More about Blades and Non-Blades

– Extended version of the paper “Zur Wirkung von Blades und Non-Blades” (About the Effects of Blades and Non-Blades), written in German for the annual meeting of the Society of Chemistry and Physics Education (GDCP) 2015 in Berlin –

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English Abstract

While in three-dimensional space every higher-dimensional object can be written as outer product of vectors (in accordance with blades in Geometric Algebra), this mathematical depiction is not always possible in four-dimensional spaces or spacetimes. In such spaces or spacetimes objects called non-blades, which cannot be spanned by vectors in an elementary way, will exist.

As our world can be interpreted as a four-dimensional spacetime, it is to be expected that apart from blades also non-blades will play a relevant role which should not be ignored in the mathematical and conceptual description of our world. The transition from classical three-dimensional to relativistic four-dimensional structures is equivalent to a transition from blades to non-blades.

It can be seen that the didactical structuring of this transition reaches far beyond the simple question, what blades and non-blades geometrically are, and the problem how blades and non-blades, interpreted as operators, effect on other geometric objects, will be central.

German Abstract

Während im dreidimensionalen Raum jedes höherdimensionale Objekt als äußeres Produkt mehrerer Vektoren (im Sinne von Blades der Geometrischen Algebra) dargestellt werden kann, ist dies in vierdimensionalen Räumen oder Raumzeiten nicht immer möglich. Wir finden hier auch Objekte (Non-Blades), die nicht elementar von Vektoren aufgespannt werden.

Da unsere Welt als vierdimensionale Raumzeit gedeutet werden kann, ist zu erwarten, dass neben Blades auch Non-Blades bei der mathematischen Beschreibung unserer Welt eine konzeptuell nicht zu vernachlässigende Rolle spielen. Der Übergang von nicht-relativistisch dreidimensionalen zu relativistisch vierdimensionalen Strukturen ist somit auch ein Übergang von Blades zu Non-Blades.

Es zeigt sich, dass die didaktische Gestaltung dieses Übergangs weit über die Frage, was Blades und Non-Blades geometrisch sind, hinausweist und stattdessen die Problematik, wie Blades und Non-Blades als Operatoren auf andere geometrische Objekte wirken, in den Vordergrund rückt.

1. Reshaping didactical structures

Even more than hundred years after the formulation of Special and General Relativity, a satisfactory conceptual and didactical understanding of this complex of theories is not yet reached. We are certainly able to construct a sufficient three-dimensional picture of our surroundings from the two-dimensional information which is send from the retina of our eyes into our human brains.

But the theory of relativity is founded on a four-dimensional description of the world we live in. Therefore we are confronted with the task to shape these four-dimensional structures didactically in a way which enables learners to gain a deeper understanding of typical four-dimensional effects.

Thus it is necessary to take into account phenomena which do not exist in three-dimensional worlds. In classical, non-relativistic everyday life, we do not experience such phenomena, and we should strive for making these phenomena accessible in modern learning processes.

The didactical aim should be to make it possible one day that our human brains (which are moulded by three-dimensionality) are able to construct a sufficient four-dimensional picture of relativity from the information which reaches our mind.

2. Objects which do not exist in three-dimensional space

In three-dimensional spaces or spacetimes every k-dimensional geometric object (k-vector) can be written as an outer product of k non-parallel vectors. In Geometric Algebra [1], [2], [3] these outer products are called blades.

As an example figure 1 shows an oriented area element which can be represented as the outer product of vectors $\mathbf{r}_1 = \sigma_x + 2\sigma_y$ and $\mathbf{r}_2 = \sigma_z$ of Geometric Algebra:

$$\mathbf{N}_1 = (\sigma_x + 2\sigma_y) \wedge \sigma_z = \sigma_x\sigma_z + 2\sigma_y\sigma_z \qquad \{1\}$$

The base vectors σ_x, σ_y, σ_z of three-dimensional space then are unit vectors

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \qquad \{2\}$$

pointing into perpendicular directions

$$\sigma_x\sigma_y = -\sigma_y\sigma_x \qquad \{3a\}$$

$$\sigma_y\sigma_z = -\sigma_z\sigma_y \qquad \{3b\}$$

$$\sigma_z\sigma_x = -\sigma_x\sigma_z \qquad \{3c\}$$

Thus these base vectors obey Pauli algebra, and they can be represented by (or identified with) Pauli matrices.

In four- or even higher-dimensional spaces or spacetimes k-vectors exist, which cannot be written in the form of an outer product. They are called non-blades. An example for non-blades is shown in figure 2 with

$$\mathbf{N}_2 = \sigma_x\sigma_y + 2\sigma_z\sigma_w \qquad \{4\}$$

The central distinguishing feature between blades and non-blades is the way their components intersect: If oriented area elements have a joint line of intersection, their sum will be a two-dimensional blade. If oriented area elements only have a joint point of intersection, their sum will be a two-dimensional non-blade – a situation which can only be realized in geometries possessing four dimensions at least.

This four-dimensionality is indicated in figure 2 by four axes. In pure four-dimensional space these axes are spacelike axes and point into the direction of the spacelike base vectors σ_w, σ_x, σ_y, σ_z which can be identified with generalized Pauli matrices:

$$\sigma_w^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \qquad \{5\}$$

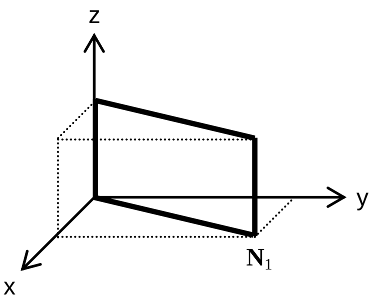

Fig.1: Example of a two-dimensional blade.

In mixed four-dimensional spacetime with one timelike and three spacelike axes the spacelike base vectors γ_x, γ_y, γ_z and the timelike base vector γ_t obey Dirac algebra. So they can be identified with Dirac matrices:

$$\gamma_t^2 = 1 \qquad \gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1 \qquad \{6\}$$

and

$$\gamma_x\gamma_t = -\gamma_t\gamma_x \qquad \gamma_x\gamma_y = -\gamma_y\gamma_x \qquad \{7a\}$$

$$\gamma_y\gamma_t = -\gamma_t\gamma_y \qquad \gamma_y\gamma_z = -\gamma_z\gamma_y \qquad \{7b\}$$

$$\gamma_z\gamma_t = -\gamma_t\gamma_z \qquad \gamma_z\gamma_x = -\gamma_x\gamma_z \qquad \{7c\}$$

The non-blade shown in figure 2 will then be

$$\mathbf{N}_4 = \gamma_x\gamma_y + 2\gamma_z\gamma_t \qquad \{8\}$$

as the fourth axis then is considered as a timelike axis.

3. Effects of blades and non-blades

Geometric objects always possess an operational ambiguity: They can be seen as objects on which is acted on (operands). And they can be seen as objects which act on other geometric objects (operators). As blades and non-blades are geometric objects, they possess this operational ambiguity too. Being interpreted as operators they will produce transformations of other geometric objects.

