Sandwich Products and Reflections

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Abstract

As reflections are an elementary part of model construction in physics, we really should look for a mathematical picture which allows for a very general description of reflections. The sandwich product delivers such a picture.

Using the mathematical language of Geometric Algebra, reflections at vectors of arbitrary dimensions and reflections at multivectors (i.e. at linear combinations of vectors of arbitrary dimensions) can be described mathematically in an astonishingly coherent picture.

1. Reflections in physics and mathematics

There is a severe difference between reflections in physics and reflections in mathematics. If we reflect an object (e.g. the arrow shown in figure 1) in a plane mirror, which is a simple optical situation in physics, the picture of the object and the original object are always supposed to have the same distance from the plane mirror, yet lying on different sides of the mirror. Thus we have to consider two kinds of information in this case: information about the position and information about the direction of object and picture.

In mathematics, we only consider directional information. Identical arrows at different positions in space represent the same vector (see figure 2). Vectors (and multivectors in general) do not possess positional information. They are mathematical objects constructed purely by directions. Thus a reflection of a vector at a plane will result in a picture of the vector with arbitrary position in space (see figure 3). An explicit value of the distance of the vector to the plane does not exist.

Thus we are usually dealing with position vectors and position multivectors when discussing problems in physics, while we are dealing with vectors and multivectors when discussing problems in mathematics.

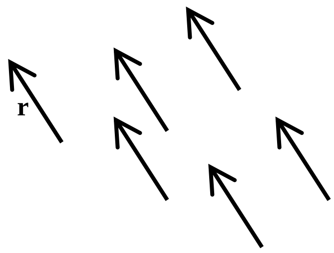

Fig.2: Some arrows which represent vector **r**.

This even gives the impression that mathematics is behaving in a more natural scientific manner compared to physics, as in science one should always change only one variable at a time when investigating problems or carrying out experiments. A mathematical reflection only changes one variable (direction) while a reflection in physics changes two variables (position and direction) at a time.

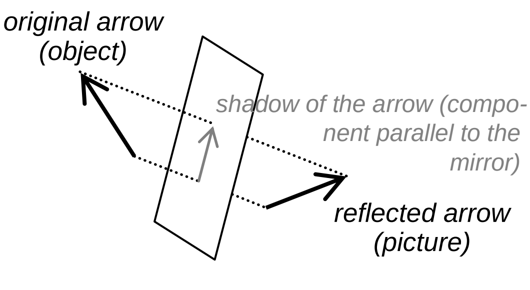

Fig.1: Reflection of an arrow at a plane mirror in physics.

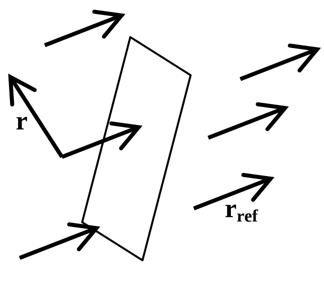

Fig.3: Reflection of vector **r** at a plane in mathematics.

Many physicists therefore tend to discuss geometric situations using conformal concepts. In such situations Conformal Geometric Algebra (CGA) is especially helpful, as positional information of three-dimensional space or four-dimensional spacetime is encoded as directional information of a five- or six-dimensional conformal spacetime [1], [2], [3, chap. 10], [4] in CGA. In this way the two different variables are condensed into one.

Therefore only mathematical reflections will be considered in the following.

2. Geometric representations of scalars

In the preceding section the difference between vectors and position vectors was discussed. A similar relation exists between scalars and position scalars.

A scalar is a dimensionless vector without direction, called a point in geometry.

We are usually dealing with points at fixed positions in physics. Often scientists represent a scalar by a very special point of a coordinate system: the origin. In this paper scalars are represented by all points of space. These mathematical scalars do not possess a position (see figure 4). Every point of space or spacetime will represent a given scalar.

Different vectors might not only have a different direction, but a different 1-dimensional volume, called length, too. In a similar way different scalars possess different 0-dimensional volumes, called value of the scalars.

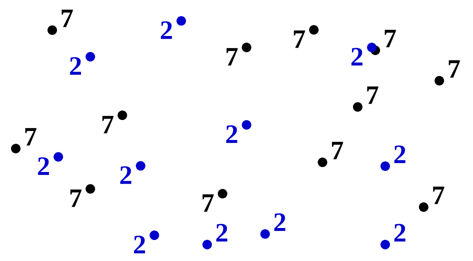

Fig.4: Geometric representation of the scalar 2 (blue points) and the scalar 7 (black points).

Thus scalars cannot be distinguished by their positions (which are not defined), but only by their values.

And as a scalar is a dimensionless object or point without direction, its picture will be the same dimensionless object without direction. Thus scalars (or points) of value k will always be reflected into the identical scalar of same value $k = k_{ref}$ (see figure 5).

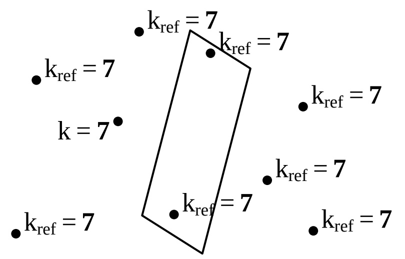

Fig.5: Reflection of scalar k = 7 at a plane in mathematics which results in $k_{ref} = 7$.

3. Basic entities of Geometric Algebra

In three-dimensional space three base vectors are required to describe all possible vectors as linear combinations of the base vectors. These base vectors can be represented by Pauli matrices σ_x, σ_y, σ_z [5, sec. 55], [6, sec. 2.6]. In higher-dimensional spaces of dimension n (some authors like to call them hyperspaces), n different base vectors are required. These base vectors can be seen as generalized Pauli matrices σ_1, σ_2, σ_3, … , σ_n as they obey Pauli algebra.

In four-dimensional spacetime four base vectors are required to describe all possible vectors as linear combinations of base vectors. These base vectors can be represented by Dirac matrices γ_t, γ_x, γ_y, γ_z [5, sec. 156], [6, chap. 5]. In higher-dimensional spaces of dimension n (some authors like to call them hyper-spacetimes), n different base vectors are required. These base vectors can be seen as generalized Dirac matrices γ_1, γ_2, γ_3, … , γ_n as they obey Dirac algebra.

Thus we live in mathematical worlds with the following mathematical ingredients. In three-dimensional space there are:

- Scalars (0-vectors) k, ℓ
  → dimensionless points,
- Vectors (1-vectors) **r**, **n**
  → oriented one-dimensional line elements,
- Bivectors (2-vectors) **A**, **N**
  → oriented two-dimensional area elements,
- Trivectors (3-vectors) **V**, **T**
  → oriented three-dimensional volume elements.

In four-dimensional spacetime there are:

- Scalars (0-vectors) k, ℓ, vectors (1-vectors) **r**, **n**, bivectors (2-vectors) **A**, **N**, trivectors (3-vectors) **V**, **T**,
- Quadvectors (4-vectors) ***Q***, **Q**
  → oriented four-dimensional hyper volume elements.

And in higher-dimensional spaces or spacetimes there are:

- Scalars (0-vectors) k, ℓ, vectors (1-vectors) **r**, **n**, bivectors (2-vectors) **A**, **N**, trivectors (3-vectors) **V**, **T**, quadvectors (4-vectors) ***Q***, **Q**,
- Pentavectors (5-vectors) ***P***, **P**
  → oriented five-dimensional hyper volume elements,
- Hexavectors (6-vectors) ***H***, **H**

→ oriented six-dimensional hyper volume elements,

- Septavectors (7-vectors) ***S***, **S**
  → oriented seven-dimensional hyper volume elements,
- … etc …
- k-dimensional hypervectors (k-vectors)
  → oriented k-dimensional hyper volume elements.

These higher-dimensional entities are required to describe Special Relativity as a four-dimensional geometry, Cosmological Special Relativity [7] or Conformal Geometric Algebra as five-dimensional geometries, Conformal Spacetime Algebra as a six-dimensional geometry and Conformal Cosmological Algebra as a seven-dimensional geometric world (see figure 6).

