma_3\sigma_4$$
$$\sigma_4 \wedge \mathbf{b} = -8\,\sigma_1\sigma_4 - 6\,\sigma_2\sigma_4 - 4\,\sigma_3\sigma_4$$
$$\mathbf{a} \wedge \sigma_1 = -5\,\sigma_1\sigma_2 - 3\,\sigma_1\sigma_3 - \sigma_1\sigma_4$$
$$\mathbf{a} \wedge \sigma_2 = 7\,\sigma_1\sigma_2 - 3\,\sigma_2\sigma_3 - \sigma_2\sigma_4$$
$$\mathbf{a} \wedge \sigma_3 = 7\,\sigma_1\sigma_3 + 5\,\sigma_2\sigma_3 - \sigma_3\sigma_4$$
$$\mathbf{a} \wedge \sigma_4 = 7\,\sigma_1\sigma_4 + 5\,\sigma_2\sigma_4 + 3\,\sigma_3\sigma_4$$

Elements of inverse matrix:

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_1 \wedge \mathbf{b}) = \frac{1}{40}\,(20 - 8\,\sigma_1\sigma_2 + 4\,\sigma_1\sigma_3 + 16\,\sigma_1\sigma_4 - 8\,\sigma_2\sigma_3 - 16\,\sigma_2\sigma_4 - 8\,\sigma_3\sigma_4)$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_2 \wedge \mathbf{b}) = \frac{1}{40}\,(0 + 14\,\sigma_1\sigma_2 - 12\,\sigma_1\sigma_3 - 18\,\sigma_1\sigma_4 + 14\,\sigma_2\sigma_3 + 28\,\sigma_2\sigma_4 - 6\,\sigma_3\sigma_4)$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_3 \wedge \mathbf{b}) = \frac{1}{40}\,(-20 - 4\,\sigma_1\sigma_2 + 12\,\sigma_1\sigma_3 - 12\,\sigma_1\sigma_4 - 4\,\sigma_2\sigma_3 - 8\,\sigma_2\sigma_4 + 36\,\sigma_3\sigma_4)$$

$$x_4 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\sigma_4 \wedge \mathbf{b}) = \frac{1}{40}\,(-40 - 2\,\sigma_1\sigma_2 - 4\,\sigma_1\sigma_3 + 14\,\sigma_1\sigma_4 - 2\,\sigma_2\sigma_3 - 4\,\sigma_2\sigma_4 - 22\,\sigma_3\sigma_4)$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_1) = \frac{1}{40}\,(-14 + 5\,\sigma_1\sigma_2 - 4\,\sigma_1\sigma_3 - 13\,\sigma_1\sigma_4 + 7\,\sigma_2\sigma_3 + 14\,\sigma_2\sigma_4 + 7\,\sigma_3\sigma_4)$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_2) = \frac{1}{40}\,(2 - 9\,\sigma_1\sigma_2 + 10\,\sigma_1\sigma_3 + 15\,\sigma_1\sigma_4 - 13\,\sigma_2\sigma_3 - 24\,\sigma_2\sigma_4 + 5\,\sigma_3\sigma_4)$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_3) = \frac{1}{40}\,(18 + 3\,\sigma_1\sigma_2 - 8\,\sigma_1\sigma_3 + 9\,\sigma_1\sigma_4 + 5\,\sigma_2\sigma_3 + 6\,\sigma_2\sigma_4 - 31\,\sigma_3\sigma_4)$$

$$y_4 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \sigma_4) = \frac{1}{40}\,(34 + \sigma_1\sigma_2 + 2\,\sigma_1\sigma_3 - 11\,\sigma_1\sigma_4 + \sigma_2\sigma_3 + 4\,\sigma_2\sigma_4 + 19\,\sigma_3\sigma_4)$$

