Posterbeitrag zur DPG-Frühjahrstagung 2015, Universität Wuppertal, DD 17.7, Di, 10. März 2015, 14:45 h – 16:15 h, Foyer Ebene G.10

Sandwich Products and Reflections

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Abstract: As reflections are an elementary part of model construction in physics, we really should look for a mathematical picture which allows for a very general description of reflections. The sandwich product delivers such a picture. Using the mathematical language of Geometric Algebra, reflections at vectors of arbitrary dimensions and reflections at multivectors (i.e. at linear combinations of vectors or blades of arbitrary dimensions) can be described mathematically in an astonishingly coherent picture.

Mathematical ingredients:

These mathematical objects are required to describe …

Scalars	(0-vectors)	k, ℓ	$\rightarrow$	dimensionless points	… 3d Geometric Algebra
Vectors	(1-vectors)	**r**, **n**	$\rightarrow$	oriented one-dimensional line elements	
Bivectors	(2-vectors)	**A**, **N**	$\rightarrow$	oriented two-dimensional area elements	
Trivectors	(3-vectors)	**V**, **T**	$\rightarrow$	oriented three-dimensional volume elements	
Quadvectors	(4-vectors)	$\boldsymbol{Q}$, **Q**	$\rightarrow$	oriented four-dimensional hyper volume elements	… 4d Spacetime Algebra of Special Relativity
Pentavectors	(5-vectors)	$\boldsymbol{P}$, **P**	$\rightarrow$	oriented five-dimensional hyper volume elements	… Conformal Geometric Algebra & 5d Cosmological Relativity
Hexavectors	(6-vectors)	$\boldsymbol{H}$, **H**	$\rightarrow$	oriented six-dimensional hyper volume elements	… Conformal Spacetime Algebra
Septavectors	(7-vectors)	$\boldsymbol{S}$, **S**	$\rightarrow$	oriented seven-dimensional hyper volume elements	… Conformal Cosmological Algebra

Sandwich products describe reflections: ***reflected operand* = ± *operator*** multiplied by ***operand*** multiplied by **operator^{-1}**

reflected mathematical object – **reflecting mathematical object** – **mathematical object, which will be reflected** – **inverse of reflecting mathematical object**

Reflection at a point (represented by scalar ℓ)

Scalars:	k_{ref}	$= \ell\, k\, \ell^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= -\ell\, \mathbf{r}\, \ell^{-1}$
Bivectors:	$\mathbf{A}_{ref}$	$= \ell\, \mathbf{A}\, \ell^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= -\ell\, \mathbf{V}\, \ell^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \ell\, \boldsymbol{Q}\, \ell^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= -\ell\, \boldsymbol{P}\, \ell^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \ell\, \boldsymbol{H}\, \ell^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= -\ell\, \boldsymbol{S}\, \ell^{-1}$

These equations are not trivial!
There is a strong conceptual difference between scalars and position scalars.
→ In the same way we distinguish vectors and position vectors.

Reflection at an axis (represented by vector **n**)

Scalars:	k_{ref}	$= \mathbf{n}\, k\, \mathbf{n}^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= \mathbf{n}\, \mathbf{r}\, \mathbf{n}^{-1}$
Bivectors:	$\mathbf{A}_{ref}$	$= \mathbf{n}\, \mathbf{A}\, \mathbf{n}^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= \mathbf{n}\, \mathbf{V}\, \mathbf{n}^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \mathbf{n}\, \boldsymbol{Q}\, \mathbf{n}^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= \mathbf{n}\, \boldsymbol{P}\, \mathbf{n}^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \mathbf{n}\, \boldsymbol{H}\, \mathbf{n}^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= \mathbf{n}\, \boldsymbol{S}\, \mathbf{n}^{-1}$

Reflection at a plane (represented by bivector **N**)

Scalars:	k_{ref}	$= \mathbf{N}\, k\, \mathbf{N}^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= -\mathbf{N}\, \mathbf{r}\, \mathbf{N}^{-1}$
Bivectors:	$\mathbf{A}_{ref}$	$= \mathbf{N}\, \mathbf{A}\, \mathbf{N}^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= -\mathbf{N}\, \mathbf{V}\, \mathbf{N}^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \mathbf{N}\, \boldsymbol{Q}\, \mathbf{N}^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= -\mathbf{N}\, \boldsymbol{P}\, \mathbf{N}^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \mathbf{N}\, \boldsymbol{H}\, \mathbf{N}^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= -\mathbf{N}\, \boldsymbol{S}\, \mathbf{N}^{-1}$