(Standard) Geometric Algebra	→ **3** dimensions
Spacetime Algebra	→ **4** dimensions
Cosmological Spacetimevelocity Algebra & Conformal Geometric Algebra	→ **5** dimensions
Conformal Spacetime Algebra	→ **6** dimensions
Conformal Cosmological Algebra	→ **7** dimensions
Higher-dimensional Conformal Geometric Algebras	→ **8 and more** dimensions

Fig.6: Dimensionalities of different Geometric Algebras.

4. Central message of this paper

Every reflection can be modeled mathematically as a threefold multiplication forming a sandwich product. Using Clifford Algebra, matrix multiplication is not required to find a reflected object.

And it makes sense to reverse this sentence: Every sandwich product can be considered as a reflection – at least in a formal way, e.g. a rotation equals a reflection at an oriented parallelogram. Thus **sandwich products describe reflections** (see figure 7). This is the central message of this paper.

This description of reflections has a tremendous didactical advantage over other mathematical constructions: Both operators and operands are modeled within the same mathematical language. They are always considered as mathematical objects of Geometric Algebra – linear combinations of vectors or linear combinations of geometric products of vectors.

The axis at which a vector is reflected at is written in the same way as the vector which is reflected. This makes it easy to find parallel and orthogonal components of these vectors (see section 7).

Of course we live in three-dimensional space. Therefore only equations {1} to {4}, {9} to {12}, {17} to {20}, and {25} to {28} are directly accessible to experiments in physics. We will never be able to hold a five-dimensional hypercube in front of a three-dimensional mirror and look at the picture of this hypercube. But we are able to generalize observations we made and mathematical ideas we formed in our very narrow, constricted three-dimensional world we live in.

These generalizations are inventions, and as long as these higher-dimensional mathematical inventions contain all known relations of three-dimensional space, they can be considered as reasonable or rational.

Such a system of higher-dimensional, reasonable relations is given in the following section 5. They help us to understand the geometry of four-dimensional spacetime, which is a very reasonable and rational invention, too. We are (at least at present) not able to hold a 3d spacetime cube with two spacelike and one timelike edges in front of a two-dimensional spacetime mirror with one spacelike and one timelike dimension and look at the picture of this 3d spacetime cube.

But physicists are able to do some experiments in 4d spacetime. And they will find the mathematical structure of the equations given in section 5 helpful to describe the outcome of these experiments in a very coherent, clear and well-structured mathematical way.

In a similar way I hope that the structure of these equations will one day help to describe and understand conformal geometric algebras in a didactically well-structured way.

Today the presentation of reflections sometimes lacks the coherence and clearness seen in equations {1} to {64}. For instance standard textbooks of Geometric Algebra often use a dual description for reflecting a vector. They describe reflections of a vector at a plane or at a hyper-plane by using the dual **n** of the plane or hyper-plane. Then equations like $\mathbf{a}_{\mathrm{ref}} = -\,\mathbf{n}\ \mathbf{a}\ \mathbf{n}$ [3, eq. 2.99, p. 40] may lead to a mathematical picture, which is confusing as it is written as a reflection at a vector, followed by the reflection at a point. In contrast to that, equations {1} to {64} are given without any reference to the dual of the operators.

$$\textbf{reflected operand} = \pm\ \textbf{operator}\ \underset{by}{\overset{multiplied}{}}\ \textbf{operand}\ \underset{by}{\overset{multiplied}{}}\ \textbf{operator}^{-1}$$

Fig.7: Sandwich products describe reflections: An operand is multiplied from left and right by an operator.

5. Overview: Reflection formulas

Reflection at a point (represented by scalar ℓ):

Scalars:	$k_{\mathrm{ref}} = \ell\, k\, \ell^{-1}$	{1}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = -\ell\, \mathbf{r}\, \ell^{-1}$	{2}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \ell\, \mathbf{A}\, \ell^{-1}$	{3}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = -\ell\, \mathbf{V}\, \ell^{-1}$	{4}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \ell\, \boldsymbol{Q}\, \ell^{-1}$	{5}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = -\ell\, \boldsymbol{P}\, \ell^{-1}$	{6}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \ell\, \boldsymbol{H}\, \ell^{-1}$	{7}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = -\ell\, \boldsymbol{S}\, \ell^{-1}$	{8}

Reflection at an axis (represented by vector **n**):

Scalars:	$k_{\mathrm{ref}} = \mathbf{n}\, k\, \mathbf{n}^{-1}$	{9}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = \mathbf{n}\, \mathbf{r}\, \mathbf{n}^{-1}$	{10}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathbf{n}\, \mathbf{A}\, \mathbf{n}^{-1}$	{11}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = \mathbf{n}\, \mathbf{V}\, \mathbf{n}^{-1}$	{12}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathbf{n}\, \boldsymbol{Q}\, \mathbf{n}^{-1}$	{13}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = \mathbf{n}\, \boldsymbol{P}\, \mathbf{n}^{-1}$	{14}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathbf{n}\, \boldsymbol{H}\, \mathbf{n}^{-1}$	{15}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = \mathbf{n}\, \boldsymbol{S}\, \mathbf{n}^{-1}$	{16}

Reflection at a plane (represented by bivector **N**):

Scalars:	$k_{\mathrm{ref}} = \mathbf{N}\, k\, \mathbf{N}^{-1}$	{17}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = -\mathbf{N}\, \mathbf{r}\, \mathbf{N}^{-1}$	{18}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathbf{N}\, \mathbf{A}\, \mathbf{N}^{-1}$	{19}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = -\mathbf{N}\, \mathbf{V}\, \mathbf{N}^{-1}$	{20}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathbf{N}\, \boldsymbol{Q}\, \mathbf{N}^{-1}$	{21}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = -\mathbf{N}\, \boldsymbol{P}\, \mathbf{N}^{-1}$	{22}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathbf{N}\, \boldsymbol{H}\, \mathbf{N}^{-1}$	{23}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = -\mathbf{N}\, \boldsymbol{S}\, \mathbf{N}^{-1}$	{24}

Reflection at a 3d space or reduced spacetime (represented by trivector **T**):

Scalars:	$k_{\mathrm{ref}} = \mathbf{T}\, k\, \mathbf{T}^{-1}$	{25}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = \mathbf{T}\, \mathbf{r}\, \mathbf{T}^{-1}$	{26}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathbf{T}\, \mathbf{A}\, \mathbf{T}^{-1}$	{27}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = \mathbf{T}\, \mathbf{V}\, \mathbf{T}^{-1}$	{28}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathbf{T}\, \boldsymbol{Q}\, \mathbf{T}^{-1}$	{29}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = \mathbf{T}\, \boldsymbol{P}\, \mathbf{T}^{-1}$	{30}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathbf{T}\, \boldsymbol{H}\, \mathbf{T}^{-1}$	{31}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = \mathbf{T}\, \boldsymbol{S}\, \mathbf{T}^{-1}$	{32}

Reflection at a 4d hyperspace or spacetime (represented by quadvector $\mathsf{\mathbf{Q}}$):

Scalars:	$k_{\mathrm{ref}} = \mathsf{\mathbf{Q}}\, k\, \mathsf{\mathbf{Q}}^{-1}$	{33}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = -\mathsf{\mathbf{Q}}\, \mathbf{r}\, \mathsf{\mathbf{Q}}^{-1}$	{34}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathsf{\mathbf{Q}}\, \mathbf{A}\, \mathsf{\mathbf{Q}}^{-1}$	{35}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = -\mathsf{\mathbf{Q}}\, \mathbf{V}\, \mathsf{\mathbf{Q}}^{-1}$	{36}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathsf{\mathbf{Q}}\, \boldsymbol{Q}\, \mathsf{\mathbf{Q}}^{-1}$	{37}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = -\mathsf{\mathbf{Q}}\, \boldsymbol{P}\, \mathsf{\mathbf{Q}}^{-1}$	{38}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathsf{\mathbf{Q}}\, \boldsymbol{H}\, \mathsf{\mathbf{Q}}^{-1}$	{39}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = -\mathsf{\mathbf{Q}}\, \boldsymbol{S}\, \mathsf{\mathbf{Q}}^{-1}$	{40}