These effects can be modeled mathematically in a very simple way by the sandwich product. For example, the right- and left-sided sandwich multiplication of vector $\mathbf{r}$ by a two-dimensional blade $\mathbf{N}$

$$\mathbf{r}_{\text{ref}} = -\mathbf{N}\,\mathbf{r}\,\mathbf{N}^{-1} \qquad \{9\}$$

will result in a reflection of vector $\mathbf{r}$ at the plane which is represented by the oriented area element $\mathbf{N}$ [4].

Figure 3 shows such a reflection at the plane represented by $\mathbf{N}_1$ {1}. The three original base vectors σ_x, σ_y, σ_z of a right handed coordinate system

$$\sigma_x\sigma_y\sigma_z = \mathbf{I} \qquad \{10\}$$

are then reflected into three different base vectors e_x, e_y, e_z (see eqs. {13}) of a left-handed coordinate system

$$e_x e_y e_z = -\mathbf{I} \qquad \{11\}$$

Together with the inverse plane $\mathbf{N}_1^{-1}$

$$\mathbf{N}_1^{-1} = \frac{\mathbf{N}_1}{\mathbf{N}_1^2} = \frac{\sigma_x\sigma_z + 2\sigma_y\sigma_z}{(\sigma_x\sigma_z + 2\sigma_y\sigma_z)^2} = -\frac{1}{5}(\sigma_x\sigma_z + 2\sigma_y\sigma_z) \ \{12\}$$

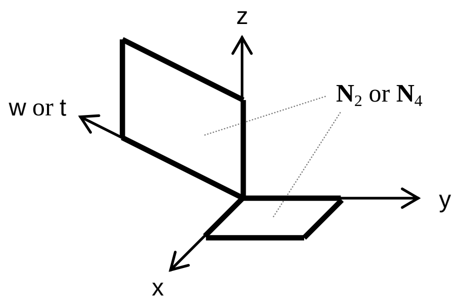

Fig.2: Example of a two-dimensional non-blade.

the new base vectors will be

$$e_x = -(\sigma_x\sigma_z + 2\,\sigma_y\sigma_z)\,\sigma_x\,(\sigma_x\sigma_z + 2\,\sigma_y\sigma_z)^{-1}$$
$$= -\frac{3}{5}\,\sigma_x + \frac{4}{5}\,\sigma_y \qquad \{13a\}$$
$$e_y = \frac{4}{5}\,\sigma_x + \frac{3}{5}\,\sigma_y \qquad \{13b\}$$
$$e_z = \sigma_z \qquad \{13c\}$$

Of course they again obey Pauli algebra:

$$e_x^{\,2} = e_y^{\,2} = e_z^{\,2} = 1 \qquad \{14\}$$

and

$$e_xe_y = -\,e_ye_x \qquad \{15a\}$$
$$e_ye_z = -\,e_ze_y \qquad \{15b\}$$
$$e_ze_x = -\,e_xe_z \qquad \{15c\}$$

Therefore the exemplary vector

$$\mathbf{r} = 1.8\,\sigma_x \qquad \{16\}$$

of figure 3 is reflected into vector

$$\mathbf{r}' = -1.08\,\sigma_x + 1.44\,\sigma_y \qquad \{17\}$$

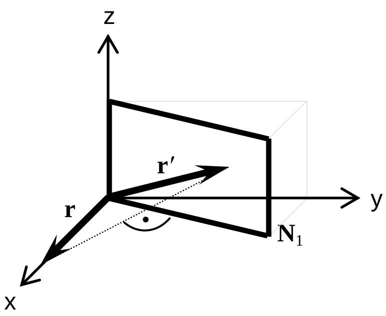

Fig.3: Example of a reflection of a vector at a blade.

In contrast to this well-known Euclidean behavior which we all experience in everyday life when we look into a plane mirror, the reflection of a vector at a non-blade shows some astonishing properties.

Firstly, in pure four-dimensional space (without time direction) the reflection will be a hyperbolic reflection. And secondly, the reflection will result in a strange change of dimensionality.

To discuss these properties the reflection of a vector **r** at a non-blade $\mathbf{N}_2$ {4} will be modeled again by the sandwich product of eq. {9}. The inverse non-blade $\mathbf{N}_2^{-1}$ equals

$$\mathbf{N}_2^{-1} = \frac{\mathbf{N}_2}{\mathbf{N}_2^{\,2}} = \frac{\sigma_x\sigma_y + 2\,\sigma_z\sigma_w}{(\sigma_x\sigma_y + 2\,\sigma_z\sigma_w)^2}$$
$$= \frac{(\sigma_x\sigma_y + 2\,\sigma_z\sigma_w)\quad(5 + 4\,\sigma_x\sigma_y\sigma_z\sigma_w)}{(-5 + 4\,\sigma_x\sigma_y\sigma_z\sigma_w)\,(5 + 4\,\sigma_x\sigma_y\sigma_z\sigma_w)}$$
$$= \frac{1}{3}\,(\sigma_x\sigma_y - 2\,\sigma_z\sigma_w) \qquad \{18\}$$

Thus the new base vectors will be linear combinations of vector and trivector parts. They are

$$e_x = -\frac{5}{3}\,\sigma_x - \frac{4}{3}\,\sigma_y\sigma_z\sigma_w \qquad \{19a\}$$
$$e_y = -\frac{5}{3}\,\sigma_y + \frac{4}{3}\,\sigma_z\sigma_w\sigma_x \qquad \{19b\}$$
$$e_z = \frac{5}{3}\,\sigma_z + \frac{4}{3}\,\sigma_w\sigma_x\sigma_y \qquad \{19c\}$$
$$e_w = \frac{5}{3}\,\sigma_w - \frac{4}{3}\,\sigma_x\sigma_y\sigma_z \qquad \{19d\}$$

They again obey Pauli algebra with

$$e_x^{\,2} = e_y^{\,2} = e_z^{\,2} = e_w^{\,2} = 1 \qquad \{20\}$$

and

$$e_xe_y = -\,e_ye_x \qquad e_xe_w = -\,e_we_x \qquad \{21a\}$$
$$e_ye_z = -\,e_ze_y \qquad e_ye_w = -\,e_we_y \qquad \{21b\}$$
$$e_ze_x = -\,e_xe_z \qquad e_ze_w = -\,e_we_z \qquad \{21c\}$$

Therefore the exemplary vector **r** {16} is now reflected in figure 4 into

$$\mathbf{r}'' = -3.00\,\sigma_x - 2.40\,\sigma_y\sigma_z\sigma_w \qquad \{22\}$$

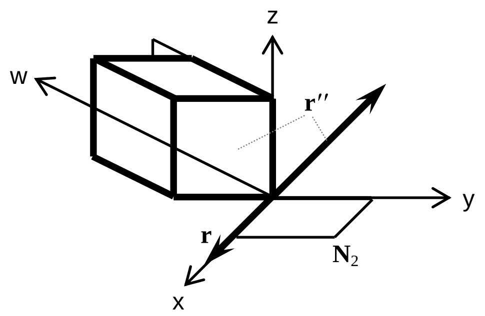

Fig.4: Example of a reflection of a vector at a non-blade.

