Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Wintersemester 2014/2015

Dr. Horn

Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A crash course of Geometric Algebra)

Stand: 31. Dez. 2014

Teaching & learning contents according to the modular description of LV 200 691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Most of this will be discussed in the standard language of the rather old-fashioned linear algebra or matrix algebra found in most textbooks of business mathematics or mathematical economics.

But as it might be helpful to get an impression of some more interesting new approaches, we will start this part of the course with a short introduction (6 x 45 min.) to Geometric Algebra.

Linear algebra is the study of linear sets of equations and their transformation properties.

> Many economic relationships can be expressed as (or approximated by) linear equations.

$$3x_1^{\,4} + \frac{12}{x_2} - 5\sqrt{x_3} = 27$$

or

$$3x_1^{\,4} + 12x_2^{\,-1} - 5x_3^{\,\frac{1}{2}} = 27$$

is **no** linear equation

$$3x_1 + 12x_2 - 5x_3 = 27$$

or

$$3x_1^{\,1} + 12x_2^{\,1} - 5x_3^{\,1} = 27$$

is a linear equation

This is an algebraic equation.

But remember Galileo Galilei!

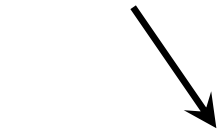
Galileo Galilei (1564 – 1642):

> "Philosophy is written in that great book which ever lies before our eyes – I mean the universe – but we cannot understand it if we do not first learn the language and characters in which it is written.
>
> This language is mathematics, and **the characters are** triangles, circles and other **geometrical figures**, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."

The characters are combined in an algebraic way, but they are geometrical figures.

Therefore we can think about the given linear equation as an equation representing a geometrical relationship in space:

$$3x_1 + 12x_2 - 5x_3 = 27$$

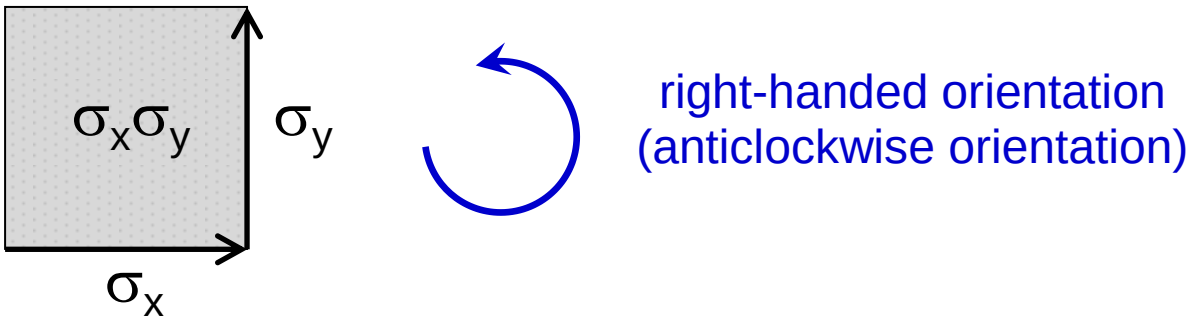


$$3x + 12y - 5z = 27$$

$$3 \text{ vector in x-direction} + 12 \text{ vector in y-direction} - 5 \text{ vector in z-direction} = 27$$

Only now we have a truly geometrical formulation.

What is space? How can we describe it?
How can we model the base vectors as basic constituents of space?



σ_x ..... base vector in x-direction
(one step parallel to x-axis)

σ_y ..... base vector in y-direction
(one step parallel to y-axis)

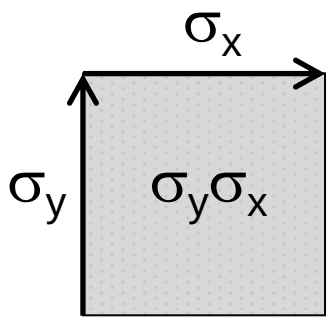
σ_z ..... base vector in z-direction
(one step parallel to z-axis)

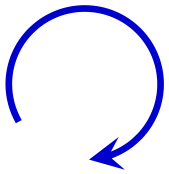
Base vectors are unit vectors:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

The multiplication of two different base vectors results in a base bivector, which represents the oriented area element (or oriented rectangle): $\sigma_x \sigma_y = \sigma_x\sigma_y$

Now we change the order of our steps:





left-handed orientation
(clockwise orientation)

σ_x ..... base vector in x-direction
(one step parallel to x-axis) } *second step*

σ_y ..... base vector in y-direction
(one step parallel to y-axis) } *first step*

σ_z ..... base vector in z-direction
(one step parallel to z-axis)

This time the multiplication of the base vectors results in a different bivector

$$\sigma_y \, \sigma_x = \sigma_y\sigma_x$$

with reversed order of the base vectors.

When multiplied the order of vectors is important! It encodes information about the orientation of the resulting area element.

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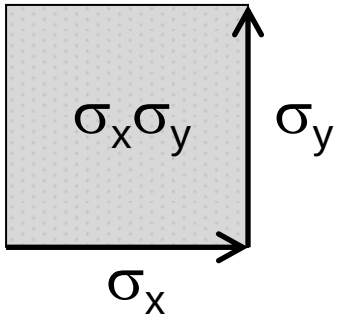
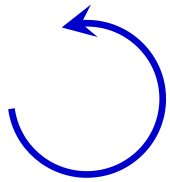
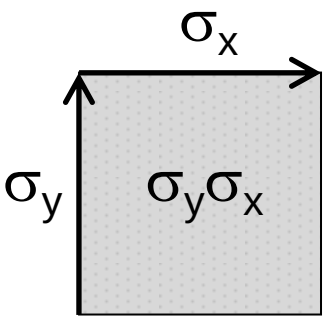
It encodes information about the orientation of the resulting area element.

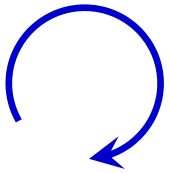




right-handed orientation
(anticlockwise orientation)

positive orientation in a right-handed coordinate system





left-handed orientation
(clockwise orientation)

negative orientation in a right-handed coordinate system

upper area element = – lower area element

$$\sigma_x\sigma_y = -\sigma_y\sigma_x$$

Similar relations can be found for the other directions:

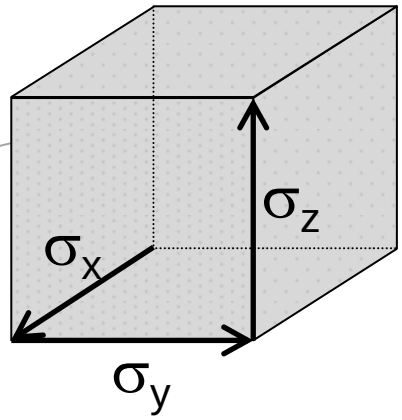
$$\sigma_y\sigma_z = -\sigma_z\sigma_y$$

$$\sigma_z\sigma_x = -\sigma_x\sigma_z$$

If the order of two different neighbouring base vectors of a multiplication is reversed, a minus sign has to be introduced.

An oriented volume element is called trivector:

$\sigma_x\sigma_y\sigma_z$



Thus in three-dimensional space
$1 + 3 + 3 + 1 = 2^3 = 8$ base elements exist.

One base scalar: 1

Three base vectors: σ_x, σ_y, σ_z

Three base bivectors: $\sigma_x\sigma_y$, $\sigma_y\sigma_z$, $\sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Anti-Commutativity

A multiplication by scalars or by trivectors (pseudoscalars) is always commutative.

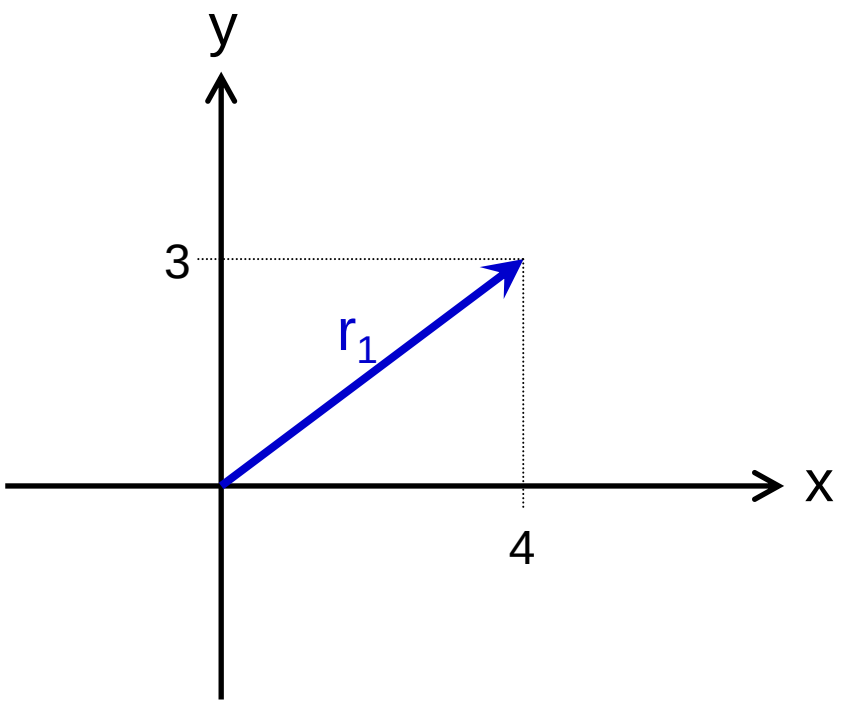
If the order of two different neighbouring base vectors of a multiplication is reversed, a minus sign has to be introduced.

$\Rightarrow$ Different base vectors anticommute.

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors.

Example: $r_1 = 4\,\sigma_x + 3\,\sigma_y$



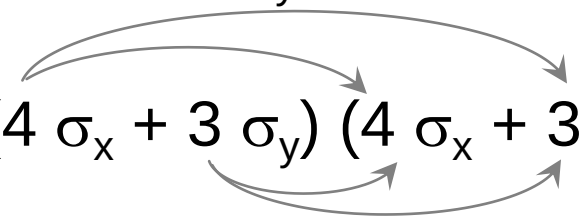
General case in three dimensions:

$$r = x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

The Square of Vectors

Example:

$$r_1^2 = (4\,\sigma_x + 3\,\sigma_y)^2$$



$$= (4\,\sigma_x + 3\,\sigma_y)\,(4\,\sigma_x + 3\,\sigma_y)$$

$$= 16\,\sigma_x^2 + 12\,\sigma_x\sigma_y + 12\,\sigma_y\sigma_x + 9\,\sigma_y^2$$

$$= 16 + 12\,\sigma_x\sigma_y - 12\,\sigma_x\sigma_y + 9$$

$$= 16 + 9 = 25$$

As all mixed terms cancel, the square of a vector always is a positive scalar:

$$r^2 = (x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)^2$$

$$= (x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)\,(x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)$$

$$= x^2 + y^2 + z^2$$

Length of Vectors

The positive square root of the square of a vector can be identified as its length:

$$|r_1| = \sqrt{r_1^2} = \sqrt{25} = 5$$

In general:

$$|r| = \sqrt{r^2} = \sqrt{x^2 + y^2 + z^2}$$

Unit Vectors

A division of a vector r by its length results in a unit vector $\hat{r}$ of length 1:

$$\hat{r}_1 = \frac{r_1}{|r_1|} = \frac{1}{5}(4\,\sigma_x + 3\,\sigma_y) = 0.8\,\sigma_x + 0.6\,\sigma_y$$

with $\hat{r}_1^2 = 0.8^2 + 0.6^2 = 0.64 + 0.36 = 1$

In general:

$$\hat{r} = \frac{r}{|r|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)$$

with $\hat{r}^2 = 1$

Inverse Vectors

A division of a vector r by its square r^2 results in the inverse r^{-1} of this vector

$$r_1^{-1} = \frac{r_1}{r_1^2} = \frac{1}{25}(4\,\sigma_x + 3\,\sigma_y) = 0.16\,\sigma_x + 0.12\,\sigma_y$$

as the product of a vector and its inverse results in the unit scalar:

$$r_1\, r_1^{-1} = r_1^{-1}\, r_1 = 1$$

In general:

$$r^{-1} = \frac{r}{r^2} = \frac{1}{x^2 + y^2 + z^2}(x\,\sigma_x + y\,\sigma_y + z\,\sigma_z)$$