Inverse matrix: $\mathbf{D}^{-1} =$

$$\frac{1}{40}\begin{pmatrix} \begin{array}{r} 20 - 8\sigma_1\sigma_2 + 4\sigma_1\sigma_3 + 16\sigma_1\sigma_4 \\ -8\sigma_2\sigma_3 - 16\sigma_2\sigma_4 - 8\sigma_3\sigma_4 \end{array} & \begin{array}{r} 14\sigma_1\sigma_2 - 12\sigma_1\sigma_3 - 18\sigma_1\sigma_4 \\ +14\sigma_2\sigma_3 + 28\sigma_2\sigma_4 - 6\sigma_3\sigma_4 \end{array} & \begin{array}{r} -20 - 4\sigma_1\sigma_2 + 12\sigma_1\sigma_3 - 12\sigma_1\sigma_4 \\ -4\sigma_2\sigma_3 - 8\sigma_2\sigma_4 + 36\sigma_3\sigma_4 \end{array} & \begin{array}{r} -40 - 2\sigma_1\sigma_2 - 4\sigma_1\sigma_3 + 14\sigma_1\sigma_4 \\ -2\sigma_2\sigma_3 - 4\sigma_2\sigma_4 - 22\sigma_3\sigma_4 \end{array} \\ \begin{array}{r} -14 + 5\sigma_1\sigma_2 - 4\sigma_1\sigma_3 - 13\sigma_1\sigma_4 \\ +7\sigma_2\sigma_3 + 14\sigma_2\sigma_4 + 7\sigma_3\sigma_4 \end{array} & \begin{array}{r} 2 - 9\sigma_1\sigma_2 + 10\sigma_1\sigma_3 + 15\sigma_1\sigma_4 \\ -13\sigma_2\sigma_3 - 24\sigma_2\sigma_4 + 5\sigma_3\sigma_4 \end{array} & \begin{array}{r} 18 + 3\sigma_1\sigma_2 - 8\sigma_1\sigma_3 + 9\sigma_1\sigma_4 \\ +5\sigma_2\sigma_3 + 6\sigma_2\sigma_4 - 31\sigma_3\sigma_4 \end{array} & \begin{array}{r} 34 + \sigma_1\sigma_2 + 2\sigma_1\sigma_3 - 11\sigma_1\sigma_4 \\ +\sigma_2\sigma_3 + 4\sigma_2\sigma_4 + 19\sigma_3\sigma_4 \end{array} \end{pmatrix}$$

Solution of production vector:

$$\mathbf{p} = \mathbf{D}^{-1}\,\mathbf{r} = \mathbf{D}^{-1}\begin{bmatrix} 2070 \\ 1530 \\ 990 \\ 450 \end{bmatrix} = \begin{bmatrix} 90 \\ 180 \end{bmatrix}$$

⇒ 90 units of the first final product E_1 and 180 units of the second final product E_2 had been produced.

- Additional example problem 3:

  To produce one unit of the first final product E_1 7 units of raw material R_1, 5 units of raw material R_2, and one unit of raw material R_4 are required.

  To produce one unit of the second final product E_2 8 units of raw material R_1, 3 units of raw material R_3, and 2 units of raw material R_4 are required.

  To produce one unit of the third final product E_3 6 units of raw material R_2, and 4 units of raw material R_3 are required.

  Find the quantities of the first, second and third final products E_1, E_2, and E_3 which were produced if at the production process exactly 2070 units of raw material R_1, 2610 units of raw material R_2, 1980 units of raw material R_3, and 450 units of raw material R_4 had been consumed.

- Solution of additional example problem 3:

Demand matrix: $\mathbf{D} = \begin{bmatrix} 7 & 8 & 0 \\ 5 & 0 & 6 \\ 0 & 3 & 4 \\ 1 & 2 & 0 \end{bmatrix}$

Production vector: $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ?$

Total demand of raw materials: $\mathbf{r} = \begin{bmatrix} 2070 \\ 2610 \\ 1980 \\ 450 \end{bmatrix}$

System of linear equations:

$$\mathbf{D}\,\mathbf{p} = \begin{bmatrix} 7 & 8 & 0 \\ 5 & 0 & 6 \\ 0 & 3 & 4 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2070 \\ 2610 \\ 1980 \\ 450 \end{bmatrix} = \mathbf{r}$$