Reflections at non-blades obviously change the dimension of the reflected object. When reflected at non-blade $\mathbf{N}_2$ one-dimensional vectors become longer one-dimensional vectors ***and*** three-dimensional volume elements. This is indeed a strange feature of a reflection in four-dimensional space. Length is transformed into volume.

4. Spacetime effects of blades and non-blades

Special Relativity is founded on four-dimensional spacetime. Therefore the properties of reflections in spacetime will be discussed next by comparing the reflection of a vector at the two-dimensional spacetime blade $\mathbf{N}_3$ {23} and at the two-dimensional spacetime non-blade $\mathbf{N}_4$ {8}.

The two-dimensional spacetime blade $\mathbf{N}_3$ will be

$$\mathbf{N}_3 = \gamma_x\gamma_y + 2\,\gamma_t\gamma_y \qquad \{23\}$$

The inverse of this spacetime plane $\mathbf{N}_3^{-1}$ then equals

$$\mathbf{N}_3^{-1} = \frac{\mathbf{N}_3}{\mathbf{N}_3^{\,2}} = \frac{\gamma_x\gamma_y + 2\gamma_t\gamma_y}{(\gamma_x\gamma_y + 2\gamma_t\gamma_y)^2}$$
$$= \frac{1}{3}\,(\gamma_x\gamma_y + 2\,\gamma_t\gamma_y) \qquad \{24\}$$

resulting in the following reflections of the four spacetime base vectors:

$$e_x = -\frac{5}{3}\gamma_x - \frac{4}{3}\gamma_t \qquad \{25a\}$$

$$e_y = \gamma_y \qquad \{25b\}$$

$$e_z = -\gamma_z \qquad \{25c\}$$

$$e_t = \frac{4}{3}\gamma_x + \frac{5}{3}\gamma_t \qquad \{25d\}$$

As expected spacetime vectors are reflected into spacetime vectors when reflected at spacetime blades.

Eqs. {25} represent hyperbolic reflections as is shown in figure 5. E.g. vector

$$\mathbf{r} = 1.8\,\gamma_x \qquad \{26\}$$

is hyperbolically reflected into vector

$$\mathbf{r}' = -3.00\,\gamma_x - 2.40\,\gamma_t \qquad \{27\}$$

This hyperbolic reflection can be considered as a Lorentz reflection, as two reflections will make up a hyperbolic rotation which is identical to a Lorentz transformation.

And it is important to note, that the two dark red lines of figure 5 are perpendicular to each other. Thus the component of the original vector **r** and the component of the reflected vector **r′** parallel to the plane of reflection (represented by blade $\mathbf{N}_3$) is given by the orange vector in figure 5.

The orthogonal components of **r** and **r′** (see red and violet vectors of figure 5) thus are orthogonal to the line of intersection of blade $\mathbf{N}_3$ and the xt-plane.

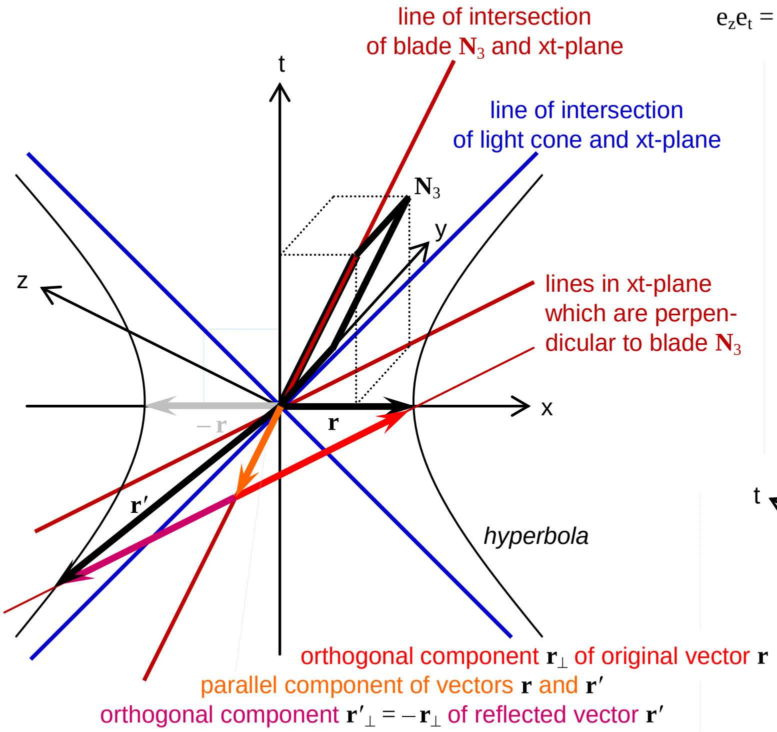

Fig.5: Spacetime example of a reflection of a vector at a blade.

This transformational behavior will be compared to the reflection of a vector at non-blade $\mathbf{N}_4$ {8} in the following. To find the sandwich product, the inverse non-blade $\mathbf{N}_4^{-1}$ is required. It equals

$$\mathbf{N}_4^{-1} = \frac{\mathbf{N}_4}{\mathbf{N}_4^{2}} = \frac{\gamma_x\gamma_y + 2\gamma_z\gamma_t}{(\gamma_x\gamma_y + 2\gamma_z\gamma_t)^2}$$

$$= \frac{(\gamma_x\gamma_y + 2\gamma_z\gamma_t)\ (3 - 4\gamma_x\gamma_y\gamma_z\gamma_t)}{(3 + 4\gamma_x\gamma_y\gamma_z\gamma_t)\ (3 - 4\gamma_x\gamma_y\gamma_z\gamma_t)}$$

$$= -\frac{1}{5}\,(\gamma_x\gamma_y - 2\gamma_z\gamma_t) \qquad \{28\}$$

And again the new base vectors will be linear combinations of vector and trivector parts – a dimensional change already observed in section 3 when the reflection of a vector in pure space was discussed. These new base vectors are

$$e_x = -\frac{3}{5}\gamma_x - \frac{4}{5}\gamma_y\gamma_z\gamma_t \qquad \{29a\}$$

$$e_y = -\frac{3}{5}\gamma_y + \frac{4}{5}\gamma_z\gamma_t\gamma_x \qquad \{29b\}$$

$$e_z = \frac{3}{5}\gamma_z + \frac{4}{5}\gamma_t\gamma_x\gamma_y \qquad \{29c\}$$

$$e_t = \frac{3}{5}\gamma_t + \frac{4}{5}\gamma_x\gamma_y\gamma_z \qquad \{29d\}$$

They still obey Dirac algebra {6} & {7} with

$$e_t^2 = 1 \qquad e_x^2 = e_y^2 = e_z^2 = -1 \qquad \{30\}$$

and

$$e_x e_t = -e_t e_x \qquad e_x e_y = -e_y e_x \qquad \{31a\}$$

$$e_y e_t = -e_t e_y \qquad e_y e_z = -e_z e_y \qquad \{31b\}$$

$$e_z e_t = -e_t e_z \qquad e_z e_x = -e_x e_z \qquad \{31c\}$$

Therefore the exemplary vector **r** of eq. {26} is now reflected into vector

$$\mathbf{r}'' = -1.08\,\gamma_x - 1.44\,\gamma_y\gamma_z\gamma_t \qquad \{32\}$$

Again, length is transformed into volume: when reflected at non-blade $\mathbf{N}_4$, one-dimensional vectors become shorter one-dimensional vectors ***and*** three-dimensional volume elements.