Reflection at a 5d hyperspace, hyperspacetime or spacetimevelocity (represented by pentavector $\mathsf{\mathbf{P}}$):

Scalars:	$k_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, k\, \mathsf{\mathbf{P}}^{-1}$	{41}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \mathbf{r}\, \mathsf{\mathbf{P}}^{-1}$	{42}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \mathbf{A}\, \mathsf{\mathbf{P}}^{-1}$	{43}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \mathbf{V}\, \mathsf{\mathbf{P}}^{-1}$	{44}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \boldsymbol{Q}\, \mathsf{\mathbf{P}}^{-1}$	{45}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \boldsymbol{P}\, \mathsf{\mathbf{P}}^{-1}$	{46}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \boldsymbol{H}\, \mathsf{\mathbf{P}}^{-1}$	{47}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = \mathsf{\mathbf{P}}\, \boldsymbol{S}\, \mathsf{\mathbf{P}}^{-1}$	{48}

Reflection at a 6d hyperspace or hyperspacetime (represented by hexavector $\mathsf{\mathbf{H}}$):

Scalars:	$k_{\mathrm{ref}} = \mathsf{\mathbf{H}}\, k\, \mathsf{\mathbf{H}}^{-1}$	{49}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = -\mathsf{\mathbf{H}}\, \mathbf{r}\, \mathsf{\mathbf{H}}^{-1}$	{50}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathsf{\mathbf{H}}\, \mathbf{A}\, \mathsf{\mathbf{H}}^{-1}$	{51}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = -\mathsf{\mathbf{H}}\, \mathbf{V}\, \mathsf{\mathbf{H}}^{-1}$	{52}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathsf{\mathbf{H}}\, \boldsymbol{Q}\, \mathsf{\mathbf{H}}^{-1}$	{53}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = -\mathsf{\mathbf{H}}\, \boldsymbol{P}\, \mathsf{\mathbf{H}}^{-1}$	{54}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathsf{\mathbf{H}}\, \boldsymbol{H}\, \mathsf{\mathbf{H}}^{-1}$	{55}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = -\mathsf{\mathbf{H}}\, \boldsymbol{S}\, \mathsf{\mathbf{H}}^{-1}$	{56}

Reflection at a 7d hyperspace or hyperspacetime (represented by septavector $\mathsf{\mathbf{S}}$):

Scalars:	$k_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, k\, \mathsf{\mathbf{S}}^{-1}$	{57}
Vectors:	$\mathbf{r}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \mathbf{r}\, \mathsf{\mathbf{S}}^{-1}$	{58}
Bivectors:	$\mathbf{A}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \mathbf{A}\, \mathsf{\mathbf{S}}^{-1}$	{59}
Trivectors:	$\mathbf{V}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \mathbf{V}\, \mathsf{\mathbf{S}}^{-1}$	{60}
Quadvectors:	$\boldsymbol{Q}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \boldsymbol{Q}\, \mathsf{\mathbf{S}}^{-1}$	{61}
Pentavectors:	$\boldsymbol{P}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \boldsymbol{P}\, \mathsf{\mathbf{S}}^{-1}$	{62}
Hexavectors:	$\boldsymbol{H}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \boldsymbol{H}\, \mathsf{\mathbf{S}}^{-1}$	{63}
Septavectors:	$\boldsymbol{S}_{\mathrm{ref}} = \mathsf{\mathbf{S}}\, \boldsymbol{S}\, \mathsf{\mathbf{S}}^{-1}$	{64}

Similar equations with the same sandwich product structure can be found for reflections at higher-dimensional hyperspaces and hyper-spacetimes.

6. Reflection of points (scalars) and at points

As points or scalars do not have any direction there will be no change of the non-existing direction, if points are reflected at an arbitrary geometrical object. And the value of the point or scalar does not change either.

Therefore equations {1}, {9}, {17}, {25}, {33}, {41}, {49}, and {57} simply state that

$$k_{\mathrm{ref}} = k \qquad \{65\}$$

because scalars commute with every other mathematical object. Therefore the operator and its inverse cancel if the operand (see figure 7) is a scalar.

A point and its reflected picture are always identical.

So it is not misleading to regard points as being somehow “parallel” to every other geometrical object. It is always possible to place them completely inside other mathematical objects.

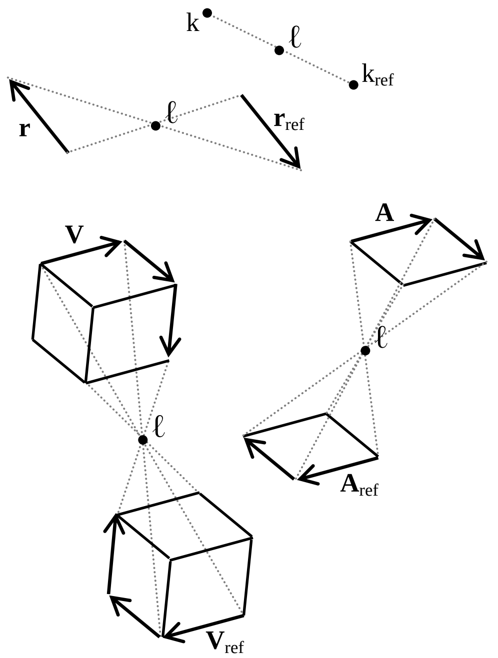

Fig.8: Reflection of scalar k, vector **r**, bivector **A**, and trivector **V** at point or scalar ℓ.

Let's use a mathematically rather illegible, but at the same time sort of well-founded physics-based definition of "parallel" and "orthogonal". If something is "parallel" to a geometric object, it will not change its direction or orientation when reflected at this geometric object. If something is "orthogonal" to a geometric object, it will reverse its direction or orientation when reflected at this geometric object.

A reflection at a point or scalar changes the direction of a vector. Thus a vector has no "parallel" component with respect to a point.

$$\mathbf{r}_{\mathrm{ref}} = -\,\ell\,\mathbf{r}\,\ell^{-1} = -\,\mathbf{r} = -\,\mathbf{r}_{\perp} \qquad \{66\}$$

As a bivector can always be written as an outer product of two vectors in three-dimensional space, these two vectors will both change their direction when reflected at a point or scalar. Thus the directional changes cancel and a bivector does not change its orientation when reflected at a point. Therefore we have to conclude, that a bivector has no "orthogonal" component with respect to a point.

$$\mathbf{A}_{\mathrm{ref}} = \ell\,\mathbf{A}\,\ell^{-1} = \mathbf{A} = \mathbf{A}_{\parallel} \qquad \{67\}$$

In three-dimensional space every trivector can be written as an outer product of three vectors. These three vectors will all change their direction when reflected at a point or scalar. Thus the orientation of a trivector is reversed when reflected at a point. Therefore a trivector has no "parallel" component with respect to a point.

$$\mathbf{V}_{\mathrm{ref}} = -\,\ell\,\mathbf{V}\,\ell^{-1} = -\,\mathbf{V} = -\,\mathbf{V}_{\perp} \qquad \{68\}$$

It is possible, to describe the reflection of higher-dimensional k-vectors at points or scalars:

$$\boldsymbol{Q}_{\mathrm{ref}} = \ \ell\,\boldsymbol{Q}\,\ell^{-1} = \ \boldsymbol{Q} = \ \boldsymbol{Q}_{\parallel} \qquad \{69\}$$
$$\boldsymbol{P}_{\mathrm{ref}} = -\,\ell\,\boldsymbol{P}\,\ell^{-1} = -\,\boldsymbol{P} = -\,\boldsymbol{P}_{\perp} \qquad \{70\}$$
$$\boldsymbol{H}_{\mathrm{ref}} = \ \ell\,\boldsymbol{H}\,\ell^{-1} = \ \boldsymbol{H} = \ \boldsymbol{H}_{\parallel} \qquad \{71\}$$
$$\boldsymbol{S}_{\mathrm{ref}} = -\,\ell\,\boldsymbol{S}\,\ell^{-1} = -\,\boldsymbol{S} = -\,\boldsymbol{S}_{\perp} \qquad \{72\}$$