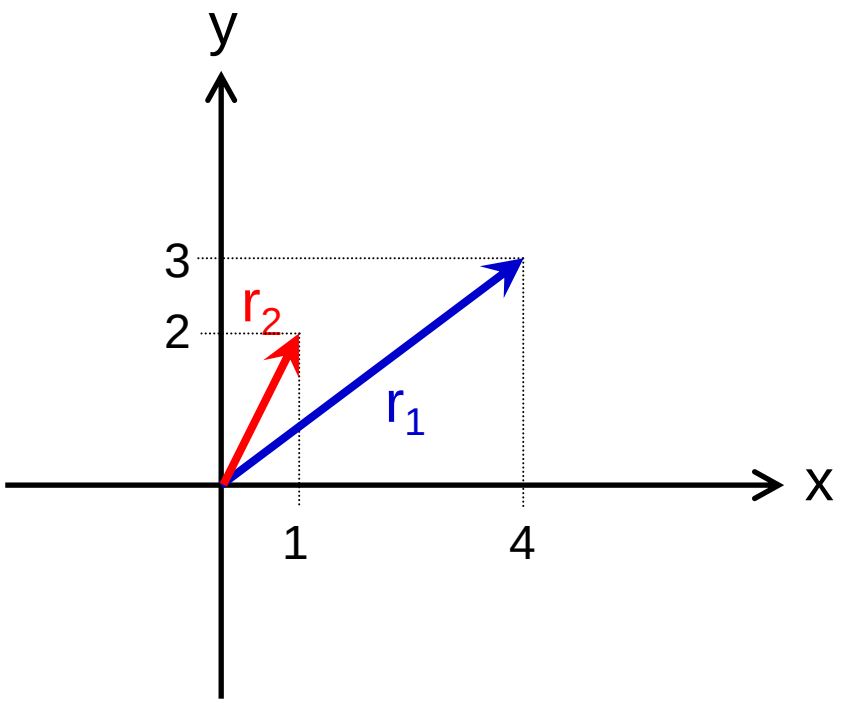
$$\Rightarrow \quad r\, r^{-1} = r^{-1}\, r = 1$$

Using inverse vectors, we are now able to divide by vectors.

Multiplication of Vectors

Example: $r_1 = 4\,\sigma_x + 3\,\sigma_y$

$r_2 = \sigma_x + 2\,\sigma_y$



$$r_1\, r_2 = (4\,\sigma_x + 3\,\sigma_y)\,(\sigma_x + 2\,\sigma_y)$$

$$= 4\,\sigma_x^{\,2} + 8\,\sigma_x\sigma_y + 3\,\sigma_y\sigma_x + 6\,\sigma_y^{\,2}$$

$$= 4 + 8\,\sigma_x\sigma_y - 3\,\sigma_x\sigma_y + 6$$

$$= 10 + 5\,\sigma_x\sigma_y$$

The product of two vectors consists of a scalar term and a bivector term.

The Inner Product

The scalar term of a product $r_1 r_2$ of two vectors can be found by adding the same product in reversed order $r_2 r_1$:

$$r_1 r_2 = 10 + 5 \sigma_x \sigma_y$$

$$r_2 r_1 = 10 - 5 \sigma_x \sigma_y$$

$$\langle r_1 r_2 \rangle_{scalar} = r_1 \bullet r_2 = \frac{1}{2}(r_1 r_2 + r_2 r_1) = 10$$

This part of a product was baptized inner product (or dot product) and a fat dot is usually used to symbolize it:

$$r_1 \bullet r_2 = \frac{1}{2}(r_1 r_2 + r_2 r_1)$$

The inner product of two vectors is a commutative product as a reversion of the order of the two vectors does not change it:

$$r_1 \bullet r_2 = r_2 \bullet r_1 = \frac{1}{2}(r_2 r_1 + r_1 r_2)$$

If the product of two vectors equals the inner product (the bivector terms cancel), the two vectors are parallel.

The Outer Product

The bivector term of a product $r_1 r_2$ of two vectors can be found by subtracting the same product in reversed order $r_2 r_1$:

$$r_1 r_2 = 10 + 5\,\sigma_x\sigma_y$$

$$r_2 r_1 = 10 - 5\,\sigma_x\sigma_y$$

$$\langle r_1 r_2 \rangle_{\text{bivector}} = r_1 \wedge r_2 = \frac{1}{2}(r_1 r_2 - r_2 r_1) = 5\,\sigma_x\sigma_y$$

This part of a product was baptized outer product (or exterior product or wedge product) and a wedge is used to symbolize it:

$$r_1 \wedge r_2 = \frac{1}{2}(r_1 r_2 - r_2 r_1)$$

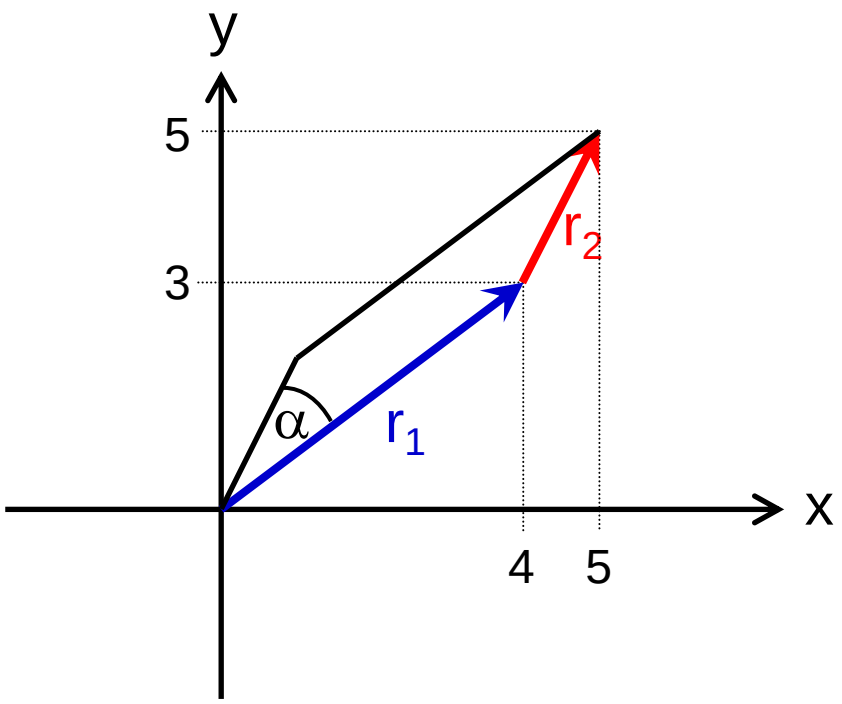
The outer product of two vectors is an anti-commutative product as a reversion of the order of the two vectors will change its sign:

$$r_2 \wedge r_1 = -\,r_1 \wedge r_2 = \frac{1}{2}(r_2 r_1 - r_1 r_2)$$

If the product of two vectors equals the outer product (the scalar terms cancel), the two vectors are orthogonal.

Interpretation of the Inner Product

Example: $r_1 r_2 = 10 + 5\,\sigma_x\sigma_y$



The product of two vectors $R = r_1 r_2$ can be visualized as oriented parallelogram.

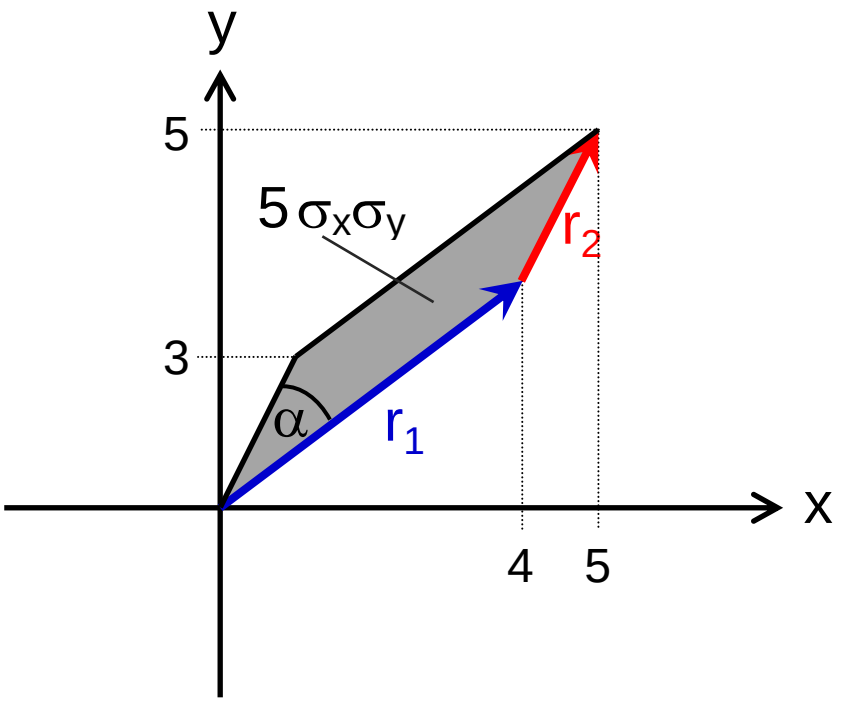
The inner product of two unit vectors $\hat{r}_1$ and $\hat{r}_2$ equals the cosine of the angle between them.

$$\cos\alpha = \hat{r}_1 \bullet \hat{r}_2 = \frac{r_1 \bullet r_2}{|r_1||r_2|}$$

Example:

$$\cos\alpha = \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}} \approx 0.8944 \quad \Rightarrow \quad \alpha = 26.57°$$

Interpretation of the Outer Product



The outer product of two vectors r_1 and r_2 is a bivector and represents the area of the oriented parallelogram.

Example: $r_1 r_2 = 10 + 5\,\sigma_x\sigma_y$

$$\Rightarrow \quad A = r_1 \wedge r_2 = 5\,\sigma_x\sigma_y$$

The parallelogram $R = r_1 r_2$ has an area of 5 unit squares.

Outlook: The oriented unit parallelogram $\hat{R} = \hat{r}_1 \hat{r}_2$ is also called rotor.

Example of a Division by a Vector

Using inverse vectors, we are able to divide by vectors.

Problem: The two vectors $r_1 = 4\,\sigma_x + 3\,\sigma_y$ and $r_2 = \sigma_x + 2\,\sigma_y$ form the parallelogram $R = r_1 r_2 = 10 + 5\,\sigma_x\sigma_y$.

Find the two vectors which represent the two heights of the parallelogram.

Answer: As the oriented area element of the parallelogram is given by

$$A = r_1 \wedge r_2 = 5\,\sigma_x\sigma_y \quad = r_1 h_1 = r_2 h_2$$

the two heights can be found by dividing A by r_1 and by r_2 respectively:

$$h_1 = r_1^{-1} A = \frac{1}{25}(4\,\sigma_x + 3\,\sigma_y)\,5\,\sigma_x\sigma_y$$

$$= -0.6\,\sigma_x + 0.8\,\sigma_y$$

$$h_2 = r_2^{-1} A = \frac{1}{5}(\sigma_x + 2\,\sigma_y)\,5\,\sigma_x\sigma_y$$

$$= -2\,\sigma_x + \sigma_y$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors.

$$A = A_{xy}\,\sigma_x\sigma_y + A_{yz}\,\sigma_y\sigma_z + A_{zx}\,\sigma_z\sigma_x$$

The Square of Bivectors

$$A^2 = (A_{xy}\,\sigma_x\sigma_y + A_{yz}\,\sigma_y\sigma_z + A_{zx}\,\sigma_z\sigma_x)^2$$

$$= -A_{xy}^{\,2} - A_{yz}^{\,2} - A_{zx}^{\,2}$$

As all mixed terms cancel, the square of a bivector always is a negative scalar.