But this time it is an Euclidean reflection.

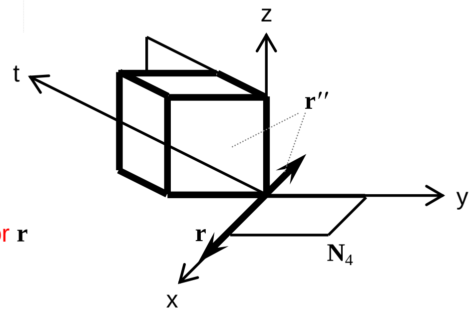

Fig.6: Spacetime example of a reflection of a vector at a non-blade.

5. Change of perspective and didactical remodeling

Geometric Algebra combines geometric and algebraic perspectives into a coherent and overall view on mathematics and physics. This supports learning processes in case students are taught who have very different views on the problem of modeling higher dimensional spaces and spacetimes.

Most students either think algebraically and look for abstract algebraic relations or they think geometrically and look for visual geometric descriptions of the mathematical and physical worlds they try to understand.

A didactical remodeling of the transformational effects of objects which do not exist in three-dimensional space should therefore deliver new insights for students of both kinds. Both perspectives – algebraic and geometric – can be considerably widened if the direct algebraic (see eqs.{19} & {29}) and direct geometric representations (see fig. 4 & 6) are didactically restructured by representing them in complex (or complex-like) planes and Argand (or Argand-like) diagrams.

In these extended Argand diagrams unit volume elements can be considered as real or complex base units which (looking at them from a geometric perspective) are perpendicular to or (looking at them from an algebraic perspective) anti-commute with the corresponding base vectors.

It is then possible to identify four different situations, which can be simplified algebraically (but complicated geometrically) with the help of the four-dimensional oriented volume elements (also called pseudo-scalars) $\mathbf{I}_s$ of pure four-dimensional space and $\mathbf{I}_{st}$ of four-dimensional spacetime:

$$\mathbf{I}_s = \sigma_x\sigma_y\sigma_z\sigma_w \qquad \mathbf{I}_s^2 = (\sigma_x\sigma_y\sigma_z\sigma_w)^2 = 1 \qquad \{33\}$$

$$\mathbf{I}_{st} = \gamma_x\gamma_y\gamma_z\gamma_t \qquad \mathbf{I}_{st}^2 = (\gamma_x\gamma_y\gamma_z\gamma_t)^2 = -1 \qquad \{34\}$$

The four different situations then are:

- **Real plane**

  In three- or four-dimensional pure space the reflection of a vector at a blade will result in a reflection which can be represented in the real plane, because all the relevant base units either square to one (e.g. see reflection at blade $\mathbf{N}_1$, eqs. {13a, b})

  $$\sigma_x^2 = 1 \quad \text{and} \quad \sigma_y^2 = 1 \qquad \{35\}$$

  or see the equivalent reflection described with generalized Dirac matrices γ_x, γ_y, γ_z, γ_w of pure four-dimensional space which then all square to minus one

  $$\gamma_x^2 = -1 \quad \text{and} \quad \gamma_y^2 = -1 \qquad \{36\}$$

  Thus a real plane possesses either two real axes or two imaginary axes.

  A real plane is shown in figure 7, and a similar mathematical behavior can be seen in pseudo-real planes (see next sub-section).

- **Pseudo-real plane**

  The reflection of a time-like vector at non-blade $\mathbf{N}_4$ (see eq. {29d}) and the reflection of a space-like vector at non-blade $\mathbf{N}_4$ (see eqs. {29a, b, c}) will result in reflections, which can be represented in pseudo-real planes, because all the relevant base units either square to one

  $$\gamma_t^2 = 1 \quad \text{and} \quad (\gamma_x\gamma_y\gamma_z)^2 = 1 \qquad \{37\}$$

  or to minus one

  $$\gamma_x^2 = -1 \quad \text{and} \quad (\gamma_y\gamma_z\gamma_t)^2 = -1 \qquad \{38a\}$$

  $$\gamma_y^2 = -1 \quad \text{and} \quad (\gamma_z\gamma_t\gamma_x)^2 = -1 \qquad \{38b\}$$

  $$\gamma_z^2 = -1 \quad \text{and} \quad (\gamma_t\gamma_x\gamma_y)^2 = -1 \qquad \{38c\}$$

  A pseudo-real plane is shown in figure 8. Such planes can be drawn with a vector-like coordinate axis and a trivectorial volume-like coordinate axis for students with a more geometric view.

  Alternatively, pseudo-real planes can be modified for students with a more algebraic view by changing the base units which represent them. So pseudo-real planes can be given either as planes which have a real axis represented by a time-like base vector (e.g. γ_t in{37}) and a pseudo-real axis is represented by a pseudo-vector, which is dual to the time-like base vector:

  $$\gamma_x\gamma_y\gamma_z = \mathbf{I}_{st}\,\gamma_t \qquad \{39\}$$

  Or pseudo-real planes can be given as planes which have an imaginary axis represented by a space-like base vector (e.g. γ_x, γ_y, γ_z in {38}) and a pseudo-imaginary axis represented by a pseudo-vector, which is dual to the corresponding space-like base vector:

  $$\gamma_y\gamma_z\gamma_t = \mathbf{I}_{st}\,\gamma_x \qquad \{40a\}$$

  $$\gamma_z\gamma_t\gamma_x = -\mathbf{I}_{st}\,\gamma_y \qquad \{40b\}$$

  $$\gamma_t\gamma_x\gamma_y = \mathbf{I}_{st}\,\gamma_z \qquad \{40c\}$$

  The algebraic implementation of imaginary and pseudo-imaginary axes in standard four-dimensional spacetime thus causes a change from a right-handed to a left-handed coordinate system, because the additional minus sign of eq. {40b} reverses the y-axis while all other axes remain unchanged compared to coordinate systems with trivectorial volume-like coordinates.

- **Complex plane**

  In spacetime, reflections at blades which mix space and time components can be represented in complex planes, because time-like base vectors square to one while the space-like base vectors square to minus one (e.g. see reflection at blade $\mathbf{N}_3$, eqs. {25a, d})

  $$\gamma_t^2 = 1 \quad \text{and} \quad \gamma_x^2 = -1 \qquad \{41\}$$

  This complex plane is shown in figure 9. A similar mathematical behavior can be seen in pseudo-complex planes, which will be discussed in the next sub-section.