Coefficient vectors:
$$\mathbf{a} = 7\,\sigma_1 + 5\,\sigma_2 + \sigma_4$$
$$\mathbf{b} = 8\,\sigma_1 + 3\,\sigma_3 + 2\,\sigma_4$$
$$\mathbf{c} = 6\,\sigma_2 + 4\,\sigma_3$$

Resulting vector:

$$\mathbf{r} = 2070\,\sigma_1 + 2610\,\sigma_2 + 1980\,\sigma_3 + 450\,\sigma_4$$

Outer products (determinant):

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = -286\,\sigma_1\sigma_2\sigma_3 - 36\,\sigma_1\sigma_2\sigma_4 - 24\,\sigma_1\sigma_3\sigma_4 - 58\,\sigma_2\sigma_3\sigma_4$$

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1} = \frac{1}{43516}\,(143\,\sigma_1\sigma_2\sigma_3 + 18\,\sigma_1\sigma_2\sigma_4 + 12\,\sigma_1\sigma_3\sigma_4 + 29\,\sigma_2\sigma_3\sigma_4)$$

Outer products (sub-determinants):

$$\sigma_1\wedge\mathbf{b}\wedge\mathbf{c} = -18\,\sigma_1\sigma_2\sigma_3 - 12\,\sigma_1\sigma_2\sigma_4 - 8\,\sigma_1\sigma_3\sigma_4$$
$$\sigma_2\wedge\mathbf{b}\wedge\mathbf{c} = -32\,\sigma_1\sigma_2\sigma_3 - 8\,\sigma_2\sigma_3\sigma_4$$
$$\sigma_3\wedge\mathbf{b}\wedge\mathbf{c} = 48\,\sigma_1\sigma_2\sigma_3 + 12\,\sigma_2\sigma_3\sigma_4$$
$$\sigma_4\wedge\mathbf{b}\wedge\mathbf{c} = 48\,\sigma_1\sigma_2\sigma_4 + 32\,\sigma_1\sigma_3\sigma_4 - 18\,\sigma_2\sigma_3\sigma_4$$

$$\mathbf{a}\wedge\sigma_1\wedge\mathbf{c} = -20\,\sigma_1\sigma_2\sigma_3 + 6\,\sigma_1\sigma_2\sigma_4 + 4\,\sigma_1\sigma_3\sigma_4$$
$$\mathbf{a}\wedge\sigma_2\wedge\mathbf{c} = 28\,\sigma_1\sigma_2\sigma_3 + 4\,\sigma_2\sigma_3\sigma_4$$
$$\mathbf{a}\wedge\sigma_3\wedge\mathbf{c} = -42\,\sigma_1\sigma_2\sigma_3 - 6\,\sigma_2\sigma_3\sigma_4$$
$$\mathbf{a}\wedge\sigma_4\wedge\mathbf{c} = -42\,\sigma_1\sigma_2\sigma_4 - 28\,\sigma_1\sigma_3\sigma_4 - 20\,\sigma_2\sigma_3\sigma_4$$

$$\mathbf{a}\wedge\mathbf{b}\wedge\sigma_1 = 15\,\sigma_1\sigma_2\sigma_3 + 10\,\sigma_1\sigma_2\sigma_4 - 3\,\sigma_1\sigma_3\sigma_4$$
$$\mathbf{a}\wedge\mathbf{b}\wedge\sigma_2 = -21\,\sigma_1\sigma_2\sigma_3 - 6\,\sigma_1\sigma_2\sigma_4 - 3\,\sigma_2\sigma_3\sigma_4$$
$$\mathbf{a}\wedge\mathbf{b}\wedge\sigma_3 = -40\,\sigma_1\sigma_2\sigma_3 - 6\,\sigma_1\sigma_3\sigma_4 - 10\,\sigma_2\sigma_3\sigma_4$$
$$\mathbf{a}\wedge\mathbf{b}\wedge\sigma_4 = -40\,\sigma_1\sigma_2\sigma_4 + 21\,\sigma_1\sigma_3\sigma_4 + 15\,\sigma_2\sigma_3\sigma_4$$