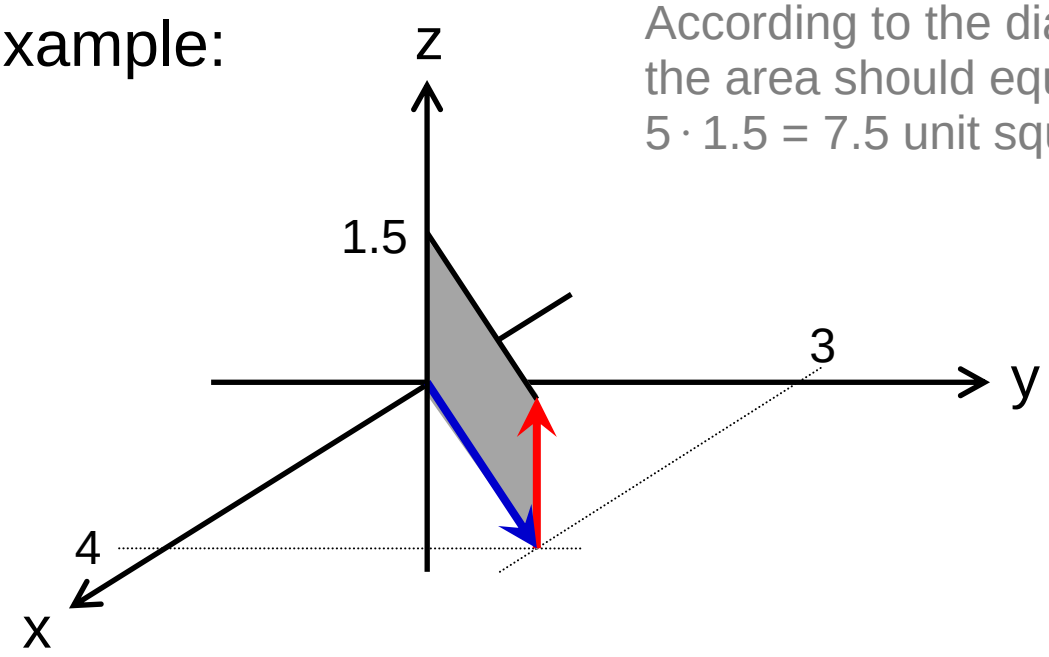
The Area of Bivectors

The square root of the square of a bivector multiplied by -1 can be identified as the area of the bivector:

$$|A| = \sqrt{-A^2} = \sqrt{A_{xy}^{\,2} + A_{yz}^{\,2} + A_{zx}^{\,2}}$$

Bivectors as Oriented Area Elements

Example:



According to the diagram the area should equal $5 \cdot 1.5 = 7.5$ unit squares.

$$A = 4.5\,\sigma_y\sigma_z - 6\,\sigma_z\sigma_x = (4\,\sigma_x + 3\,\sigma_y)\,1.5\,\sigma_z$$

$$A^2 = (4.5\,\sigma_y\sigma_z - 6\,\sigma_z\sigma_x)^2$$

$$= 20.25\,(\sigma_y\sigma_z)^2 - 27\,\sigma_y\sigma_z\sigma_z\sigma_x - 27\,\sigma_z\sigma_x\sigma_y\sigma_z + 36\,(\sigma_z\sigma_x)^2$$

$$= -20.25 + 27\,\sigma_x\sigma_y - 27\,\sigma_x\sigma_y - 36$$

$$= -56.25$$

Area of the bivector:

$$|A| = \sqrt{-A^2} = \sqrt{-(-56.25)^2} = 7.5$$

Mapping vectors to vectors

- Dilations

  The multiplication of a vector by a scalar ℓ maps this vector to a dilated vector:

$$r = x\sigma_x + y\sigma_y + z\sigma_z \longrightarrow r\ell = \ell x\sigma_x + \ell y\sigma_y + \ell z\sigma_z$$

- ???

  The multiplication of a vector r by another vector (e.g. the base vector σ_x) does not map the vector r to a vector, but to a scalar and a bivector:

$$r = x\sigma_x + y\sigma_y + z\sigma_z \longrightarrow r\sigma_x = x - \underbrace{y\sigma_x\sigma_y + z\sigma_z\sigma_x}_{\text{bivector}}$$

scalar

- How can we design a multiplication of a vector by another vector, which maps the vector r to a vector again?

  $\Rightarrow$ This special multiplication is named after a British earl.

The Sandwich Product

If vector r is multiplied by another vector from the left and from the right in a sandwich-like manner, vector r will be mapped to a resulting vector r':

$$r' = \sigma_x \, r \, \sigma_x = \quad x \, \sigma_x - y \, \sigma_y - z \, \sigma_z$$

$$r'' = \sigma_y \, r \, \sigma_y = - x \, \sigma_x + y \, \sigma_y - z \, \sigma_z$$

$$r''' = \sigma_z \, r \, \sigma_z = - x \, \sigma_x - y \, \sigma_y + z \, \sigma_z$$

These formulae describe reflections!

If r is reflected at a vector pointing into the direction of the x-axis, the x-coordinate is unchanged while the y- and z-coordinates will change their signs. Thus r is mapped to r' by a reflection at the x-axis.

r is mapped to r'' by a reflection at the y-axis.
r is mapped to r''' by a reflection at the z-axis.

The sandwich product of a vector with a unit vector results in a reflection.

Reflections

The sandwich product of a vector with an arbitrary vector n (which is no unit vector) results in a reflection **and** a dilation.

To suppress the dilation and to get a pure reflection, it is necessary to divide by n^2. Thus a reflection at an axis which points into the direction of vector n, has to be written as

$$r_{ref} = \frac{1}{n^2} n\, r\, n$$

or

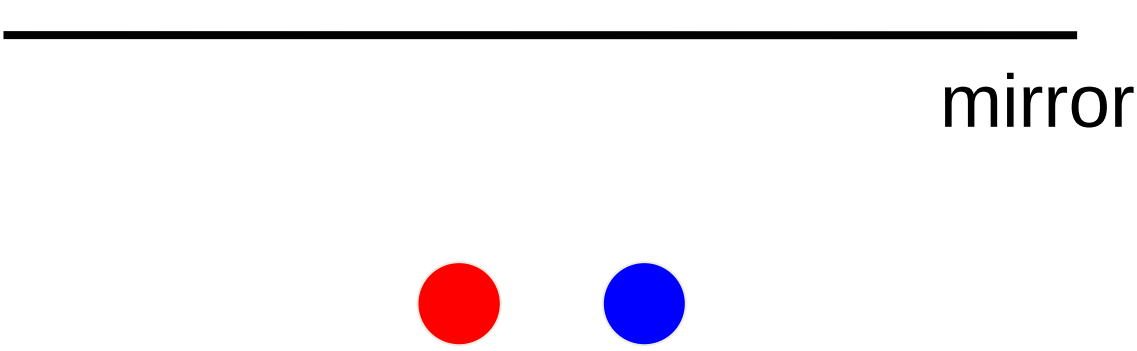
$$r_{ref} = n\, r\, n^{-1}$$

Reflections are important operations, as they conserve the length of vectors:

$$|r_{ref}| = |r|$$

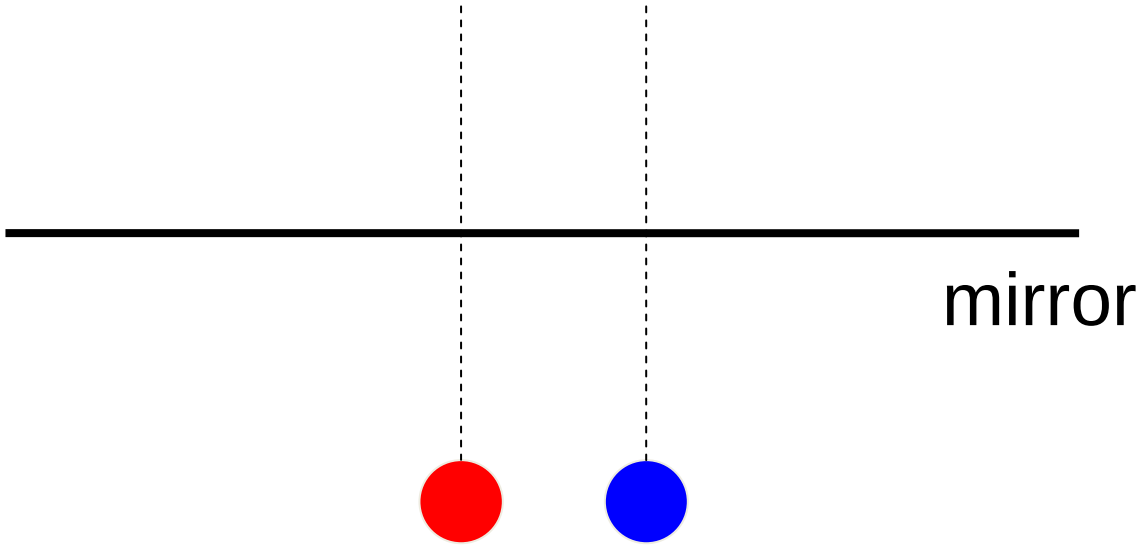
Lasenby, Doran: "This formula is already more compact than can be written down without the geometric product … The compression afforded by the geometric product becomes increasingly impressive as reflections are compounded together."

Reflection at a mirror #1



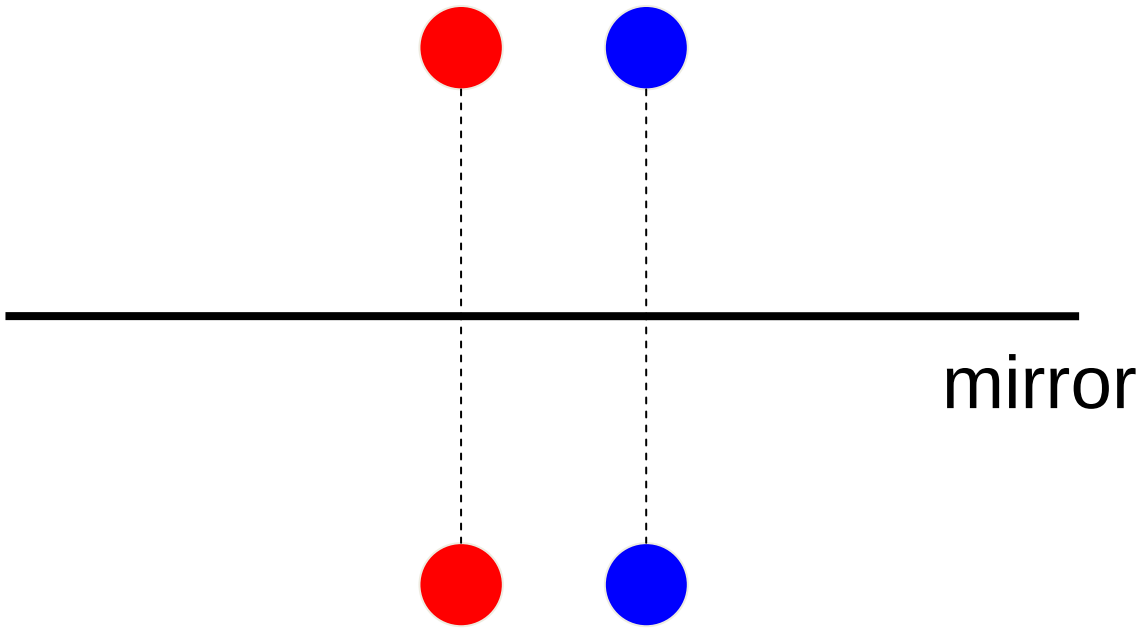
Two different objects will be reflected at a plane mirror.