- **Pseudo-complex plane**

The reflection of vectors in pure four-dimensional space at non-blade $\mathbf{N}_2$ (see eqs. {19}) will result in reflections, which can be represented in pseudo-complex planes because one base unit squares to one and the other base unit squares to minus one, if base vectors of pure space are described by generalized Pauli matrices:

$$\sigma_x^2 = 1 \quad \text{and} \quad (\sigma_y\sigma_z\sigma_w)^2 = -1 \qquad \{42a\}$$

$$\sigma_y^2 = 1 \quad \text{and} \quad (\sigma_z\sigma_w\sigma_x)^2 = -1 \qquad \{42b\}$$

$$\sigma_z^2 = 1 \quad \text{and} \quad (\sigma_w\sigma_x\sigma_y)^2 = -1 \qquad \{42c\}$$

$$\sigma_w^2 = 1 \quad \text{and} \quad (\sigma_x\sigma_y\sigma_z)^2 = -1 \qquad \{42d\}$$

The same reflections can also be represented in pseudo-complex planes if base vectors are described by generalized Dirac matrices:

$$\gamma_x^2 = -1 \quad \text{and} \quad (\gamma_y\gamma_z\gamma_w)^2 = 1 \qquad \{43a\}$$

$$\gamma_y^2 = -1 \quad \text{and} \quad (\gamma_z\gamma_w\gamma_x)^2 = 1 \qquad \{43b\}$$

$$\gamma_z^2 = -1 \quad \text{and} \quad (\gamma_w\gamma_x\gamma_y)^2 = 1 \qquad \{43c\}$$

$$\gamma_w^2 = -1 \quad \text{and} \quad (\gamma_x\gamma_y\gamma_z)^2 = 1 \qquad \{43d\}$$

Such a pseudo-complex plane is shown in figure 10, and it can be drawn as a plane of a vector-like coordinate axis and a trivectorial volume-like coordinate axis for students with a more geometric view.

Alternatively, pseudo-complex plane can be modified for students with a more algebraic view by changing the base units which represent them. So pseudo-complex planes can be given either as planes which have a real axis and a pseudo-imaginary axis as the base vectors representing the trivectorial coordinate axes can be written as

$$\sigma_y\sigma_z\sigma_w = -\mathbf{I}_s\,\sigma_x \qquad \{44a\}$$

$$\sigma_z\sigma_w\sigma_x = \mathbf{I}_s\,\sigma_y \qquad \{44b\}$$

$$\sigma_w\sigma_x\sigma_y = -\mathbf{I}_s\,\sigma_z \qquad \{44c\}$$

$$\sigma_x\sigma_y\sigma_z = \mathbf{I}_s\,\sigma_w \qquad \{44d\}$$

Or pseudo-complex planes can be given as planes which have an imaginary axis and a pseudo-real axis as the base vectors representing trivectorial coordinate axes can be written with the modified four-dimensional volume element

$$\mathbf{I}_{pc} = \gamma_x\gamma_y\gamma_z\gamma_w \qquad \mathbf{I}_{pc}^2 = (\gamma_x\gamma_y\gamma_z\gamma_w)^2 = 1 \qquad \{45\}$$

as

$$\gamma_y\gamma_z\gamma_w = \mathbf{I}_{pc}\,\gamma_x \qquad \{45a\}$$

$$\gamma_z\gamma_w\gamma_x = -\mathbf{I}_{pc}\,\gamma_y \qquad \{45b\}$$

$$\gamma_w\gamma_x\gamma_y = \mathbf{I}_{pc}\,\gamma_z \qquad \{45c\}$$

$$\gamma_x\gamma_y\gamma_z = -\mathbf{I}_{pc}\,\gamma_w \qquad \{45d\}$$

This time the algebraic implementation of real and pseudo-imaginary axes or of imaginary and pseudo-real axes will cause no change from a right-handed to a left-handed coordinate system, because always two {44a,c}, {44b,d} of the four axes reverse their direction which compensates the directional reversal.

Obviously we live in a world with four different dimensions. And we surely experience three spatial dimensions and only one time dimension in everyday life and in experiments of physics.

An analysis of the different spacetime situations described by pseudo-real and pseudo-complex planes clearly shows, that four-dimensional pseudo-complex situations are inextricably connected with spacetimes of an even number (0 or 2 or 4) of time-like dimensions. Using the mathematics of pseudo-complex planes makes only sense in pure four-dimensional space, pure four-dimensional time or a mixed four-dimensional spacetime with two space-like and two time-like dimensions.

And it only makes sense to describe the world we live in by pseudo-real planes: The geometry of mixed spacetime with one time-like and three space-like is inextricably connected with pseudo-real situations. Only there base vectors are represented by Dirac matrices via eqs. {37}, {38}, {39}, & {40}.

Therefore it should be expected that pseudo-complex relativity does not make sense. Relativity of one time and three space dimensions follows the geometry of pseudo-reality.

6. These are indeed reflections!

Should the sandwich product of a vector and a non-blade really be called a reflection? At first glance, figures 4 and 6 do not look familiar to us: We never observe a long, one-dimensional string to show a picture which consists of a one-dimensional string and a three-dimensional parallelepiped when we look into mirrors existing in the world we all live in. So do figures 4 and 6 really describe reflections?

This question can only be answered sufficiently when we change our perspective and remodel the mathematical situation didactically by using trivectorial coordinates or Argand-like diagrams discussed in the previous section.

Comparing fig. 7, which shows a reflection in a real plane and fig. 8, which shows a reflection in a pseudo-real plane, both reflections have an identical structure: A vector is reflected into a vector at blade $\mathbf{N}_1$, indicated in fig. 7 by the intersecting line of $\mathbf{N}_1$ and the xy-plane. And a vector is reflected into a vector plus pseudo-vector at non-blade $\mathbf{N}_4$. This non-blade is indicated by the intersecting pseudo-line of non-blade $\mathbf{N}_4$, now written as

$$\mathbf{N}_4 = \gamma_x\gamma_y + 2\,\gamma_z\gamma_t = (\gamma_y + 2\,\gamma_z\gamma_t\gamma_x)\,(-\gamma_t) \qquad \{46\}$$

and the pseudo-real y(ztx)-plane.

The components parallel to blade $\mathbf{N}_1$ (see fig. 7) and the components parallel to non-blade $\mathbf{N}_4$ (see fig. 8) remain unchanged, while the components orthogonal to blade $\mathbf{N}_1$ (fig. 7) and orthogonal to $\mathbf{N}_4$ (fig. 8) change their direction.

Thus the algebraic remodeling is connected with a remodeling of the geometric picture of the sandwich product.

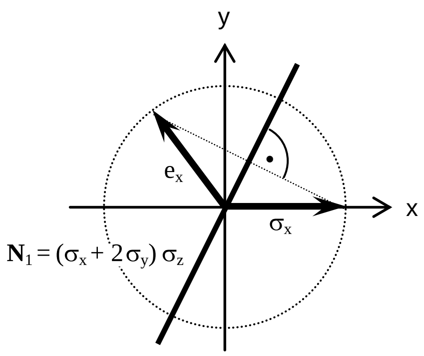

Fig.7: Reflection {13a} shown in a real plane.