Reflection at a 3d space or spacetime
(represented by trivector **T**)

Scalars:	k_{ref}	$= \mathbf{T}\, k\, \mathbf{T}^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= \mathbf{T}\, \mathbf{r}\, \mathbf{T}^{-1}$
Bivectors:	A_{ref}	$= \mathbf{T}\, A\, \mathbf{T}^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= \mathbf{T}\, V\, \mathbf{T}^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \mathbf{T}\, \boldsymbol{Q}\, \mathbf{T}^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= \mathbf{T}\, \boldsymbol{P}\, \mathbf{T}^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \mathbf{T}\, \boldsymbol{H}\, \mathbf{T}^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= \mathbf{T}\, \boldsymbol{S}\, \mathbf{T}^{-1}$

Reflection at a 4d space or spacetime
(represented by quadvector **Q**)

Scalars:	k_{ref}	$= \mathbf{Q}\, k\, \mathbf{Q}^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= -\mathbf{Q}\, \mathbf{r}\, \mathbf{Q}^{-1}$
Bivectors:	$\mathbf{A}_{ref}$	$= \mathbf{Q}\, \mathbf{A}\, \mathbf{Q}^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= -\mathbf{Q}\, \mathbf{V}\, \mathbf{Q}^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \mathbf{Q}\, \boldsymbol{Q}\, \mathbf{Q}^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= -\mathbf{Q}\, \boldsymbol{P}\, \mathbf{Q}^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \mathbf{Q}\, \boldsymbol{H}\, \mathbf{Q}^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= -\mathbf{Q}\, \boldsymbol{S}\, \mathbf{Q}^{-1}$

Reflection at a 5d space or spacetime
(represented by pentavector **P**)

Scalars:	k_{ref}	$= \mathbf{P}\, k\, \mathbf{P}^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= \mathbf{P}\, \mathbf{r}\, \mathbf{P}^{-1}$
Bivectors:	$\mathbf{A}_{ref}$	$= \mathbf{P}\, \mathbf{A}\, \mathbf{P}^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= \mathbf{P}\, \mathbf{V}\, \mathbf{P}^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \mathbf{P}\, \boldsymbol{Q}\, \mathbf{P}^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= \mathbf{P}\, \boldsymbol{P}\, \mathbf{P}^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \mathbf{P}\, \boldsymbol{H}\, \mathbf{P}^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= \mathbf{P}\, \boldsymbol{S}\, \mathbf{P}^{-1}$

Reflection at a 6d space or spacetime
(represented by hexavector **H**)

Scalars:	k_{ref}	$= \mathbf{H}\, k\, \mathbf{H}^{-1}$
Vectors:	$\mathbf{r}_{ref}$	$= -\mathbf{H}\, \mathbf{r}\, \mathbf{H}^{-1}$
Bivectors:	$\mathbf{A}_{ref}$	$= \mathbf{H}\, \mathbf{A}\, \mathbf{H}^{-1}$
Trivectors:	$\mathbf{V}_{ref}$	$= -\mathbf{H}\, \mathbf{V}\, \mathbf{H}^{-1}$
Quadvectors:	$\boldsymbol{Q}_{ref}$	$= \mathbf{H}\, \boldsymbol{Q}\, \mathbf{H}^{-1}$
Pentavectors:	$\boldsymbol{P}_{ref}$	$= -\mathbf{H}\, \boldsymbol{P}\, \mathbf{H}^{-1}$
Hexavectors:	$\boldsymbol{H}_{ref}$	$= \mathbf{H}\, \boldsymbol{H}\, \mathbf{H}^{-1}$
Septavectors:	$\boldsymbol{S}_{ref}$	$= -\mathbf{H}\, \boldsymbol{S}\, \mathbf{H}^{-1}$

⇒ Every reflection can be modeled mathematically as a threefold multiplication forming a sandwich product. Using Clifford Algebra, matrix multiplication is not required to find a reflected object.