Reflection at a mirror #2



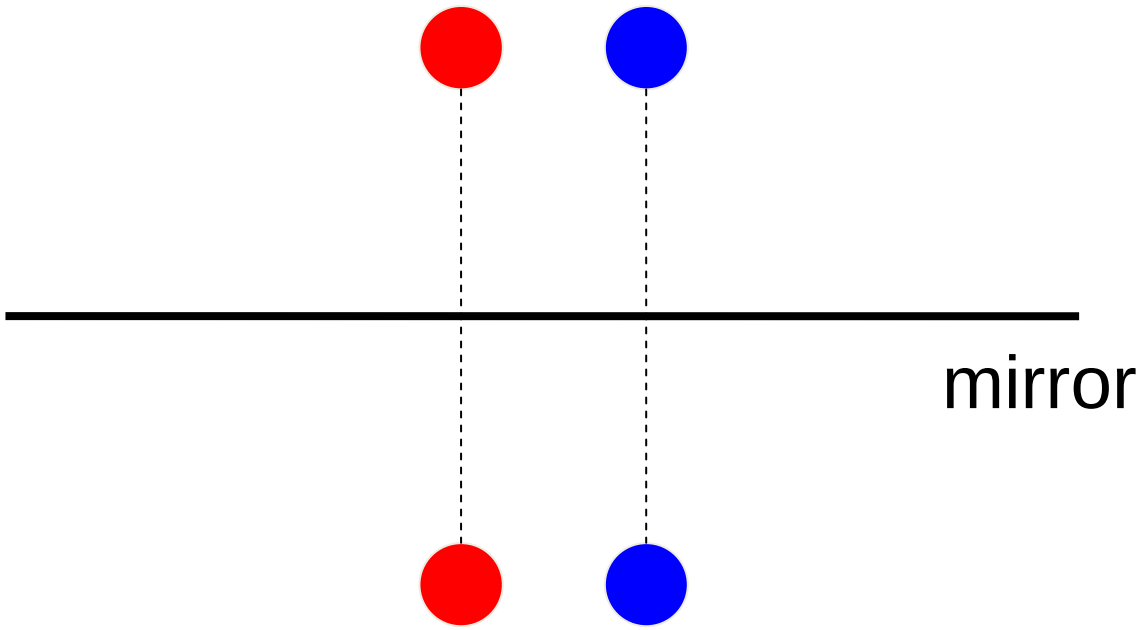
Two different objects will be reflected at a plane mirror.
The position of the images of these objects can be found with the orthogonal lines.

Reflection at a mirror #3



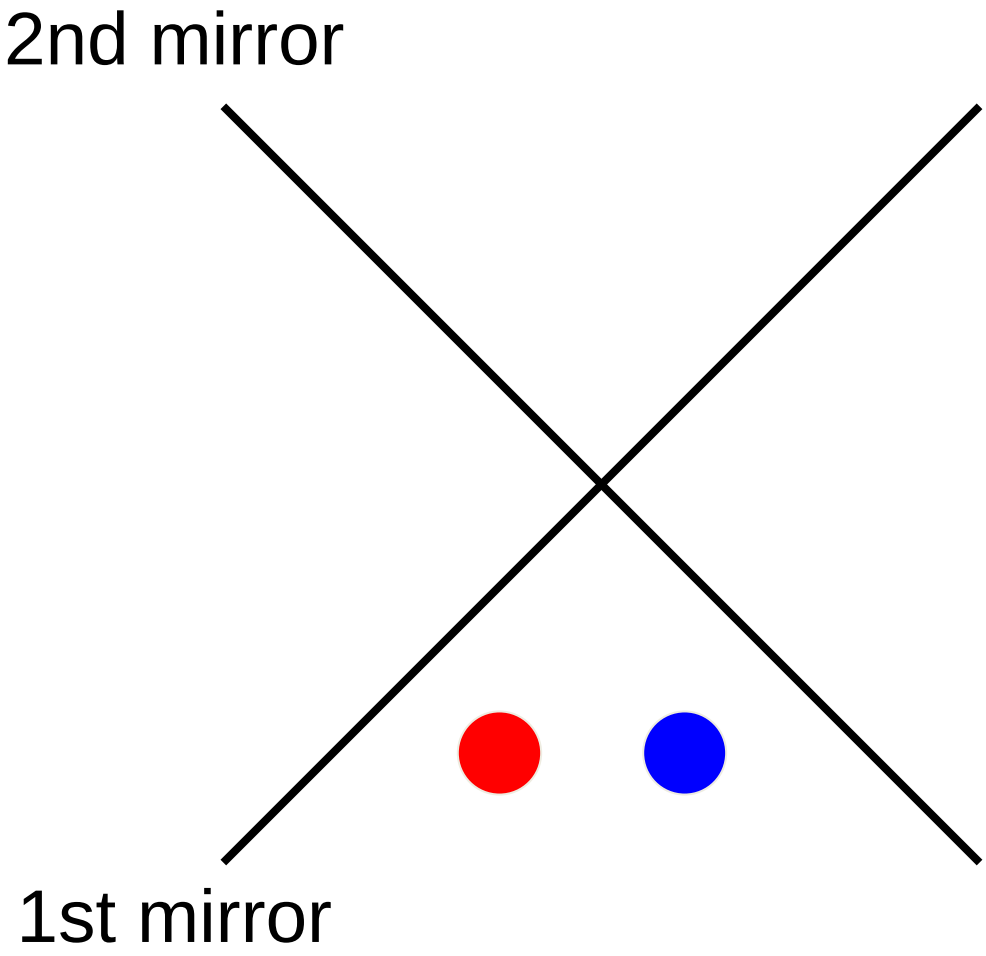
Two different objects will be reflected at a plane mirror.
The position of the images of these objects can be found with the orthogonal lines.
Original objects and images have the same distance from the plane mirror.

Reflection at a mirror #4



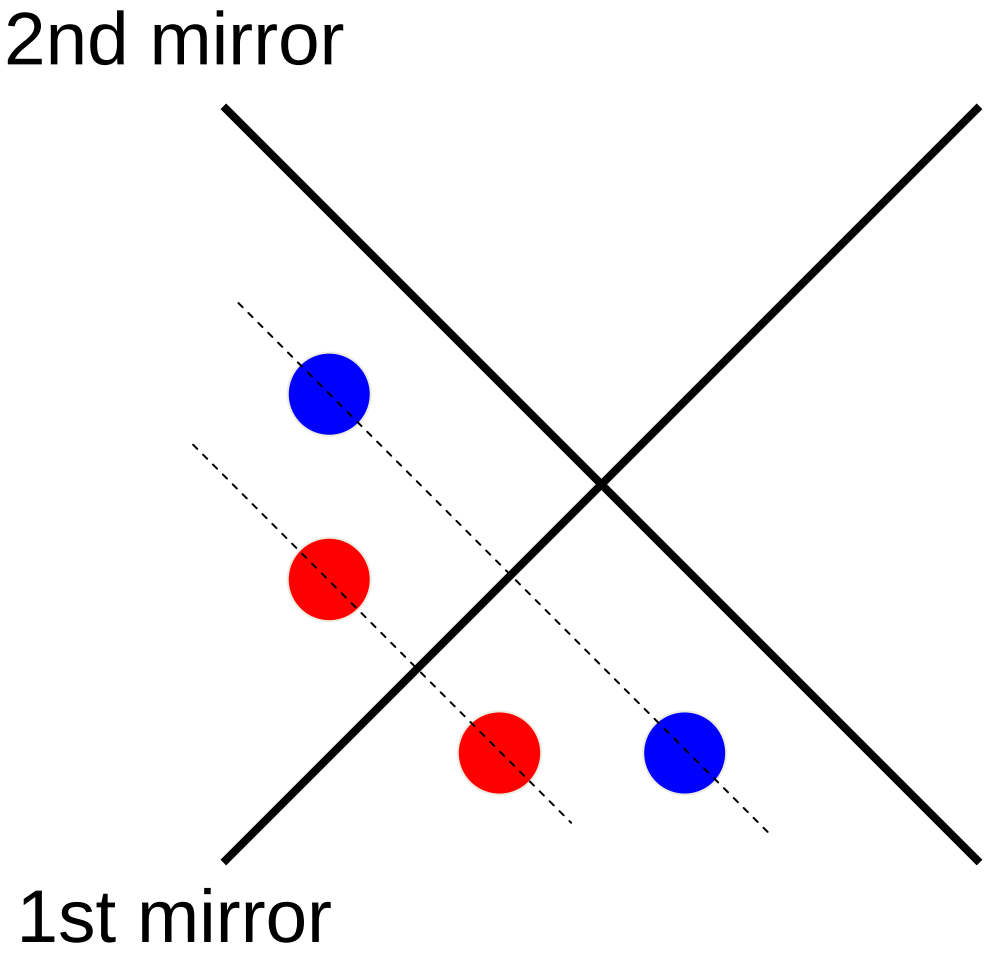
This is a reflection. The image of the red object lies opposite to the red object, and the image of the blue object lies opposite to the blue object.

Double reflection at two mirrors #1



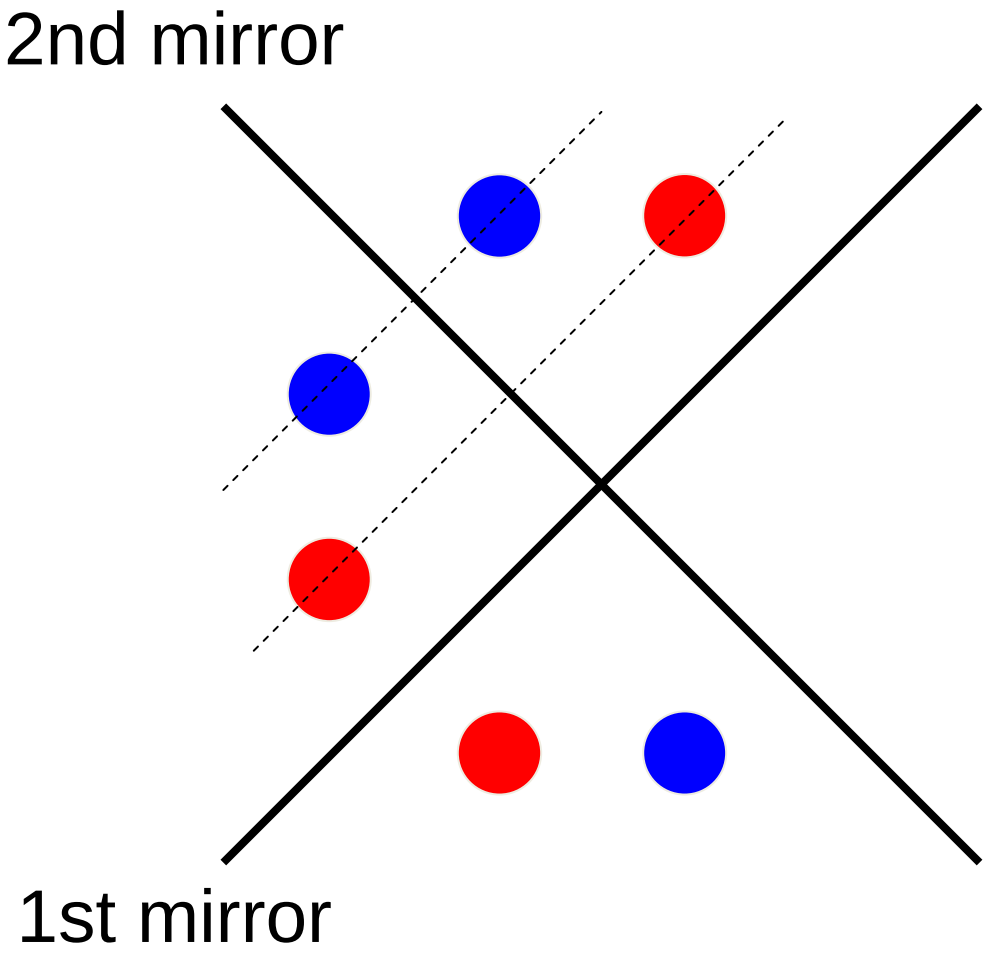
Now the two different objects will be reflected at two plane mirrors which make an angle of 90° with each other.