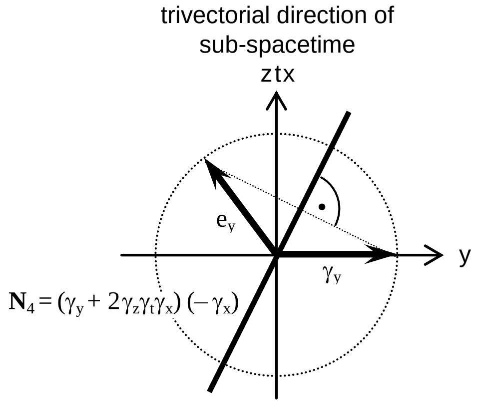

Fig.8: Reflection {29b} shown in a pseudo-real plane.

In a similar way figures 9 and 10 can be compared. Figure 9, which shows a reflection in a complex plane and fig. 10, which shows a reflection in a pseudo-complex plane, have an identical structure: A vector is reflected into a vector at blade $\mathbf{N}_3$, indicated in fig. 9 by the intersecting line of $\mathbf{N}_3$ and the xt-plane.

And a vector is reflected into a vector plus pseudo-vector at non-blade $\mathbf{N}_2$. This non-blade is indicated by the intersecting pseudo-line of non-blade $\mathbf{N}_2$, now written as

$$\mathbf{N}_2 = \sigma_x\sigma_y + 2\,\sigma_z\sigma_w = (\sigma_x + 2\,\sigma_y\sigma_z\sigma_w)\,\sigma_y \quad \{47\}$$

and the pseudo-real x(yzw)-plane.

The components parallel to blade $\mathbf{N}_3$ (see fig. 9) and the components parallel to non-blade $\mathbf{N}_2$ (see fig. 10) remain unchanged, while the components orthogonal to blade $\mathbf{N}_3$ (fig. 9) and orthogonal to $\mathbf{N}_2$ (fig. 10) change their direction.

Thus the algebraic remodeling is connected with a remodeling of the geometric picture of the sandwich product. The dimensional-changes of figures 4 & 6 are now clearly identified as reflections. And by the way: Fig. 9 is equivalent to fig. 5 with a more reluctant use of color.

7. Hyperbolic rotations

Lorentz transformations are understood and modeled as spacetime rotations in Special Relativity. On the other hand rotations can be understood and modeled as two succeeding reflections. Thus Lorentz transformations can be directly described by the reflections discussed in the previous sections.

As a first example let's have a look at a reflection at blade $\mathbf{N}_3$, followed by a second reflection at blade

$$\mathbf{N}_5 = \mathbf{N}_5^{-1} = \gamma_y\gamma_t \qquad \{48\}$$

The rotor is then given by

$$\mathbf{N}_5\mathbf{N}_3 = \gamma_y\gamma_t\,(\gamma_x\gamma_y + 2\,\gamma_t\gamma_y) = \gamma_x\gamma_t - 2 \qquad \{49\}$$

$$(\mathbf{N}_5\mathbf{N}_3)^{-1} = \mathbf{N}_3^{-1}\mathbf{N}_5^{-1} = -\frac{1}{3}\,(\gamma_x\gamma_t + 2) \qquad \{50\}$$

This constitutes a four-dimensional rotation, which takes place in the xt-plane. Thus the components of vectors parallel to the xt-plane are not rotated around a one-dimensional axis, but they are rotated around a two-dimensional axis represented by bivector $\gamma_y\gamma_z$. In this way vectors are rotated around the yz-plane. Components parallel to the yz-plane remain unchanged.

The angle of rotation equals twice the angle between

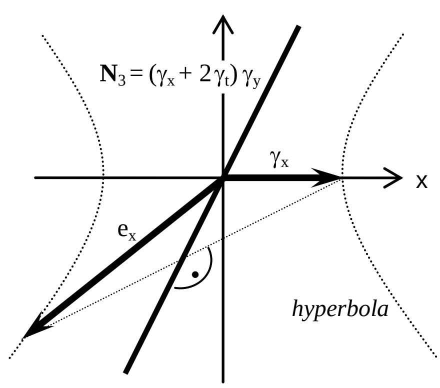

Fig.9: Reflection {25a} shown in a complex plane.

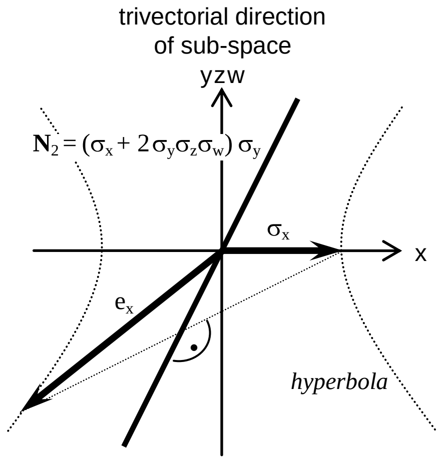

Fig.10: Reflection {19a} shown in a pseudo-complex plane.

the two blades $\mathbf{N}_3$ and $\mathbf{N}_5$. The rotated base vectors then are:

$$e_x = \frac{5}{3}\gamma_x - \frac{4}{3}\gamma_t \qquad \{51a\}$$

$$e_y = \gamma_y \qquad \{51b\}$$

$$e_z = \gamma_z \qquad \{51c\}$$

$$e_t = -\frac{4}{3}\gamma_x + \frac{5}{3}\gamma_t \qquad \{51d\}$$

This is a relativistic boost equivalent to a Lorentz transformation which takes place in the xt-plane. The old coordinate axes into x- and t-direction are transformed into coordinate axes which point into

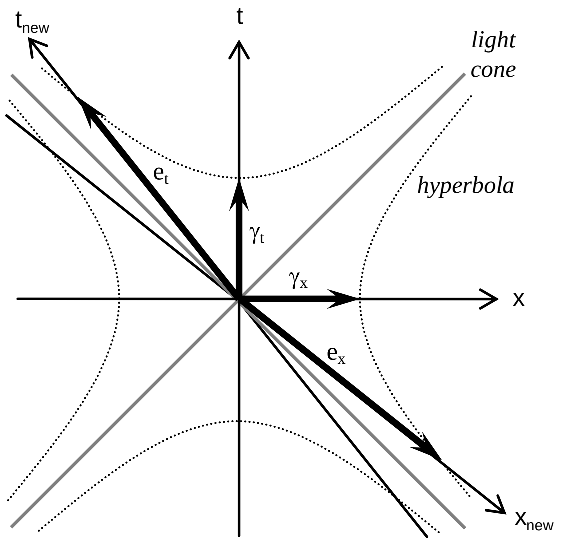

Fig.11: Lorentz transformations {51a, d} shown in a complex plane.

the direction of base vectors e_x {51a} and e_t {51d} (see figure 11). The new coordinate system then moves with velocity

$$v = -\frac{4}{5}c = -0.8\,c \qquad \{52\}$$

with respect to the old coordinate system. And the inverse transformations, which rotate the new base vectors back into the original coordinate system, are

$$\gamma_x = \frac{5}{3}e_x + \frac{4}{3}e_t \qquad \{53a\}$$

$$\gamma_y = e_y \qquad \{53b\}$$

$$\gamma_z = e_z \qquad \{53c\}$$

$$\gamma_t = \frac{4}{3}e_x + \frac{5}{3}e_t \qquad \{53d\}$$

In a similar way hyperbolic rotations around spatial non-blades can be modeled in pure four-dimensional space.