⇒ And it makes sense to reverse this sentence: Every sandwich product can be considered as a reflection – at least in a formal way, e.g.: A rotation equals a reflection at an oriented parallelogram.

Reflections at multivectors

⇒ Linear combinations of vectors (or blades) of different dimensions are called multivectors. A sandwich product of a mathematical object sandwiched between a multivector and its inverse equals a reflection of the mathematical object at this multivector.

⇒ As an example reflections at linear combinations of scalars and vectors ($\ell + \mathbf{n}$) and of scalars and bivectors ($\ell + \mathbf{N}$) will be discussed in the following.

Example I: Hyperbolic rotations

$$\mathbf{M_1} = \ell + \mathbf{n} = \sqrt{\ell^2 - \mathbf{n}^2}\,(\cosh\alpha + \sinh\alpha\,\hat{\mathbf{n}})$$

$$\mathbf{M_1}^{-1} = \frac{\ell - \mathbf{n}}{\ell^2 - \mathbf{n}^2} = \frac{\cosh\alpha - \sinh\alpha\,\hat{\mathbf{n}}}{\sqrt{\ell^2 - \mathbf{n}^2}} \qquad \hat{\mathbf{n}}^2 = \mathbf{1}$$

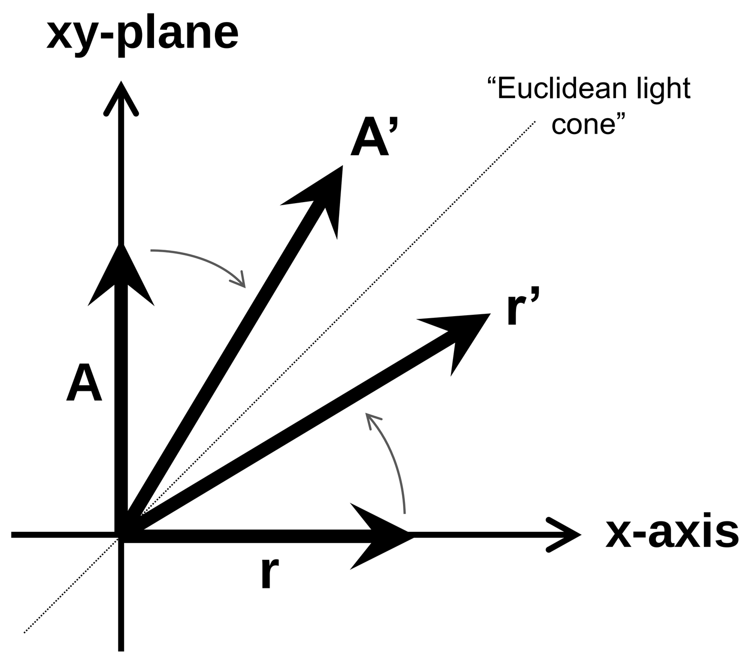
$$\text{and} \quad \mathbf{r} = \mathbf{r_n} + \mathbf{r}_\perp \qquad \mathbf{r_n} \parallel \mathbf{n} \qquad \mathbf{r}_\perp \perp \mathbf{n}$$

$$\Rightarrow \quad \mathbf{M_1\, r\, M_1}^{-1} = \mathbf{r_n} + \cosh(2\alpha)\,\mathbf{r}_\perp + \sinh(2\alpha)\,\hat{\mathbf{n}}\,\mathbf{r}_\perp$$

Thus hyperbolic rotations can be modeled as reflections in Euclidean space. This is shown in the following figure with $\mathbf{M_1} = 3 + \sigma_x$ and $\mathbf{M_1}^{-1} = ⅛(3 - \sigma_x)$

$\mathbf{r} = \sigma_y \quad \Rightarrow \quad \mathbf{r}' = \mathbf{M_1\, r\, M_1}^{-1} = 1{,}25\,\sigma_y + 0{,}75\,\sigma_x\sigma_y$

$\mathbf{A} = \sigma_x\sigma_y \quad \Rightarrow \quad \mathbf{A}' = \mathbf{M_1\, A\, M_1}^{-1} = 0{,}75\,\sigma_y + 1{,}25\,\sigma_x\sigma_y$



This hyperbolic rotation changes the geometric quality of k-vectors: Line elements are transformed into area elements and area elements into line elements.