7. Other reflections in three-dimensional space

The idea that sandwich products describe reflections builds on the behavior of vectors in three-dimensional space. The base vectors σ_x, σ_y, σ_z act as base reflections, changing the sign of components perpendicular to them:

$$\begin{aligned}\mathbf{r}' &= \sigma_x\,\mathbf{r}\,\sigma_x = \sigma_x\,(x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)\,\sigma_x \\ &= +\,x\,\sigma_x - y\,\sigma_y - z\,\sigma_z \qquad \{73\}\end{aligned}$$
$$\mathbf{r}'' = \sigma_y\,\mathbf{r}\,\sigma_y = -\,x\,\sigma_x + y\,\sigma_y - z\,\sigma_z \qquad \{74\}$$
$$\mathbf{r}''' = \sigma_z\,\mathbf{r}\,\sigma_z = -\,x\,\sigma_x - y\,\sigma_y + z\,\sigma_z \qquad \{75\}$$

The first sandwich product {73} represents a reflection at an axis pointing into x-direction. Equation {74} represents a reflection at an axis pointing into y-direction, and equation {75} represents a reflection at an axis pointing into z-direction.

These equations can be generalized for reflections at an axis pointing into the direction of vector **n** {10}. The component of **r** and $\mathbf{r}_{\mathrm{ref}}$ parallel to the axis of reflection can be found by adding these vectors.

$$\begin{aligned}\mathbf{r}_{\parallel} &= \frac{1}{2}(\mathbf{r} + \mathbf{r}_{\mathrm{ref}}) \\ &= \frac{1}{2\mathbf{n}^2}(\mathbf{r\,n\,n} + \mathbf{n\,r\,n}) = \frac{1}{2\mathbf{n}^2}(\mathbf{r\,n} + \mathbf{n\,r})\,\mathbf{n} \\ &= \frac{1}{\mathbf{n}^2}(\mathbf{r} \bullet \mathbf{n})\,\mathbf{n} \qquad \{76\}\end{aligned}$$

The component of **r** (and likewise $-\mathbf{r}_{\mathrm{ref}}$) perpendicular to the axis of reflection can be found by subtracting them (see figure 9).

$$\begin{aligned}\mathbf{r}_{\perp} &= \frac{1}{2}(\mathbf{r} - \mathbf{r}_{\mathrm{ref}}) \\ &= \frac{1}{2\mathbf{n}^2}(\mathbf{r\,n\,n} - \mathbf{n\,r\,n}) = \frac{1}{2\mathbf{n}^2}(\mathbf{r\,n} - \mathbf{n\,r})\,\mathbf{n} \\ &= \frac{1}{\mathbf{n}^2}(\mathbf{r} \wedge \mathbf{n})\,\mathbf{n} \qquad \{77\}\end{aligned}$$

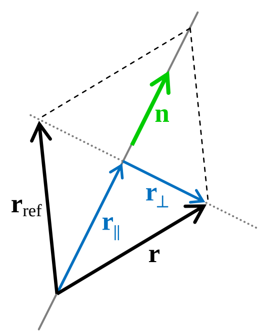

Fig.9: Reflection of vector **r** at an axis pointing into the direction of vector **n**.

Thus the reflected vector $\mathbf{r}_{\text{ref}}$ is the composition of the parallel component $\mathbf{r}_{\|}$ plus the reversed orthogonal component $-\mathbf{r}_{\perp}$:

$$\begin{aligned}\mathbf{r}_{\text{ref}} &= \mathbf{n}\,\mathbf{r}\,\mathbf{n}^{-1} = \frac{1}{\mathbf{n}^2}\,\mathbf{n}\,\mathbf{r}\,\mathbf{n}\\ &= \frac{1}{\mathbf{n}^2}(\mathbf{n}\wedge\mathbf{r} + \mathbf{n}\bullet\mathbf{r})\,\mathbf{n}\\ &= \frac{1}{\mathbf{n}^2}(-\,\mathbf{r}\wedge\mathbf{n} + \mathbf{r}\bullet\mathbf{n})\,\mathbf{n} = -\,\mathbf{r}_{\perp} + \mathbf{r}_{\|} \qquad \{78\}\end{aligned}$$

In a similar way the reflection of a vector at a plane, which is represented by bivector $\mathbf{N}$ (see figure 10), can be modeled. This time a minus sign has to be taken into account again as the base bivectors $\sigma_x\sigma_y$, $\sigma_y\sigma_z$, $\sigma_z\sigma_x$ are considered as base reflections, changing the sign of components perpendicular to them:

$$\begin{aligned}\mathbf{r}' &= -\,\sigma_y\sigma_z\,\mathbf{r}\,\sigma_z\sigma_y\\ &= -\,\sigma_y\sigma_z\,(x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)\,\sigma_z\sigma_y\\ &= -\,x\,\sigma_x + y\,\sigma_y + z\,\sigma_z \qquad \{79\}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'' &= -\,\sigma_z\sigma_x\,\mathbf{r}\,\sigma_x\sigma_z\\ &= +\,x\,\sigma_x - y\,\sigma_y + z\,\sigma_z \qquad \{80\}\end{aligned}$$

$$\begin{aligned}\mathbf{r}''' &= -\,\sigma_x\sigma_y\,\mathbf{r}\,\sigma_y\sigma_x\\ &= +\,x\,\sigma_x + y\,\sigma_y - z\,\sigma_z \qquad \{81\}\end{aligned}$$

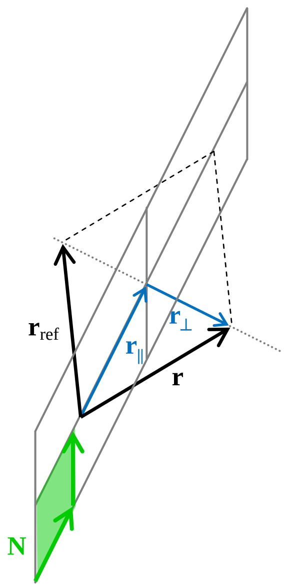

Fig.10: Reflection of vector **r** at a plane represented by bivector **N**.

These additional minus signs are a consequence of the symmetry of inner and outer products. While the inner product of two vectors is commutative, the inner product of a vector and a bivector is anti-commutative. And while the outer product of two vectors is anti-commutative, the outer product of a vector and a bivector is commutative [3, sec. 2.4.1].

We therefore get

$$\begin{aligned}\mathbf{r}_{\text{ref}} &= -\,\mathbf{N}\,\mathbf{r}\,\mathbf{N}^{-1} = -\,\frac{1}{\mathbf{N}^2}\,\mathbf{N}\,\mathbf{r}\,\mathbf{N}\\ &= -\,\frac{1}{\mathbf{N}^2}(\mathbf{N}\wedge\mathbf{r} + \mathbf{N}\bullet\mathbf{r})\,\mathbf{N}\\ &= \frac{1}{\mathbf{N}^2}(-\,\mathbf{r}\wedge\mathbf{N} + \mathbf{r}\bullet\mathbf{N})\,\mathbf{N} = -\,\mathbf{r}_{\perp} + \mathbf{r}_{\|} \qquad \{82\}\end{aligned}$$

with

$$\begin{aligned}\mathbf{r}_{\|} &= \frac{1}{2}(\mathbf{r} + \mathbf{r}_{\text{ref}})\\ &= \frac{1}{2\mathbf{N}^2}(\mathbf{r}\,\mathbf{N}\,\mathbf{N} - \mathbf{N}\,\mathbf{r}\,\mathbf{N}) = \frac{1}{2\mathbf{N}^2}(\mathbf{r}\,\mathbf{N} - \mathbf{N}\,\mathbf{r})\,\mathbf{N}\\ &= \frac{1}{\mathbf{N}^2}(\mathbf{r}\bullet\mathbf{N})\,\mathbf{N} \qquad \{83\}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{\perp} &= \frac{1}{2}(\mathbf{r} - \mathbf{r}_{\text{ref}})\\ &= \frac{1}{2\mathbf{N}^2}(\mathbf{r}\,\mathbf{N}\,\mathbf{N} + \mathbf{N}\,\mathbf{r}\,\mathbf{N}) = \frac{1}{2\mathbf{N}^2}(\mathbf{r}\,\mathbf{N} + \mathbf{N}\,\mathbf{r})\,\mathbf{N}\\ &= \frac{1}{\mathbf{N}^2}(\mathbf{r}\wedge\mathbf{N})\,\mathbf{N} \qquad \{84\}\end{aligned}$$

If a vector which is "parallel" to a plane, is reflected at this plane, this vector will not change its direction. In a similar way we can think about reflections in a three-dimensional mirror in three-dimensional space.