As an example let's have a look at a reflection at non-blade $\mathbf{N}_2$, followed by a second reflection at the blade

$$\mathbf{N}_6 = \mathbf{N}_6^{-1} = \sigma_z\sigma_w \qquad \{54\}$$

The rotor is then given by

$$\mathbf{N}_6\mathbf{N}_2 = \sigma_z\sigma_w(\sigma_x\sigma_y + 2\sigma_z\sigma_w) = \sigma_x\sigma_y\sigma_z\sigma_w - 2 \qquad \{55\}$$

$$(\mathbf{N}_6\mathbf{N}_2)^{-1} = \mathbf{N}_2^{-1}\mathbf{N}_6^{-1} = \sigma_x\sigma_y\sigma_z\sigma_w + 2 \qquad \{56\}$$

This constitutes a four-dimensional rotation, which takes place in all four pseudo-complex planes simultaneously. Now vectors are not rotated into pure vectors, but they are rotated into linear combinations of vectors and trivectors. These rotated pseudo-base vectors are:

$$e_x = \frac{5}{3}\sigma_x + \frac{4}{3}\sigma_y\sigma_z\sigma_w \qquad \{57a\}$$

$$e_y = \frac{5}{3}\sigma_y - \frac{4}{3}\sigma_z\sigma_w\sigma_x \qquad \{57b\}$$

$$e_z = \frac{5}{3}\sigma_z + \frac{4}{3}\sigma_w\sigma_x\sigma_y \qquad \{57c\}$$

$$e_w = \frac{5}{3}\sigma_w - \frac{4}{3}\sigma_x\sigma_y\sigma_z \qquad \{57d\}$$

And the inverse transformations rotate the new base vectors back into the original coordinate system with

$$\sigma_x = \frac{5}{3}e_x - \frac{4}{3}e_ye_ze_w \qquad \{58a\}$$

$$\sigma_y = \frac{5}{3}e_y + \frac{4}{3}e_ze_we_x \qquad \{58b\}$$

$$\sigma_z = \frac{5}{3}e_z - \frac{4}{3}e_we_xe_y \qquad \{58c\}$$

$$\sigma_w = \frac{5}{3}e_w + \frac{4}{3}e_xe_ye_z \qquad \{58d\}$$

As all these transformations happen in a Galilean world, time remains unchanged.

Now let's suppose that there are physicists living in the new coordinate system who are only able to measure the lengths of objects. If they haven't invented methods to measure trivectors, they will only be able to measure the one-dimensional "shadow" of objects which have length and volume.

If a rod which has a length of $\ell = 1$ cm in the original coordinate system is now measured in the rotated coordinate system by these physicists with unit rulers e_x, e_y, e_z, e_w, the new length of the rod will be shorter.

As every unit ruler is expanded by 5/3 compared to the original unit rulers, the rod now has a length of

$$\ell_{new} = \frac{3}{5}\ell = 0.6 \text{ new cm} \qquad \{59\}$$

in the rotated coordinate system.

If these physicists are able to measure lengths, they will be also able to measure positions of a moving rod. Thus they should observe a different inertia of the rod, as it will not move with the same velocity compared to a rod without trivector part.

These physicists, who are not recognizing the correct cause for the unusual inertia of the rod, might be tempted to interpret all this as caused by a field which changes the motion of the rod.

8. Euclidean rotations

As a third example let's have a look at a reflection at blade $\mathbf{N}_1$, followed by a second reflection at blade

$$\mathbf{N}_7 = -\mathbf{N}_7^{-1} = \sigma_y\sigma_z \quad \{60\}$$

The rotor is then given by

$$\mathbf{N}_7\mathbf{N}_1 = \sigma_y\sigma_z\,(\sigma_x\sigma_z + 2\,\sigma_y\sigma_z) = \sigma_x\sigma_y - 2 \quad \{61\}$$

$$(\mathbf{N}_7\mathbf{N}_1)^{-1} = \mathbf{N}_1^{-1}\mathbf{N}_7^{-1} = -\frac{1}{5}\,(\sigma_x\sigma_y + 2) \quad \{62\}$$

This constitutes a spatial rotation, which takes place in the xy-plane. Thus the components of vectors parallel to the xy-plane are rotated in an Euclidean way, while all components orthogonal to the xy-plane (and therefore parallel to the wz-plane as the two-dimensional rotation axis) remain unchanged.

The angle of rotation equals twice the angle between the two blades $\mathbf{N}_1$ and $\mathbf{N}_7$ and the rotated base vectors will be

$$e_x = \frac{3}{5}\,\sigma_x + \frac{4}{5}\,\sigma_y \quad \{63a\}$$

$$e_y = -\frac{4}{5}\,\sigma_x + \frac{3}{5}\,\sigma_y \quad \{63b\}$$

$$e_z = \sigma_z \quad \{63c\}$$

$$e_w = \sigma_w \quad \{63d\}$$

The inverse transformations are

$$\sigma_x = \frac{3}{5}\,e_x - \frac{4}{5}\,e_y \quad \{64a\}$$

$$\sigma_y = \frac{4}{5}\,e_x + \frac{3}{5}\,e_y \quad \{64b\}$$

$$\sigma_z = e_z \quad \{64c\}$$

$$\sigma_w = e_w \quad \{64d\}$$

In a similar way Euclidean rotations around spacetime non-blades can be modeled in four-dimensional spacetime.

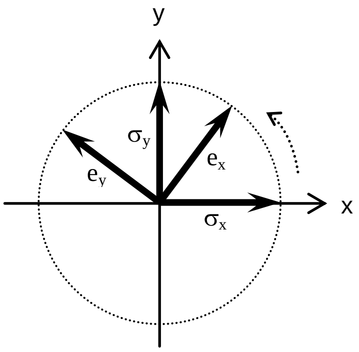

Fig.12: Spatial rotations {63a, b} shown in a real plane.