Double reflection at two mirrors #2



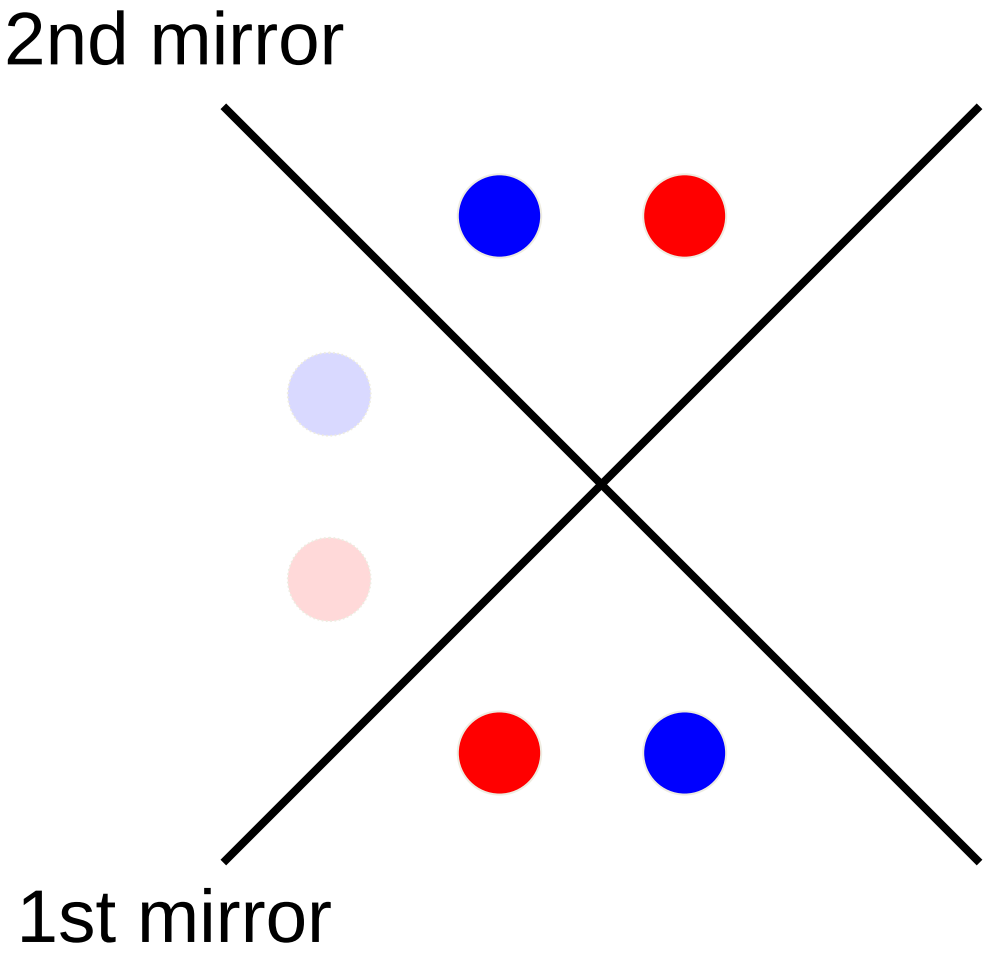
Now the two different objects will be reflected at two plane mirrors which make an angle of 90° with each other.
Again orthogonal lines help to find the two images of the objects.

Double reflection at two mirrors #3



Now the two different objects will be reflected at two plane mirrors which make an angle of 90° with each other.
Again orthogonal lines help to find the two images of the objects.

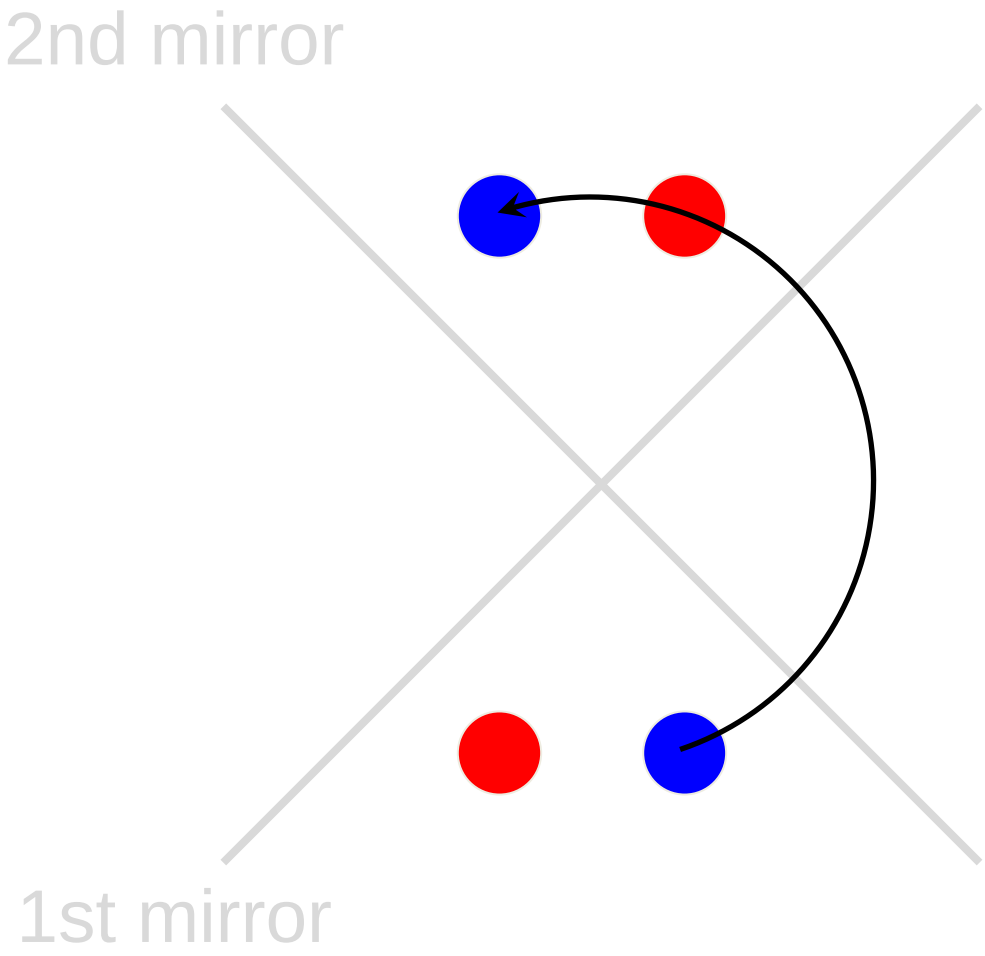
Double reflection at two mirrors #4



This is a reflection at two plane mirrors which are perpendicular to each other.

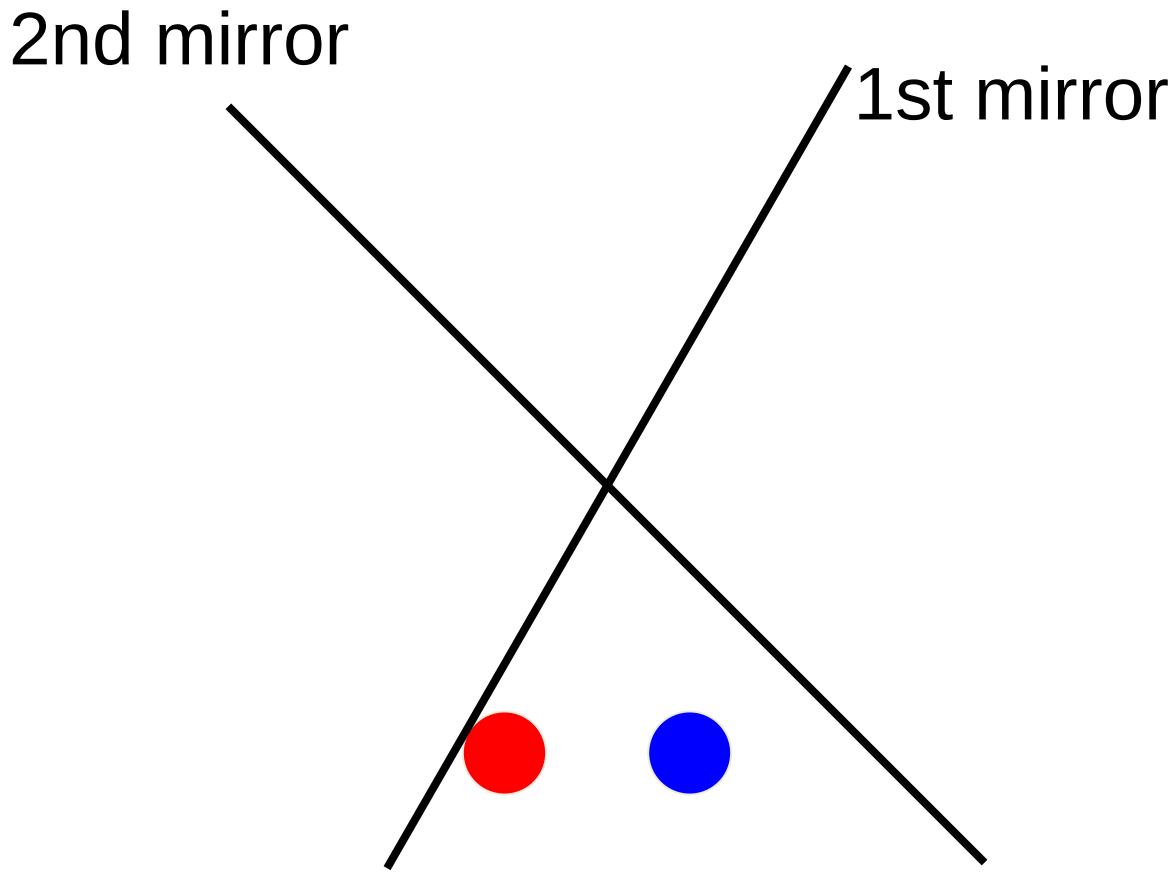
Now the image of the red object lies opposite to the blue object, and the image of the blue object lies opposite to the red object.

Double reflection at two mirrors #5



This is a rotation. Reflecting objects at two plane mirrors results in a rotation of these objects.