Elements of inverse matrix:

$$x_1 = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\sigma_1\wedge\mathbf{b}\wedge\mathbf{c}) = \frac{1}{43516}\,(2886 - 232\,\sigma_1\sigma_2 + 348\,\sigma_1\sigma_3 - 522\,\sigma_1\sigma_4 - 928\,\sigma_2\sigma_4 + 1392\,\sigma_3\sigma_4)$$

$$x_2 = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\sigma_2\wedge\mathbf{b}\wedge\mathbf{c}) = \frac{1}{43516}\,(4808 + 96\,\sigma_1\sigma_2 - 144\,\sigma_1\sigma_3 + 216\,\sigma_1\sigma_4 + 384\,\sigma_2\sigma_4 - 576\,\sigma_3\sigma_4)$$

$$x_3 = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\sigma_3\wedge\mathbf{b}\wedge\mathbf{c}) = \frac{1}{43516}\,(-7212 - 144\,\sigma_1\sigma_2 + 216\,\sigma_1\sigma_3 - 324\,\sigma_1\sigma_4 - 576\,\sigma_2\sigma_4 + 864\,\sigma_3\sigma_4)$$

$$x_4 = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\sigma_4\wedge\mathbf{b}\wedge\mathbf{c}) = \frac{1}{43516}\,(-726 + 1144\,\sigma_1\sigma_2 - 1716\,\sigma_1\sigma_3 + 2574\,\sigma_1\sigma_4 + 4576\,\sigma_2\sigma_4 - 6864\,\sigma_3\sigma_4)$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \sigma_1 \wedge \mathbf{c})$$
$$= \frac{1}{43516} (2704 + 116\,\sigma_1\sigma_2 - 174\,\sigma_1\sigma_3 - 580\,\sigma_1\sigma_4 + 812\,\sigma_2\sigma_4 - 1218\,\sigma_3\sigma_4)$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \sigma_2 \wedge \mathbf{c})$$
$$= \frac{1}{43516} (-4120 - 48\,\sigma_1\sigma_2 + 72\,\sigma_1\sigma_3 + 240\,\sigma_1\sigma_4 - 336\,\sigma_2\sigma_4 + 504\,\sigma_3\sigma_4)$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \sigma_3 \wedge \mathbf{c})$$
$$= \frac{1}{43516} (6180 + 72\,\sigma_1\sigma_2 - 108\,\sigma_1\sigma_3 - 360\,\sigma_1\sigma_4 + 504\,\sigma_2\sigma_4 - 756\,\sigma_3\sigma_4)$$

$$y_4 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \sigma_4 \wedge \mathbf{c})$$
$$= \frac{1}{43516} (1672 - 572\,\sigma_1\sigma_2 + 858\,\sigma_1\sigma_3 + 2860\,\sigma_1\sigma_4 - 4004\,\sigma_2\sigma_4 + 6006\,\sigma_3\sigma_4)$$

$$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \sigma_1)$$
$$= \frac{1}{43516} (-2289 - 87\,\sigma_1\sigma_2 - 290\,\sigma_1\sigma_3 + 435\,\sigma_1\sigma_4 + 174\,\sigma_2\sigma_3 - 609\,\sigma_2\sigma_4 - 1160\,\sigma_3\sigma_4)$$

$$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \sigma_2)$$
$$= \frac{1}{43516} (3198 + 36\,\sigma_1\sigma_2 + 120\,\sigma_1\sigma_3 - 180\,\sigma_1\sigma_4 - 72\,\sigma_2\sigma_3 + 252\,\sigma_2\sigma_4 + 480\,\sigma_3\sigma_4)$$

$$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \sigma_3)$$
$$= \frac{1}{43516} (6082 - 54\,\sigma_1\sigma_2 - 180\,\sigma_1\sigma_3 + 270\,\sigma_1\sigma_4 + 108\,\sigma_2\sigma_3 - 378\,\sigma_2\sigma_4 - 720\,\sigma_3\sigma_4)$$

$$z_4 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \sigma_4)$$
$$= \frac{1}{43516} (33 + 429\,\sigma_1\sigma_2 + 1430\,\sigma_1\sigma_3 - 2145\,\sigma_1\sigma_4 - 858\,\sigma_2\sigma_3 + 3003\,\sigma_2\sigma_4 + 5720\,\sigma_3\sigma_4)$$