In our world which obviously has three spatial dimensions all vectors are inside this world: Vectors therefore are always "parallel" to trivectors. They do not change when reflected at three-dimensional space which is represented by trivector $\mathbf{T}$.

$$\mathbf{r}_{\text{ref}} = \mathbf{T}\,\mathbf{r}\,\mathbf{T}^{-1} = \mathbf{r} \qquad \{85\}$$

For the same reason ever other geometrical object which exists in three-dimensional space, is "parallel" to three-dimensional space and will not change its direction or orientation when reflected at three-dimensional space.

$$k_{\text{ref}} = \mathbf{T}\,k\,\mathbf{T}^{-1} = k \qquad \{86\}$$

$$\mathbf{A}_{\text{ref}} = \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1} = \mathbf{A} \qquad \{87\}$$

$$\mathbf{V}_{\text{ref}} = \mathbf{T}\,\mathbf{V}\,\mathbf{T}^{-1} = \mathbf{V} \qquad \{88\}$$

Having now discussed all possible basic reflections of scalars (see eq. {65}) and vectors in three-dimensional space, similar relations for basic reflections of bivectors and trivectors can be found in three-dimensional space.

As in three-dimensional space every bivector can be written as an outer product of two different vectors and every trivector as an outer product to three different vectors, for example we will get as reflection of bivector $\mathbf{A} = \mathbf{r}_1 \wedge \mathbf{r}_2$ and trivector $\mathbf{V} = \mathbf{r}_1 \wedge \mathbf{r}_2 \wedge \mathbf{r}_3$ at an axis pointing into the direction of vector $\mathbf{n}$:

$$\begin{aligned}\mathbf{A}_{\text{ref}} &= \mathbf{r}_{1\text{ref}} \wedge \mathbf{r}_{2\text{ref}} = (\mathbf{n}\,\mathbf{r}_1\,\mathbf{n}^{-1}) \wedge (\mathbf{n}\,\mathbf{r}_2\,\mathbf{n}^{-1}) \qquad \{89\}\\ &= \mathbf{n}\,(\mathbf{r}_1 \wedge \mathbf{r}_2)\,\mathbf{n}^{-1} = \mathbf{n}\,\mathbf{A}\,\mathbf{n}^{-1}\end{aligned}$$

$$\begin{aligned}\mathbf{V}_{\text{ref}} &= \mathbf{r}_{1\text{ref}} \wedge \mathbf{r}_{2\text{ref}} \wedge \mathbf{r}_{3\text{ref}} \qquad \{90\}\\ &= \mathbf{n}\,(\mathbf{r}_1 \wedge \mathbf{r}_2 \wedge \mathbf{r}_3)\,\mathbf{n}^{-1} = \mathbf{n}\,\mathbf{V}\,\mathbf{n}^{-1} = -\,\mathbf{V}\end{aligned}$$

8. The rotational perspective on reflections

Every reflection has something of a rotation in it. This rotational viewpoint of reflections will be discussed in the following at the example of a reflection at an axis, which is represented by a vector **n** of the x_2x_4-plane

$$\mathbf{n} = a\,\sigma_2 + b\,\sigma_4 \qquad \{91\}$$

in four-dimensional space. Vectors can then be written as

$$\mathbf{r} = x_1\,\sigma_1 + x_2\,\sigma_2 + x_3\,\sigma_3 + x_4\,\sigma_4 \qquad \{92\}$$

The axis of reflection **n** can be split into two trigonometrical parts (a, b are scalars) with the definitions

$$\left.\begin{aligned} \frac{a}{\sqrt{a^2+b^2}} &= \cos\alpha \qquad \{93\} \\ \frac{b}{\sqrt{a^2+b^2}} &= \sin\alpha \qquad \{94\} \end{aligned}\right\} \cos^2\alpha + \sin^2\alpha = 1$$

and therefore

$$\mathbf{n} = \sqrt{a^2+b^2}\,(\cos\alpha\,\sigma_2 + \sin\alpha\,\sigma_4) \qquad \{95\}$$

$$\begin{aligned} \mathbf{n}^{-1} &= \frac{1}{a^2+b^2}\,(a\,\sigma_2 + b\,\sigma_4) \\ &= \frac{1}{\sqrt{a^2+b^2}}\,(\cos\alpha\,\sigma_2 + \sin\alpha\,\sigma_4) \qquad \{96\} \end{aligned}$$

The reflection at **n** is then given by

$$\begin{aligned} \mathbf{r}_{ref} &= \mathbf{n}\,\mathbf{r}\,\mathbf{n}^{-1} \qquad \{97\} \\ &= (\cos\alpha\,\sigma_2 + \sin\alpha\,\sigma_4)\,\mathbf{r}\,(\cos\alpha\,\sigma_2 + \sin\alpha\,\sigma_4) \\ &= -x_1\,\sigma_1 + (x_2\cos(2\alpha) + x_4\sin(2\alpha))\,\sigma_2 \\ &\quad - x_3\,\sigma_3 + (x_2\sin(2\alpha) - x_4\cos(2\alpha))\,\sigma_4 \end{aligned}$$

This is clearly a reflection at an axis which points into the direction of **n** (see figure 11).

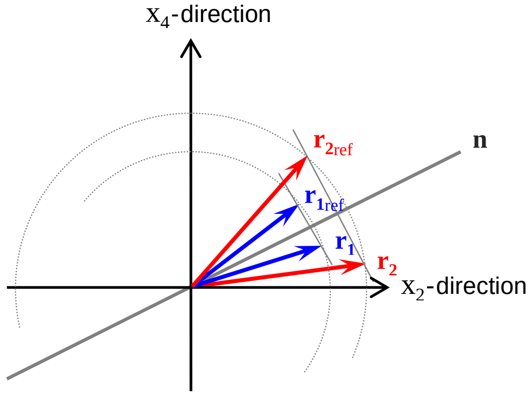

Fig.11: Reflections of some vectors at axis **n**.

Nevertheless figure 11 shows some rotations too. The blue vector $\mathbf{r}_1$ can be thought of being rotated about a small angle into vector $\mathbf{r}_{1ref}$. The red vector $\mathbf{r}_2$ is rotated about a much bigger angle into vector $\mathbf{r}_{2ref}$.

These are rotations about different angles of rotation, but this can be fixed with a simple trick: We only have to reflect the reflected vectors at another vector. If this second reflection vector is parallel to the x_4-axis, the double reflection will result in the situation shown in figure 12.

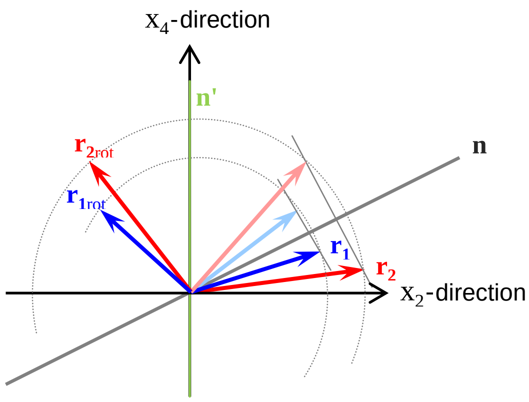

Fig.12: Two reflections result in a rotation.