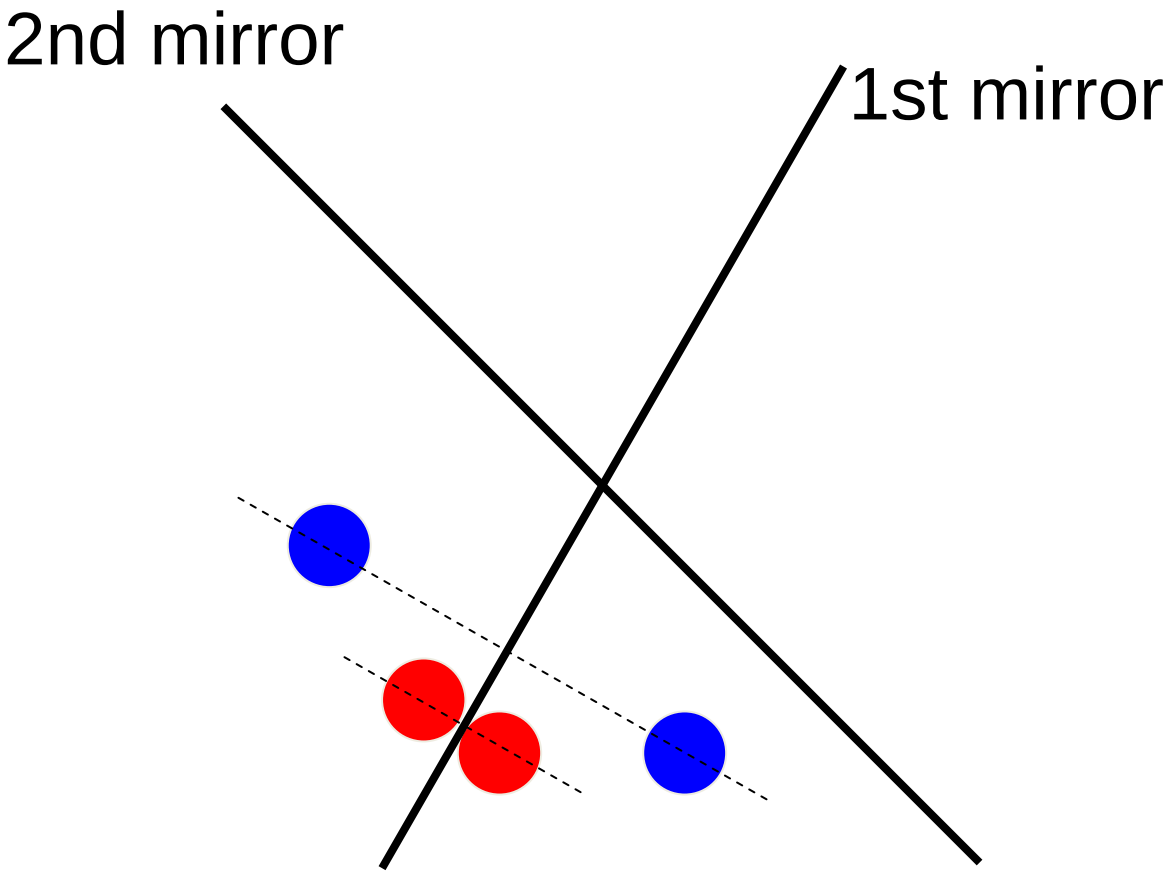
Double reflection at two mirrors #6



Two succeeding reflections result in a rotation.

This works with two plane mirrors which make a different angle as well.

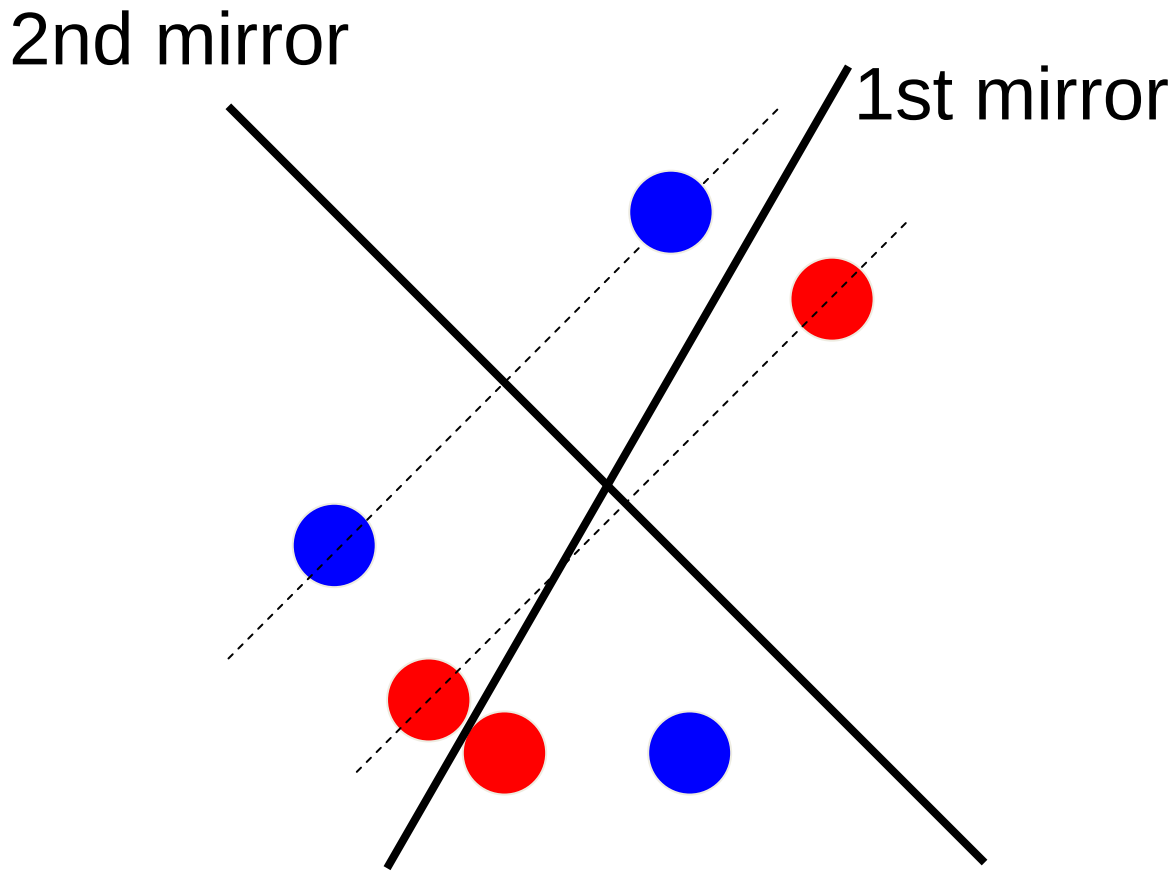
Double reflection at two mirrors #7



Two succeeding reflections result in a rotation.

This works with two plane mirrors which make a different angle as well.

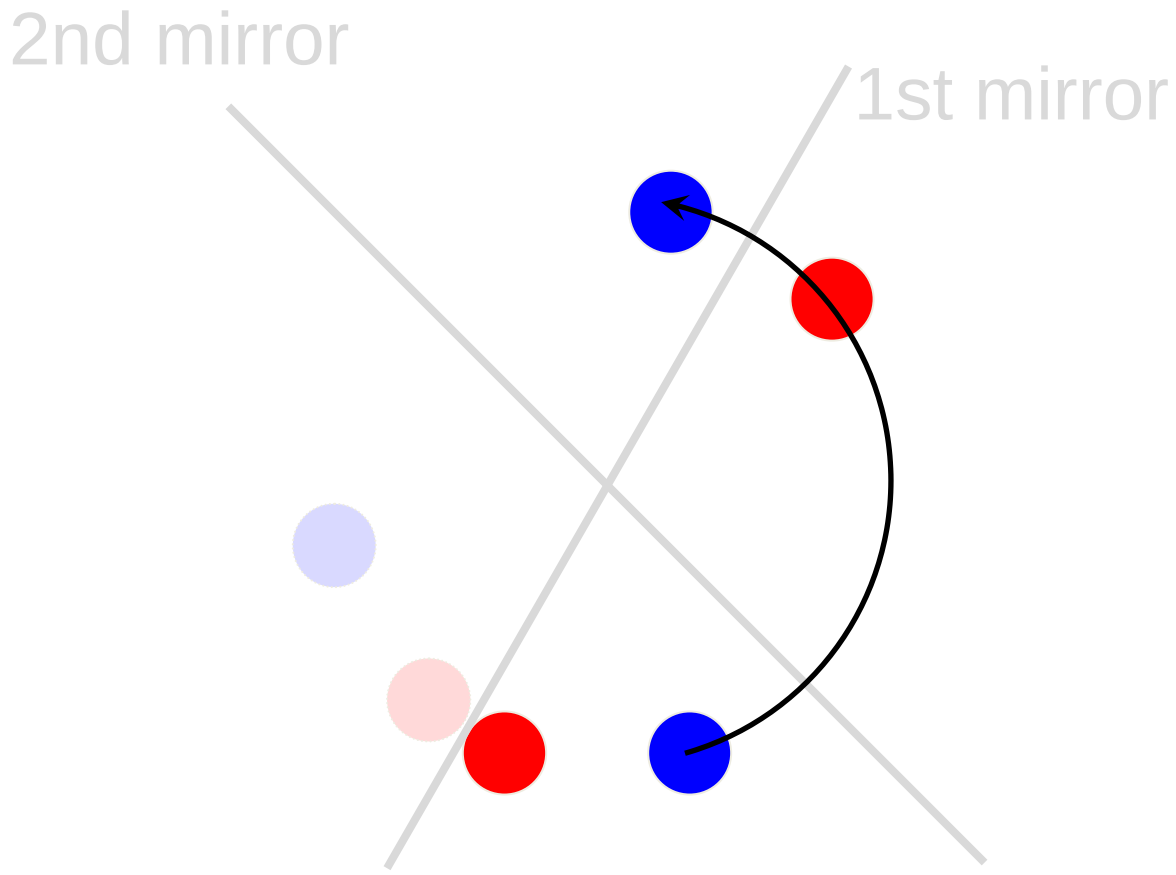
Double reflection at two mirrors #8



Two succeeding reflections result in a rotation.

This works with two plane mirrors which make a different angle as well.