Inverse matrix: $\mathbf{D}^{-1} =$

$$\frac{1}{43516}\begin{pmatrix} 2886 - 232\sigma_1\sigma_2 + 348\sigma_1\sigma_3 - 522\sigma_1\sigma_4 - 928\sigma_2\sigma_4 + 1392\sigma_3\sigma_4 & 4808 + 96\sigma_1\sigma_2 - 144\sigma_1\sigma_3 + 216\sigma_1\sigma_4 + 384\sigma_2\sigma_4 - 576\sigma_3\sigma_4 & -7212 - 144\sigma_1\sigma_2 + 216\sigma_1\sigma_3 - 324\sigma_1\sigma_4 - 576\sigma_2\sigma_4 + 864\sigma_3\sigma_4 & -726 - 1144\sigma_1\sigma_2 - 1716\sigma_1\sigma_3 - 2574\sigma_1\sigma_4 + 4576\sigma_2\sigma_4 - 6864\sigma_3\sigma_4 \\ 2704 + 116\sigma_1\sigma_2 - 174\sigma_1\sigma_3 - 580\sigma_1\sigma_4 + 812\sigma_2\sigma_4 - 1218\sigma_3\sigma_4 & -4120 - 48\sigma_1\sigma_2 + 72\sigma_1\sigma_3 + 240\sigma_1\sigma_4 - 336\sigma_2\sigma_4 + 504\sigma_3\sigma_4 & 6180 + 72\sigma_1\sigma_2 - 108\sigma_1\sigma_3 - 360\sigma_1\sigma_4 + 504\sigma_2\sigma_4 - 756\sigma_3\sigma_4 & 1672 - 572\sigma_1\sigma_2 + 858\sigma_1\sigma_3 + 2860\sigma_1\sigma_4 - 4004\sigma_2\sigma_4 + 6006\sigma_3\sigma_4 \\ -2289 - 87\sigma_1\sigma_2 - 290\sigma_1\sigma_3 + 435\sigma_1\sigma_4 + 174\sigma_2\sigma_3 - 609\sigma_2\sigma_4 - 1160\sigma_3\sigma_4 & 3198 + 36\sigma_1\sigma_2 + 120\sigma_1\sigma_3 - 180\sigma_1\sigma_4 - 72\sigma_2\sigma_3 + 252\sigma_2\sigma_4 + 480\sigma_3\sigma_4 & 6082 - 54\sigma_1\sigma_2 - 180\sigma_1\sigma_3 + 270\sigma_1\sigma_4 + 108\sigma_2\sigma_3 - 378\sigma_2\sigma_4 - 720\sigma_3\sigma_4 & 33 + 429\sigma_1\sigma_2 + 1430\sigma_1\sigma_3 - 2145\sigma_1\sigma_4 - 858\sigma_2\sigma_3 + 3003\sigma_2\sigma_4 + 5720\sigma_3\sigma_4 \end{pmatrix}$$

Solution of production vector:

$$\mathbf{p} = \mathbf{D}^{-1}\,\mathbf{r} = \mathbf{D}^{-1}\begin{bmatrix} 2070 \\ 2610 \\ 1980 \\ 450 \end{bmatrix} = \frac{1}{43516}\begin{bmatrix} 3916440 \\ 7832880 \\ 15665760 \end{bmatrix} = \begin{bmatrix} 90 \\ 180 \\ 360 \end{bmatrix}$$

$\Rightarrow$ 90 units of the first final product E_1, 180 units of the second final product E_2, and 360 units of the third final product E_3 had been produced.