Thus the rotation

$$\mathbf{r}_{rot} = \mathbf{n}_2\,\mathbf{n}_1\,\mathbf{r}\,\mathbf{n}_1^{-1}\,\mathbf{n}_2^{-1} \qquad \{98\}$$

is identical with a double reflection at the two axes $\mathbf{n}_1$ and $\mathbf{n}_2$. But the same reflection can be found by a reflection at the oriented parallelogram

$$\mathbf{N}_{21} = \mathbf{n}_2\,\mathbf{n}_1 = \underbrace{\mathbf{n}_2 \bullet \mathbf{n}_1}_{\text{scalar}} + \underbrace{\mathbf{n}_2 \wedge \mathbf{n}_1}_{\text{bivector}} \qquad \{99\}$$

with

$$\mathbf{r}_{para} = -\,\mathbf{N}_{21}\,\mathbf{r}\,\mathbf{N}_{21}^{-1} = -\,\mathbf{n}_2\,\mathbf{n}_1\,\mathbf{r}\,\mathbf{n}_1^{-1}\,\mathbf{n}_2^{-1} \qquad \{100\}$$

followed by a reflection at a scalar

$$\mathbf{r}_{rot} = -\,\ell\,\mathbf{r}_{para}\,\ell^{-1} = \mathbf{n}_2\,\mathbf{n}_1\,\mathbf{r}\,\mathbf{n}_1^{-1}\,\mathbf{n}_2^{-1} \qquad \{101\}$$

And there are still more possibilities to describe the same reflection {98}. It can also be found by a reflection at the oriented parallelepiped

$$\begin{aligned} \mathbf{T}_{321} = \mathbf{n}_3\,\mathbf{n}_2\,\mathbf{n}_1 &= \mathbf{n}_3\,(\mathbf{n}_2 \bullet \mathbf{n}_1 + \mathbf{n}_2 \wedge \mathbf{n}_1) \qquad \{102\} \\ &= \underbrace{\mathbf{n}_3\,(\mathbf{n}_2 \bullet \mathbf{n}_1)}_{\text{vector}} + \underbrace{\mathbf{n}_3\,(\mathbf{n}_2 \wedge \mathbf{n}_1)}_{\text{vector + trivector}} \end{aligned}$$

with

$$\mathbf{r}_{epi} = \mathbf{T}_{321}\,\mathbf{r}\,\mathbf{T}_{321}^{-1} = \mathbf{n}_3\,\mathbf{n}_2\,\mathbf{n}_1\,\mathbf{r}\,\mathbf{n}_1^{-1}\,\mathbf{n}_2^{-1}\,\mathbf{n}_3^{-1} \qquad \{103\}$$

followed by a reflection at the third vector $\mathbf{n}_3$:

$$\mathbf{r}_{rot} = \mathbf{n}_3\,\mathbf{r}_{epi}\,\mathbf{n}_3^{-1} = \mathbf{n}_2\,\mathbf{n}_1\,\mathbf{r}\,\mathbf{n}_1^{-1}\,\mathbf{n}_2^{-1} \qquad \{104\}$$

Taking this geometric point of view, rotations can be considered as multiple reflections of linear combinations of vectors of arbitrary dimension. Rotations are reflections of parallelograms, parallelepipeds, hyper-parallelepipeds, which are followed by a second reflection. It therefore makes sense to say that rotations are combinations of reflections.

9. Hyperbolic rotations

A reflection of the Euclidean vector **r** {92} at a linear combination of a scalar and a bivector {99}, {100} will result in a Euclidean rotation {98}, {101}. In a similar way hyperbolic rotations of Euclidean vectors can be modeled.

Now the Euclidean vector **r** {92} is reflected at a linear combination of a scalar and a vector

$$\mathbf{M} = \ell + \mathbf{n} \qquad \{105\}$$

This multivector **M** can be split into two trigonometrical parts again with $n = \sqrt{\mathbf{n}^2}$ and $\hat{\mathbf{n}} = \mathbf{n}/n$

$$\left.\begin{array}{ll} \dfrac{\ell}{\sqrt{\ell^2 - n^2}} = \cosh\alpha & \{106\} \\ \dfrac{n}{\sqrt{\ell^2 - n^2}} = \sinh\alpha & \end{array}\right\} \cosh^2\alpha - \sinh^2\alpha = 1 \quad (\text{if } \ell > n) \qquad \{107\}$$

and therefore

$$\mathbf{M} = \sqrt{\ell^2 - n^2}\,(\cosh\alpha + \sinh\alpha\,\hat{\mathbf{n}}) \qquad \{108\}$$

$$\mathbf{M}^{-1} = \frac{1}{\ell^2 - n^2}(\ell - \mathbf{n}) = \frac{1}{\sqrt{\ell^2 - n^2}}(\cosh\alpha - \sinh\alpha\;\hat{\mathbf{n}}) \qquad \{109\}$$

The reflection of **r** (with equations {76} & {77})

$$\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp} \qquad \{110\}$$

at the geometric object represented by **M** therefore results in

$$\mathbf{r}_{\text{ref}} = \pm\,\mathbf{M}\,\mathbf{r}\,\mathbf{M}^{-1} \qquad \{111\}$$

This is an interesting situation. If the vector part **n** decreases and finally disappears ($\mathbf{n} \to \mathbf{0}$), the reflection should equal a reflection of a vector at a point {2} with negative sign. If the scalar part ℓ decreases and finally disappears ($\ell \to 0$), the reflection should equal a reflection of a vector at an axis {10} with positive sign.

Even if there is no direct graphic interpretation of multivector **M** {105} in a diagram, it will be of some importance for physics (see section 11). The mathematical result of this reflection is impressing. Choosing the positive alternative of equation {111} it is

$$\begin{aligned} \mathbf{r}_{\text{ref}} &= (\ell + \mathbf{n})\,(\mathbf{r}_{\parallel} + \mathbf{r}_{\perp})\,(\ell + \mathbf{n})^{-1} \\ &= (\ell^2 - n^2)\,\mathbf{r}_{\parallel} + (\ell^2 + n^2)\,\mathbf{r}_{\perp} + 2\,\mathbf{n}\,\ell\,\mathbf{r}_{\perp} \\ &= \mathbf{r}_{\parallel} + \cosh(2\alpha)\,\mathbf{r}_{\perp} + \sinh(2\alpha)\,\hat{\mathbf{n}}\,\mathbf{r}_{\perp} \qquad \{112\} \end{aligned}$$

Obviously, this reflection changes the geometrical quality of the reflected vector. While the original vector **r** is a pure 1-vector, the result of the reflection $\mathbf{r}_{\text{ref}}$ is a linear combination of a 1-vector and a 2-vector (see figure 13). This hyperbolic reflection rotates one-dimensional space partly into two-dimensional space.

And it rotates two-dimensional space partly into one-dimensional space as the oriented parallelogram

$$\mathbf{A} = \hat{\mathbf{n}}\,\mathbf{r} = \hat{\mathbf{n}}\,(\mathbf{r}_{\parallel} + \mathbf{r}_{\perp}) \qquad \{113\}$$

is rotated into

$$\begin{aligned} \mathbf{A}_{\text{ref}} &= (\ell + \mathbf{n})\,\hat{\mathbf{n}}\,(\mathbf{r}_{\parallel} + \mathbf{r}_{\perp})\,(\ell + \mathbf{n})^{-1} \\ &= (\ell^2 - n^2)\;\hat{\mathbf{n}}\,\mathbf{r}_{\parallel} + (\ell^2 + n^2)\;\hat{\mathbf{n}}\,\mathbf{r}_{\perp} + 2\,n\,\ell\,\mathbf{r}_{\perp} \\ &= \hat{\mathbf{n}}\,\mathbf{r}_{\parallel} + \cosh(2\alpha)\;\hat{\mathbf{n}}\,\mathbf{r}_{\perp} + \sinh(2\alpha)\;\mathbf{r}_{\perp} \qquad \{114\} \end{aligned}$$

when reflected at multivector **M** {105}.