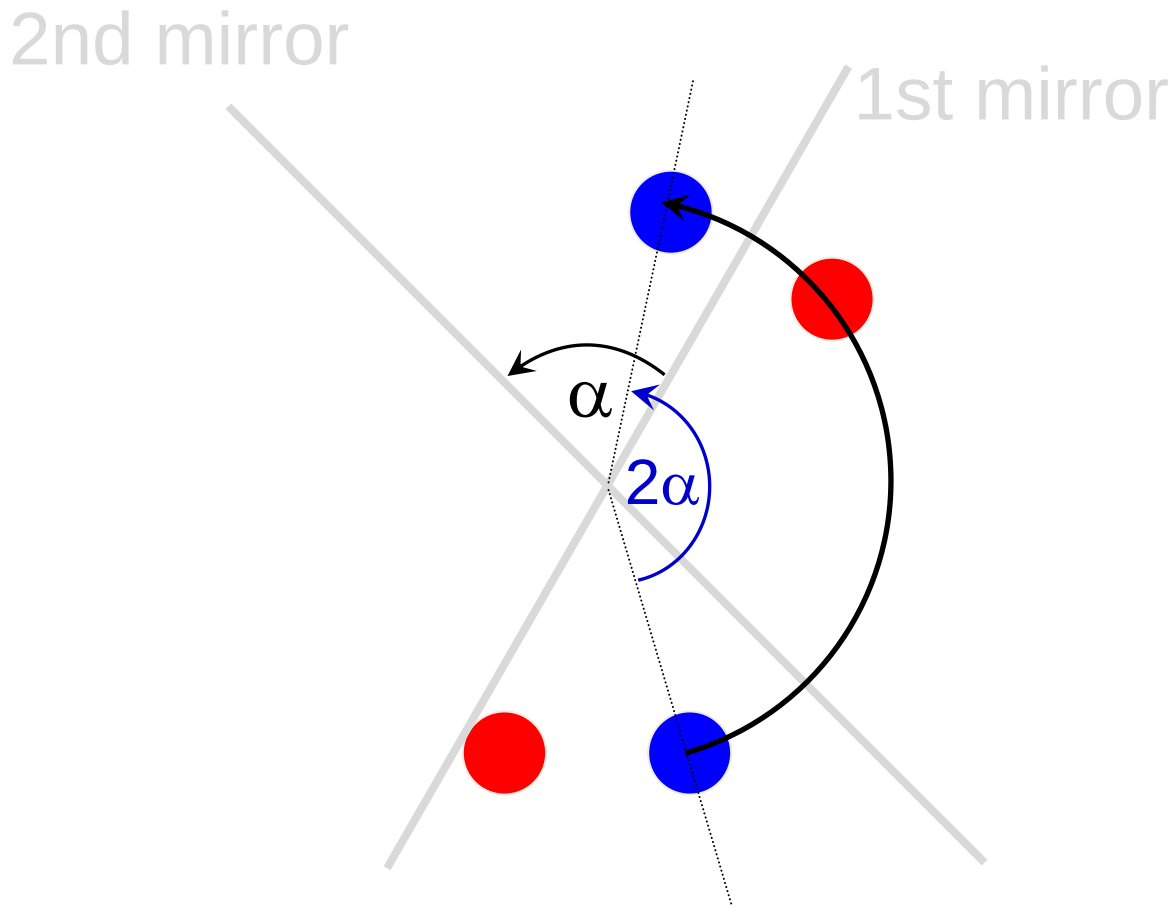
Double reflection at two mirrors #9



Two succeeding reflections result in a rotation.

This works with two plane mirrors which make a different angle as well.

Two reflections = one rotation #10



Two succeeding reflections result in a rotation.

If the angle of the two mirrors is α, the rotation angle will be 2α.

Rotations

A reflection at an axis which points into the direction of vector n, can be written as

$$r_{ref} = n\, r\, n^{-1}$$

A rotation in the plane which is spanned by the two vectors n and m can then be written as:

$$r_{rot} = m\, r_{ref}\, m^{-1} = m\, n\, r\, n^{-1}\, m^{-1}$$

The vector r is rotated about twice the angle between n and m.

Again the length of the vector is conserved:

$$|r_{rot}| = |r|$$

Lasenby, Doran: "This is starting to look extremely simple! … The rule also works for any grade of multivector!"

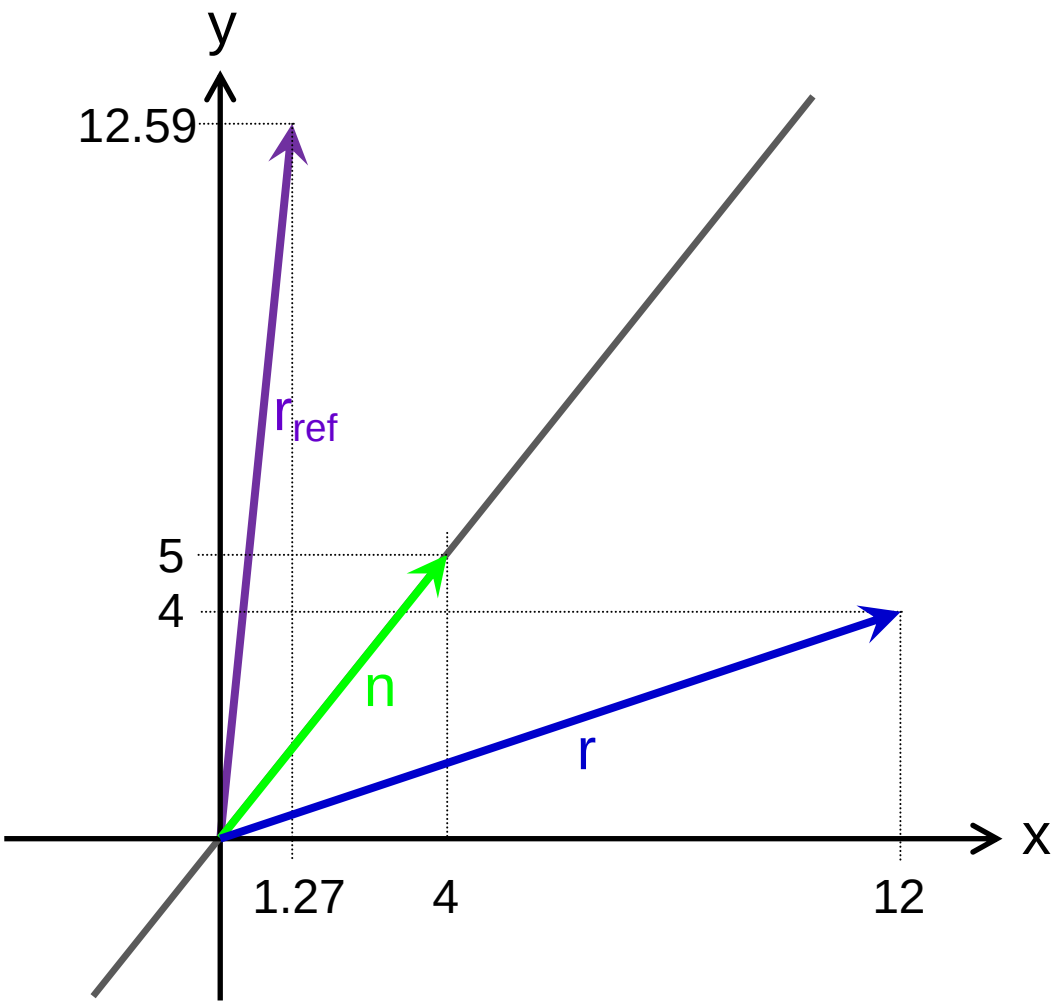
(Chris Doran, Anthony Lasenby: Geometric Algebra for Physicists, Cambridge University Press, Cambridge 2003)

Whiteboard Example of a Rotation

– from Lesson at Nov. 20th, 2014 –

1st step: Vector $r = 12\,\sigma_x + 4\,\sigma_y$ is reflected at an axis which points into the direction of vector $n = 4\,\sigma_x + 5\,\sigma_y$.

2nd step: The reflected Vector r_{ref} is now reflected at an axis which points into the direction of vector $m = 4\,\sigma_x + 6\,\sigma_y$.



First reflection at n

First reflection at an axis which points into the direction of n:

$$r = 12\,\sigma_x + 4\,\sigma_y \quad \Rightarrow \quad r^2 = 160$$

$$n = 4\,\sigma_x + 5\,\sigma_y \quad \Rightarrow \quad n^2 = 41$$

$$r_{ref} = n\,r\,n^{-1}$$

$$= \frac{1}{41}(4\sigma_x + 5\sigma_y)\,(12\sigma_x + 4\sigma_y)\,(4\sigma_x + 5\sigma_y)$$

$$= \frac{1}{41}(68 - 44\sigma_x\sigma_y)\,(4\sigma_x + 5\sigma_y)$$

$$= \frac{1}{41}(52\,\sigma_x + 516\,\sigma_y)$$

$$\approx 1.27\,\sigma_x + 12.59\,\sigma_y$$

Checking the result:

$$r_{ref}{}^2 = \frac{52^2 + 516^2}{41^2} = \frac{268960}{1681} = 160 = r^2$$

Second reflection at an axis which points into the direction of m:

$$r_{ref} = \frac{1}{41}(52\,\sigma_x + 516\,\sigma_y) \quad \Rightarrow \quad r_{ref}^{\,2} = 160$$

$$m = 4\,\sigma_x + 6\,\sigma_y \quad \Rightarrow \quad m^2 = 52$$

$$r_{rot} = m\, r_{ref}\, m^{-1}$$

$$= \frac{1}{52}(4\sigma_x+6\sigma_y)\frac{1}{41}(52\sigma_x+516\sigma_y)(4\sigma_x+6\sigma_y)$$

$$= \frac{1}{13\cdot 41}(2\sigma_x+3\sigma_y)(52\sigma_x+516\sigma_y)(2\sigma_x+3\sigma_y)$$

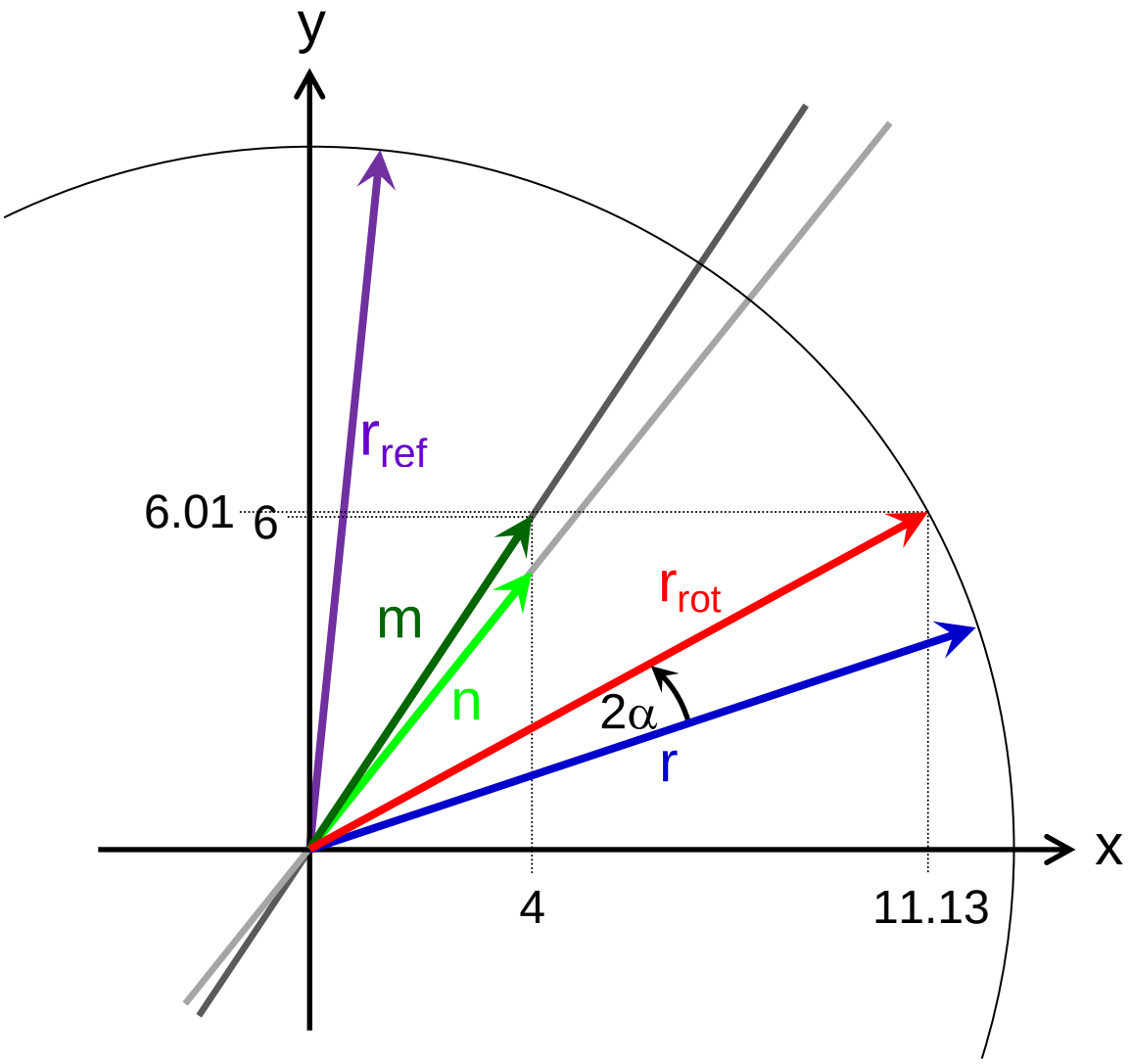
$$= \frac{1}{533}(1652 + 876\,\sigma_x\sigma_y)(2\,\sigma_x + 3\,\sigma_y)$$