As an example let's have a look at a reflection at non-blade $\mathbf{N}_4$ {8}, followed by a second reflection at blade $\mathbf{N}_5 = \gamma_y\gamma_t$ {48}. The rotor is then given by

$$\mathbf{N}_5\mathbf{N}_4 = \gamma_y\gamma_t\,(\gamma_x\gamma_y + 2\,\gamma_z\gamma_t) = \gamma_x\gamma_t - 2\,\gamma_y\gamma_z \quad \{65\}$$

$$(\mathbf{N}_5\mathbf{N}_4)^{-1} = \mathbf{N}_4^{-1}\mathbf{N}_5^{-1} = \frac{1}{5}\,(\gamma_x\gamma_t + 2\,\gamma_y\gamma_z) \quad \{66\}$$

This constitutes a four-dimensional rotation, which takes place in all four pseudo-real planes simultaneously. Now vectors are not rotated into pure vectors, but they are rotated into linear combinations of vectors and trivectors. These rotated pseudo-base vectors are:

$$e_x = \frac{3}{5}\,\gamma_x + \frac{4}{5}\,\gamma_y\gamma_z\gamma_t \quad \{67a\}$$

$$e_y = -\frac{3}{5}\,\gamma_y + \frac{4}{5}\,\gamma_z\gamma_t\gamma_x \quad \{67b\}$$

$$e_z = -\frac{3}{5}\,\gamma_z - \frac{4}{5}\,\gamma_t\gamma_x\gamma_y \quad \{67c\}$$

$$e_t = \frac{3}{5}\,\gamma_t + \frac{4}{5}\,\gamma_x\gamma_y\gamma_z \quad \{67d\}$$

These transformations indeed look like rotations as the four base trivectors $\gamma_x\gamma_y\gamma_z$, $\gamma_y\gamma_z\gamma_t$, $\gamma_z\gamma_t\gamma_x$, $\gamma_t\gamma_x\gamma_y$ transform according to

$$\mathbf{V}_{rot} = \mathbf{N}_5\mathbf{N}_4\,\mathbf{V}\,\mathbf{N}_4^{-1}\mathbf{N}_5^{-1} \quad \{68\}$$

into the following rotated pseudo-base trivectors:

$$e_xe_ye_z = \frac{3}{5}\,\gamma_x\gamma_y\gamma_z - \frac{4}{5}\,\gamma_t \quad \{69a\}$$

$$e_ye_ze_t = \frac{3}{5}\,\gamma_y\gamma_z\gamma_t - \frac{4}{5}\,\gamma_x \quad \{69b\}$$

$$e_ze_te_x = -\frac{3}{5}\,\gamma_z\gamma_t\gamma_x - \frac{4}{5}\,\gamma_y \quad \{69c\}$$

$$e_te_xe_y = -\frac{3}{5}\,\gamma_t\gamma_x\gamma_y + \frac{4}{5}\,\gamma_z \quad \{69d\}$$

All these rotations are shown in figures 13, 14, 15, and 16. They rotate para-vectors[1] about 60° (fig. 13), 120° (fig. 14), 240° (fig.15), and again 60° (fig. 16). The inverse transformations can now easily be found by multiplying eqs. {67} & {69} first by the original base vectors and then by the rotated base vectors, e.g. see the back-rotation of eq. {67a}.

$$\gamma_x\,e_x = -\frac{3}{5} + \frac{4}{5}\,\gamma_x\gamma_y\gamma_z\gamma_t = -\frac{3}{5} + \frac{4}{5}\,e_xe_ye_ze_t \quad \{70\}$$

$$\Rightarrow \quad \gamma_x = \frac{3}{5}\,e_x - \frac{4}{5}\,e_ye_ze_t \quad \{71a\}$$

$$\gamma_y = -\frac{3}{5}\,e_y - \frac{4}{5}\,e_ze_te_x \quad \{71b\}$$

$$\gamma_z = -\frac{3}{5}\,e_z + \frac{4}{5}\,e_te_xe_y \quad \{71c\}$$

$$\gamma_t = \frac{3}{5}\,e_t - \frac{4}{5}\,e_xe_ye_z \quad \{71d\}$$

[1] Para-vector: Linear combination of a vector and a trivector.

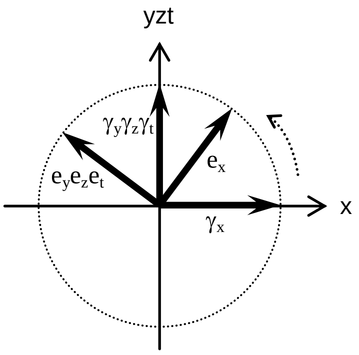

Fig.13: Rotations {67a} and {69b} shown in a pseudo-real plane.

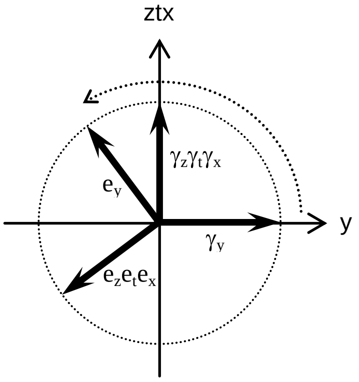

Fig.14: Rotations {67b} and {69c} shown in a pseudo-real plane.

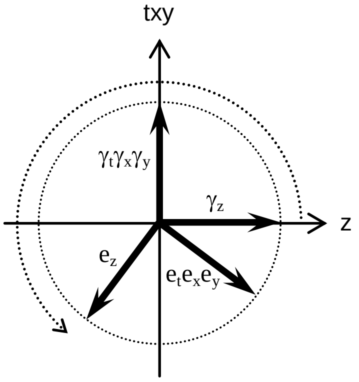

Fig.15: Rotations {67c} and {69d} shown in a pseudo-real plane.

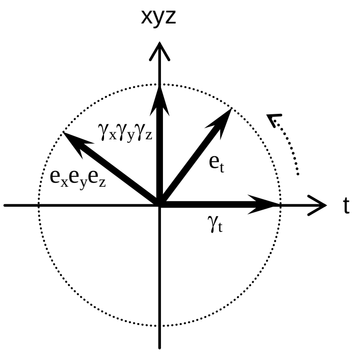

Fig.16: Rotations {67d} and {69a} shown in a pseudo-real plane.

$$\gamma_x\gamma_y\gamma_z = \frac{3}{5}\, e_xe_ye_z + \frac{4}{5}\, e_t \tag{72a}$$

$$\gamma_y\gamma_z\gamma_t = \frac{3}{5}\, e_ye_ze_t + \frac{4}{5}\, e_x \tag{72b}$$

$$\gamma_z\gamma_t\gamma_x = -\frac{3}{5}\, e_ze_te_x + \frac{4}{5}\, e_y \tag{72c}$$

$$\gamma_t\gamma_x\gamma_y = -\frac{3}{5}\, e_te_xe_y - \frac{4}{5}\, e_z \tag{72d}$$

Again physicists who live in such a transformed world and who are only able to measure distances might tell colleagues that the reason for the strange different kinematical behavior of objects should be a field.

9. Outlook

The transition from non-relativistic three-dimensional to relativistic four-dimensional structures must also be thought as a transition from blades to non-blades: One-dimensional lines can be transformed into three-dimensional space or spacetime elements. And three-dimensional space or spacetime elements can be transformed into one-dimensional lines.

And we might be forced to rethink, what it means when we measure distances or volumes, as a distance in the first coordinate system might be partly a volume in another coordinate system. Our rulers not only change the scale, but they also change the dimension – and it is true, that “the algebra of the space that we do observe contains so many wonders that are not yet generally appreciated” [5, p. 1200]. We should think about theses wonders. And we should discuss them with our students.

10. Literature

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