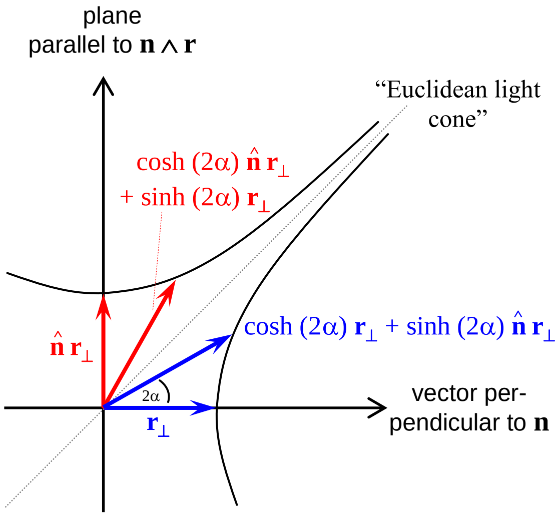

Fig.13: Reflections at multivector M correspond to hyperbolic rotations.

Such changes of the dimensional quality of geometric objects are an astonishing mathematical feature of transformations in Geometric Algebra.

As oriented plane elements like $\hat{\mathbf{n}}\,\mathbf{r}_{\perp}$ have negative squares, they can be identified as imaginary directions of a coordinate system. Therefore reflections {112} and {114} can be visualized by Argand diagrams (see figure 13).

Similar effects can be found in higher-dimensional space or spacetimes, when the reflections at blades and at non-blades are compared.

10. Comparing blades and non-blades

Blades of grade k are k-vectors which can be written as outer products of k vectors. Non-blades are k-vectors which cannot be expressed as an outer product of vectors.

The step from a three-dimensional, purely spacelike Newtonian world to our modern four-dimensional spacetime world of Special Relativity is always a step from blades to non-blades. In three dimensions only blades exist. In spaces or spacetimes of four dimensions there are blades and non-blades. It is interesting to compare their operational behavior.

As an example the operational behavior of the two bivectors

$$\begin{aligned} \mathbf{N}_1 &= a\,\sigma_1\sigma_2 + b\,\sigma_1\sigma_4 \qquad \{115\} \\ &= \sigma_1 \wedge (a\,\sigma_2 + b\,\sigma_4) = \sigma_1 \wedge \mathbf{n} = \sigma_1\,\mathbf{n} \end{aligned}$$

$$\mathbf{N_2} = c\,\sigma_1\sigma_2 + d\,\sigma_3\sigma_4 \qquad \{116\}$$

of a four-dimensional space {92} will be compared in the following.

Bivector $\mathbf{N_1}$ is a blade because it can be written as an outer product of unit vector σ_1 and the reflection vector **n** of equation {91}. Bivector $\mathbf{N_2}$ cannot be written as an outer product.

Therefore the squares

$$\mathbf{N_1}^2 = -a^2 - b^2 \qquad \{117\}$$

$$\mathbf{N_2}^2 = -c^2 - d^2 + 2\,c\,d\,\sigma_1\sigma_2\sigma_3\sigma_4 \qquad \{118\}$$

and consequently the inverses of both bivectors $\mathbf{N_1}$, $\mathbf{N_2}$ possess a different structure. With equations {93} and {94} and with

$$\left.\begin{aligned} \frac{c}{\sqrt{c^2-d^2}} &= \cosh\alpha \qquad \{119\} \\ \frac{d}{\sqrt{c^2-d^2}} &= \sinh\alpha \end{aligned}\right\} \cosh^2\alpha - \sinh^2\alpha = 1 \;\; (\text{if } c > d) \qquad \{120\}$$

these bivector operators and their inverses are

$$\mathbf{N_1} = \sqrt{a^2+b^2}\,(\cos\alpha\,\sigma_1\sigma_2 + \sin\alpha\,\sigma_1\sigma_4) \qquad \{121\}$$

$$\mathbf{N_1}^{-1} = -\frac{1}{a^2+b^2}\,(a\,\sigma_1\sigma_2 + b\,\sigma_1\sigma_4) \qquad \{122\}$$

$$= -\frac{1}{\sqrt{a^2+b^2}}\,(\cos\alpha\,\sigma_1\sigma_2 + \sin\alpha\,\sigma_1\sigma_4)$$

$$\mathbf{N_2} = \sqrt{c^2-d^2}\,(\cosh\alpha\,\sigma_1\sigma_2 + \sinh\alpha\,\sigma_3\sigma_4) \qquad \{123\}$$

$$\mathbf{N_2}^{-1} = -\frac{1}{c^2-d^2}\,(c\,\sigma_1\sigma_2 - d\,\sigma_3\sigma_4) \qquad \{124\}$$

$$= -\frac{1}{\sqrt{c^2-d^2}}\,(\cosh\alpha\,\sigma_1\sigma_2 - \sinh\alpha\,\sigma_3\sigma_4)$$

The reflections of vector **r** {92} of four-dimensional space at these bivectors then result in

$$\mathbf{r}_{1ref} = -\mathbf{N_1}\,\mathbf{r}\,\mathbf{N_1}^{-1} = -\sigma_1\,\mathbf{n}\,\mathbf{r}\,\mathbf{n}^{-1}\,\sigma_1 \qquad \{125\}$$

$$= (\cos\alpha\,\sigma_2 + \sin\alpha\,\sigma_4)\,\sigma_1\,\mathbf{r}\,\sigma_1\,(\cos\alpha\,\sigma_2 + \sin\alpha\,\sigma_4)$$

$$= x_1\,\sigma_1 + (x_2\cos(2\alpha) + x_4\sin(2\alpha))\,\sigma_2 - x_3\,\sigma_3 + (x_2\sin(2\alpha) - x_4\cos(2\alpha))\,\sigma_4$$

$$\mathbf{r}_{2ref} = -\mathbf{N_2}\,\mathbf{r}\,\mathbf{N_2}^{-1} \qquad \{126\}$$

$$= (\cosh(2\alpha) - \sinh(2\alpha)\,\sigma_1\sigma_2\sigma_3\sigma_4)\,(x_1\,\sigma_1 + x_2\,\sigma_2 - x_3\,\sigma_3 - x_4\,\sigma_4)$$

$$\begin{aligned} = \;& (+\cosh(2\alpha)\,\sigma_1 + \sinh(2\alpha)\,\sigma_2\sigma_3\sigma_4)\,x_1 \\ &+ (+\cosh(2\alpha)\,\sigma_2 - \sinh(2\alpha)\,\sigma_3\sigma_4\sigma_1)\,x_2 \\ &+ (-\cosh(2\alpha)\,\sigma_3 - \sinh(2\alpha)\,\sigma_4\sigma_1\sigma_2)\,x_3 \\ &+ (-\cosh(2\alpha)\,\sigma_4 + \sinh(2\alpha)\,\sigma_1\sigma_2\sigma_3)\,x_4 \end{aligned}$$

Transformation {125} describes a reflection at plane $\mathbf{N_1}$, which can be identified geometrically as a first reflection at an axis pointing into the direction of **n**, followed by a second reflection at an axis pointing into the direction of σ_1, and then followed by a third reflection at a point.

This all results in an anti-rotation: a rotation followed by a reflection at a point.

This geometric situation corresponds to the usual situation in three-dimensional space: $\mathbf{N_1}$ is the sum of two planes, which possess a line of intersection pointing into the direction of σ_1. Therefore the two planes $a\,\sigma_1\sigma_2$ and $b\,\sigma_1\sigma_4$ can be combined into the one plane $\mathbf{N_1}$.

Transformation {126} describes a reflection at the non-blade $\mathbf{N_2}$. This non-blade is a sum of two planes which only possess one point of intersection[1]. Therefore the reflection at $\mathbf{N_2}$ cannot be considered as a series of several succeeding reflections, but as only one immediate reflection at the two planes and their common point of intersection.