Example II: Euclidean rotations

$$\mathbf{M_2} = \ell + \mathbf{N} = \sqrt{\ell^2 - \mathbf{N}^2}\,(\cos\alpha + \sin\alpha\,\hat{\mathbf{N}})$$

$$\mathbf{M_2}^{-1} = \frac{\ell - \mathbf{N}}{\ell^2 - \mathbf{N}^2} = \frac{\cos\alpha - \sin\alpha\,\hat{\mathbf{N}}}{\sqrt{\ell^2 - \mathbf{N}^2}} \qquad \hat{\mathbf{N}}^2 = -\mathbf{1}$$

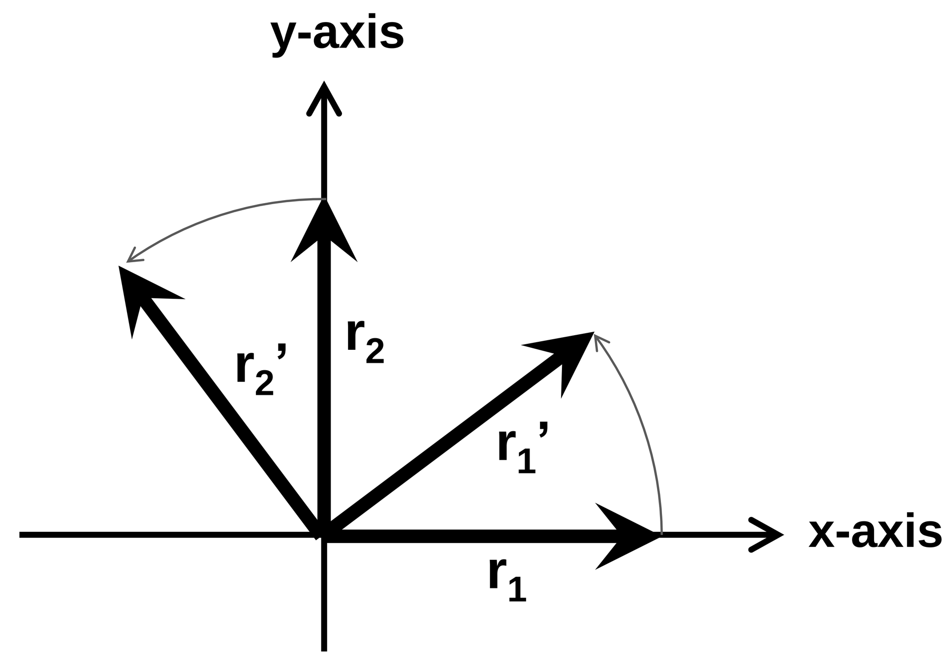
$$\text{and} \quad \mathbf{r} = \mathbf{r_N} + \mathbf{r}_\perp \qquad \mathbf{r_N} \parallel \mathbf{N} \qquad \mathbf{r}_\perp \perp \mathbf{N}$$

$$\Rightarrow \quad \mathbf{M_2\, r\, M_2}^{-1} = \mathbf{r}_\perp + \cos(2\alpha)\,\mathbf{r_N} + \sin(2\alpha)\,\hat{\mathbf{N}}\,\mathbf{r_N}$$

Thus Euclidean rotations can be modeled as reflections at parallelograms. This is shown in the following figure with $\mathbf{M_2} = 3 + \sigma_y\sigma_x = 3 - \sigma_x\sigma_y$ and $\mathbf{M_2}^{-1} = \tfrac{1}{10}(3 + \sigma_x\sigma_y)$

$\mathbf{r_1} = \sigma_x \quad \Rightarrow \quad \mathbf{r_1}' = \mathbf{M_2\, r_1\, M_2}^{-1} = 0{,}8\,\sigma_x + 0{,}6\,\sigma_y$

$\mathbf{r_2} = \sigma_y \quad \Rightarrow \quad \mathbf{r_2}' = \mathbf{M_2\, r_2\, M_2}^{-1} = -0{,}6\,\sigma_x + 0{,}8\,\sigma_y$



The geometric quality (dimension) of k-vectors is conserved.