This equals a hyperbolic anti-rotation and rotates pure one-dimensional vector components into linear combinations of one-dimensional vectors and three-dimensional trivectors.

Spacelike unit trivectors square to minus one. Algebraically they can be seen as imaginary units. Therefore the whole situation can be visualized by Argand diagrams again (see figure 14).

A complete description of all these rotations can be found with the help of the four-dimensional pseudoscalar

$$\mathbf{I_4} = \sigma_1\sigma_2\sigma_3\sigma_4 \qquad \{127\}$$

when trivector

$$\mathbf{V} = x_1\,\sigma_1\,\mathbf{I_4} + x_2\,\sigma_2\,\mathbf{I_4} + x_3\,\sigma_3\,\mathbf{I_4} + x_4\,\sigma_4\,\mathbf{I_4} \qquad \{128\}$$

$$= x_1\,\sigma_2\sigma_3\sigma_4 - x_2\,\sigma_3\sigma_4\sigma_1 + x_3\,\sigma_4\sigma_1\sigma_2 - x_4\,\sigma_1\sigma_2\sigma_3$$

is reflected at the same bivector $\mathbf{N_2}$ {116}. It results in the hyperbolic anti-rotation

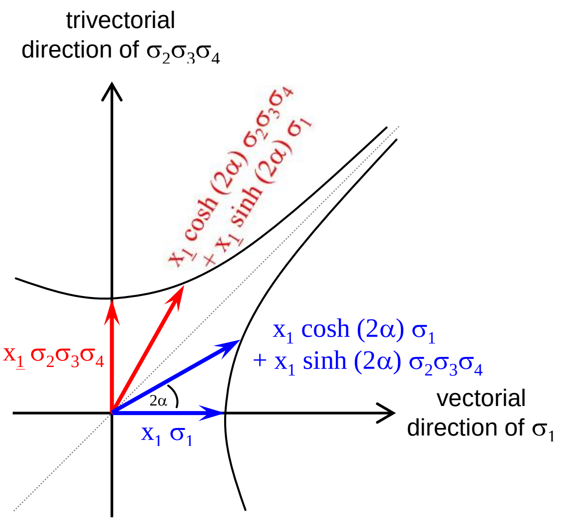

Fig.14: The reflections of vector $x_1\,\sigma_1$ and trivector $x_1\,\sigma_2\sigma_3\sigma_4$ at bivector $c\,\sigma_1\sigma_2 + d\,\sigma_3\sigma_4$ correspond to hyperbolic anti-rotations.

[1] Such a geometric situation is only possible in spaces or spacetimes of four or more dimensions.

$$
\begin{aligned}
\mathbf{V}_{2\mathrm{ref}} &= -\,\mathbf{N_2}\,\mathbf{V}\,\mathbf{N_2}^{-1} && \{129\}\\
&= (\cosh(2\alpha) - \sinh(2\alpha)\,\sigma_1\sigma_2\sigma_3\sigma_4)\\
&\qquad (x_{\underline{1}}\,\sigma_1 + x_{\underline{2}}\,\sigma_2 - x_{\underline{3}}\,\sigma_3 - x_{\underline{4}}\,\sigma_4)\,\mathbf{I_4}\\
&= \quad (+\cosh(2\alpha)\,\sigma_2\sigma_3\sigma_4 + \sinh(2\alpha)\,\sigma_1)\,x_{\underline{1}}\\
&\quad + (-\cosh(2\alpha)\,\sigma_3\sigma_4\sigma_1 + \sinh(2\alpha)\,\sigma_2)\,x_{\underline{2}}\\
&\quad + (-\cosh(2\alpha)\,\sigma_4\sigma_1\sigma_2 - \sinh(2\alpha)\,\sigma_3)\,x_{\underline{3}}\\
&\quad + (+\cosh(2\alpha)\,\sigma_1\sigma_2\sigma_3 - \sinh(2\alpha)\,\sigma_4)\,x_{\underline{4}}
\end{aligned}
$$

Thus the complete reflection of a multivector

$$
\begin{aligned}
\mathbf{M_2} &= \mathbf{r} + \mathbf{V} && \{130\}\\
&= x_1\,\sigma_1 + x_{\underline{1}}\,\sigma_2\sigma_3\sigma_4 + x_2\,\sigma_2 - x_{\underline{2}}\,\sigma_3\sigma_4\sigma_1\\
&\quad + x_3\,\sigma_3 + x_{\underline{3}}\,\sigma_4\sigma_1\sigma_2 + x_4\,\sigma_4 - x_{\underline{4}}\,\sigma_1\sigma_2\sigma_3
\end{aligned}
$$

at the non-blade $\mathbf{N_2}$ may be modeled as

$$
\mathbf{M}_{2\mathrm{ref}} = -\,\mathbf{N_2}\,\mathbf{M_2}\,\mathbf{N_2}^{-1} \qquad \{131\}
$$

11. Worlds behind our world

There is this standard mathematical world which reflects or rotates vectors into vectors (see section 8 or the reflection at blades in section 10). This coincides with our everyday experience of reflections or rotations.

Long objects (like a straw or a broomstick) will remain long objects when reflected in a plane mirror or rotated about an axis. Pictures of such (nearly) one-dimensional objects like straws or broomsticks will always be “one-dimensional”. They will never transform into something broad and “two-dimensional”. Straws and broomsticks never become plates or carpets as they do not undergo a dimensional change in the world we live in.

But there are mathematical worlds behind this standard world of mathematics, which allow dimensional changes. One-dimensional objects can be transformed into linear combinations of one- and two-dimensional objects by transformations. In these worlds, a one-dimensional broomstick grows into a longer one-dimensional broomstick and a two-dimensional carpet, when reflected at geometric objects like (ℓ + $\mathbf{n}$) {105}. And a “two-dimensional” carpet grows into the sum of a broader, bigger carpet and and a broomstick.

Broomsticks become carpets, and carpets become broomsticks.

And “one-dimensional” broomsticks become longer one-dimensional broomsticks and “three-dimensional” cupboards, while “three-dimensional” cupboards become sums of even bigger cupboards and broomsticks, when reflected at non-blades {116}.

At first sight it seems that these transformations are artificial mathematical speculations – mathematical operations which are not relevant for physics.

At a second sight yet, it might make sense to consider the philosophical position of Dirac and others: Mathematical laws seem to be inventions, but whatever we invent in mathematics (provided it is beautiful) will be found in physics as a law of nature one day[2]. Following this philosophical position, dimensional changes should exist, because they are beautiful.

And they are of tremendous structural and conceptual importance already today as the foundation of relativity is firmly build on them. For example, the electric vector is rotated into a magnetic bivector, while the magnetic bivector is rotated into an electric vector when an electromagnetic field moves forward relativistically [9, sec. VIII], [10, sec. 6].

Magnetic and electric fields can be transformed into each other even though they are encoded mathematically by structures of different dimensionality.

Similar effects should be expected when we describe our (three-dimensional) world in a conformal way. The geometry of a three-dimensional world can be encoded geometrically as a five-dimensional world using Conformal Geometric Algebra (CGA).

The basic entities of Conformal Geometric Algebra are blades and can be constructed as outer products of points [12, sec.4.3.7]. The mathematics of this CGA world is tremendously effective in solving computer graphics problems [13] or other engineering problems in computer science [12].

As dimensional changes are not very relevant for computer graphics, computer scientists mainly discuss the geometry of blades or of versors and their transformation properties.

But this might be different for physicists: relativity shows us that dimensional changes are side effects of relativistic transformations. Therefore it should be expected that reflections at non-blades will come to life in CGA too, if physical problems are modeled in a conformal way.

The sandwich product thus will not only help us to understand the mathematics of conformal worlds, it will even help us to understand the physics of such conformal worlds one day.

12. Literature

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[2] As Dirac states, “one may describe the situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen [11].

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