$$= \frac{1}{533}(5932\,\sigma_x + 3204\,\sigma_y)$$

$$\approx 11.13\,\sigma_x + 6.01\,\sigma_y$$

Checking the result:

$$r_{rot}^{\,2} = \frac{5932^2 + 3204^2}{533^2} = 160 = r_{ref}^{\,2} = r^2$$



Composition of a rotation: Two succeeding reflections at n and m

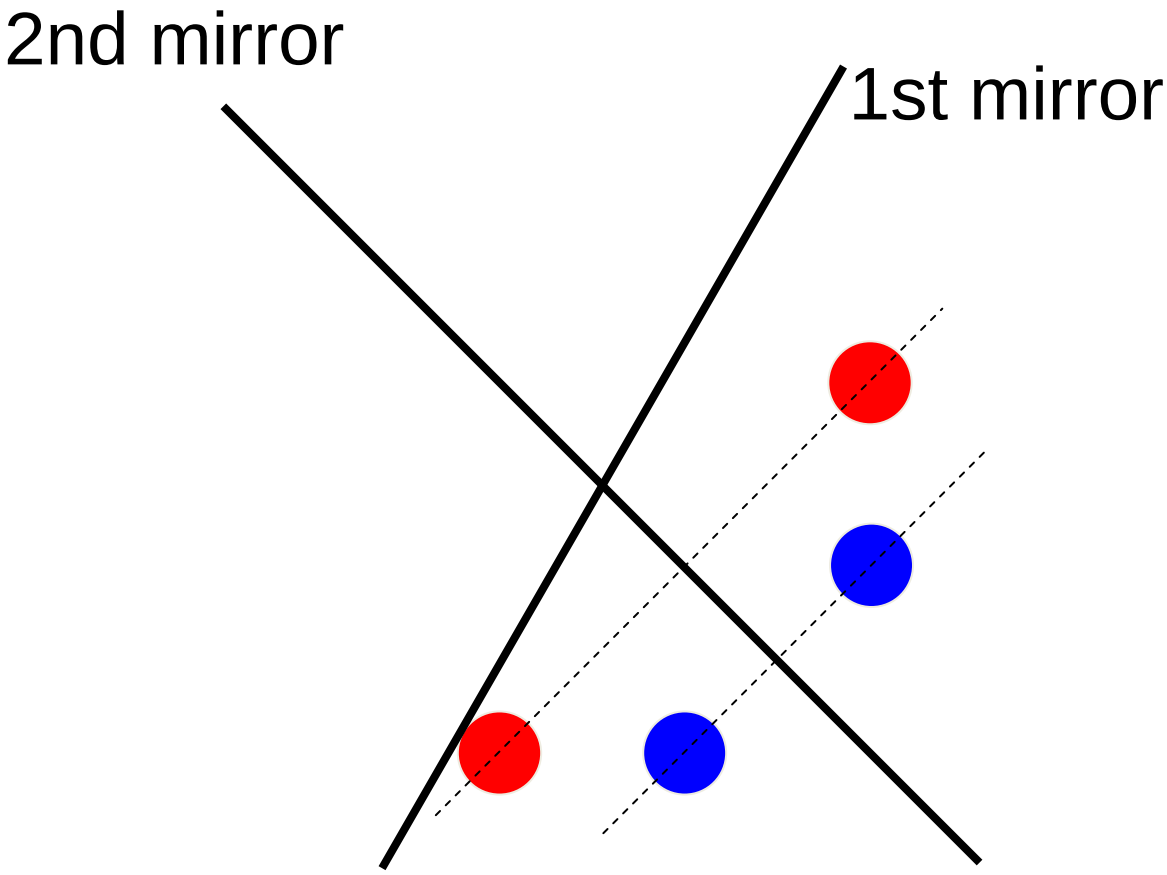
Angle of rotation:

$$\cos\alpha = \frac{n \bullet m}{|n||m|} = \frac{16+30}{\sqrt{41 \cdot 52}} = 0.9962$$

$$\Rightarrow \quad \alpha = 4.97°$$

$$\Rightarrow \quad 2\alpha = 9.94°$$

Double reflection at two mirrors #11

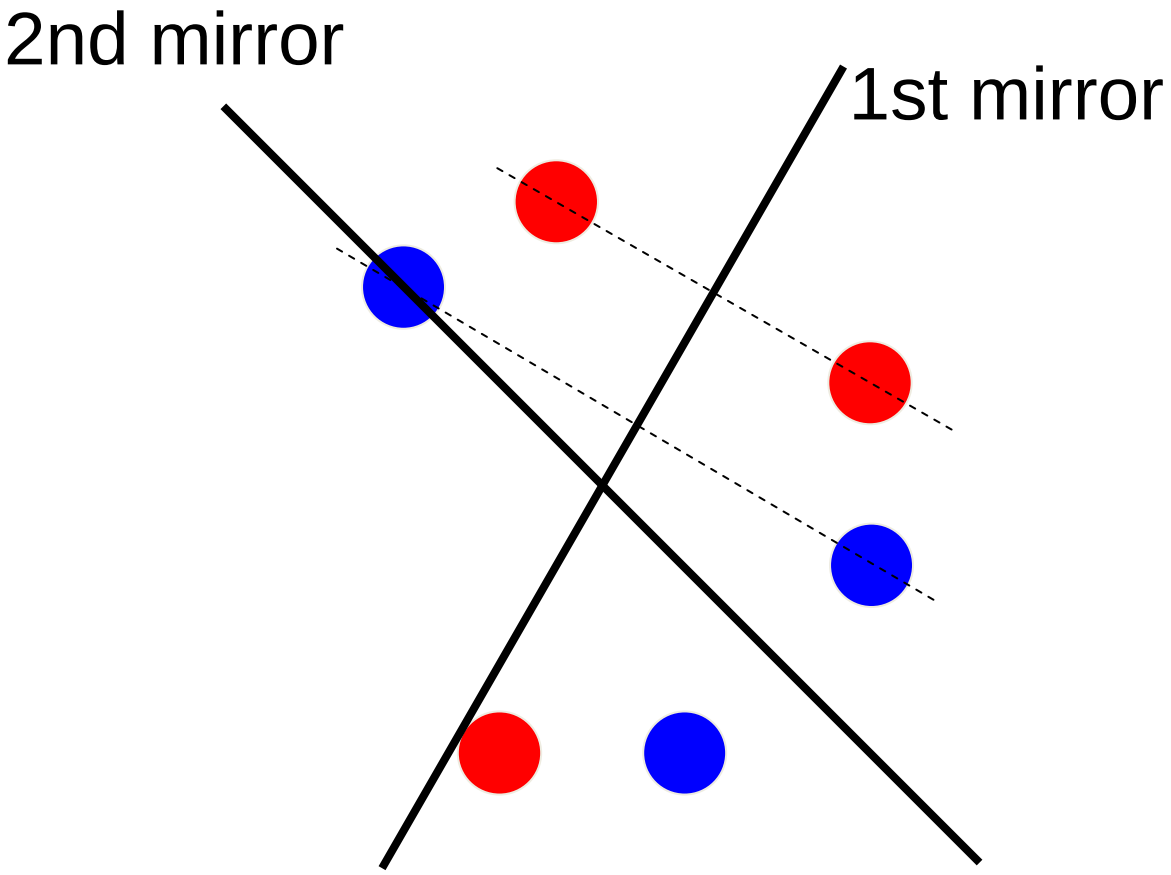


By the way: Reflections are not commutative.

Slides #7 to #10 show a reflection at the 1st mirror, followed by a second reflection at the 2nd mirror.

The following slides show, that a reflection at the 2nd mirror …

Double reflection at two mirrors #12

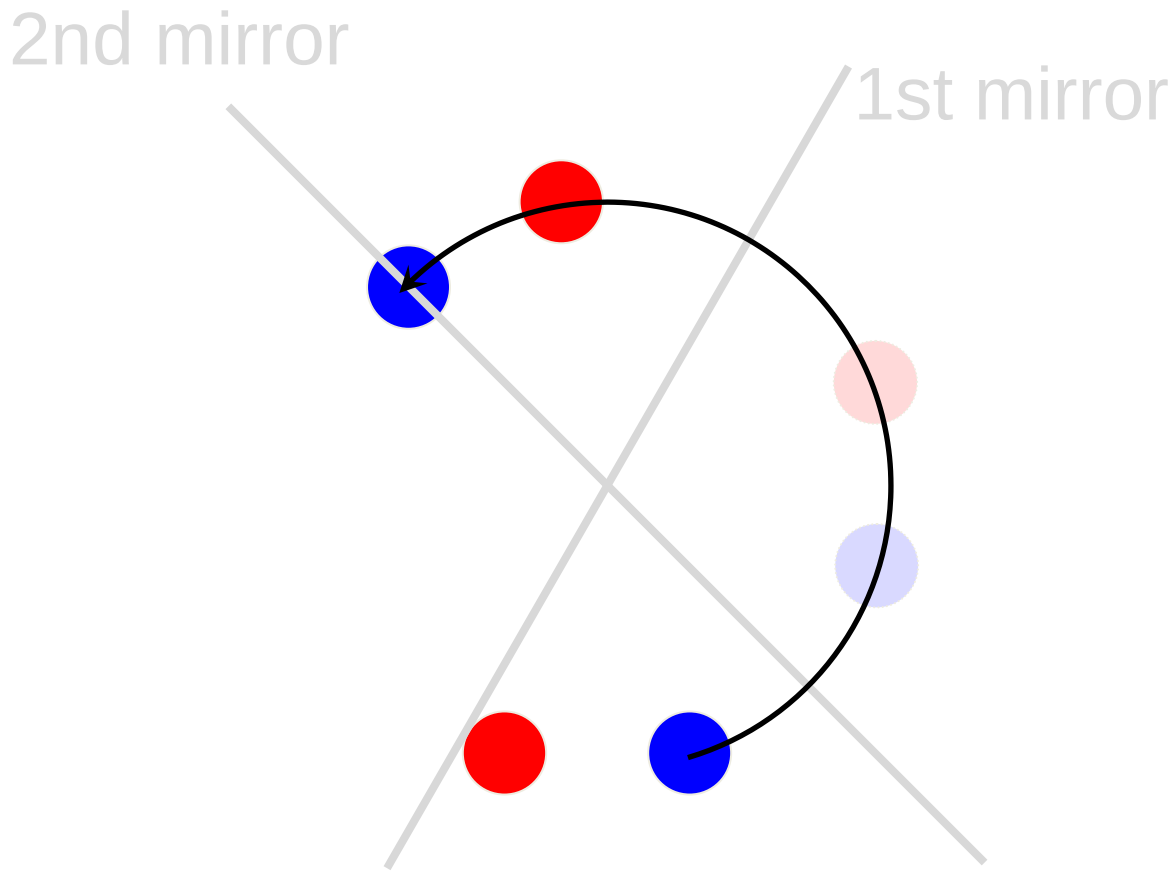


By the way: Reflections are not commutative.

Slides #7 to #10 show a reflection at the 1st mirror, followed by a second reflection at the 2nd mirror.

Slides #11 to #14 show, that a reflection at the 2nd mirror, followed by a reflection at the 1st mirror …

Double reflection at two mirrors #13

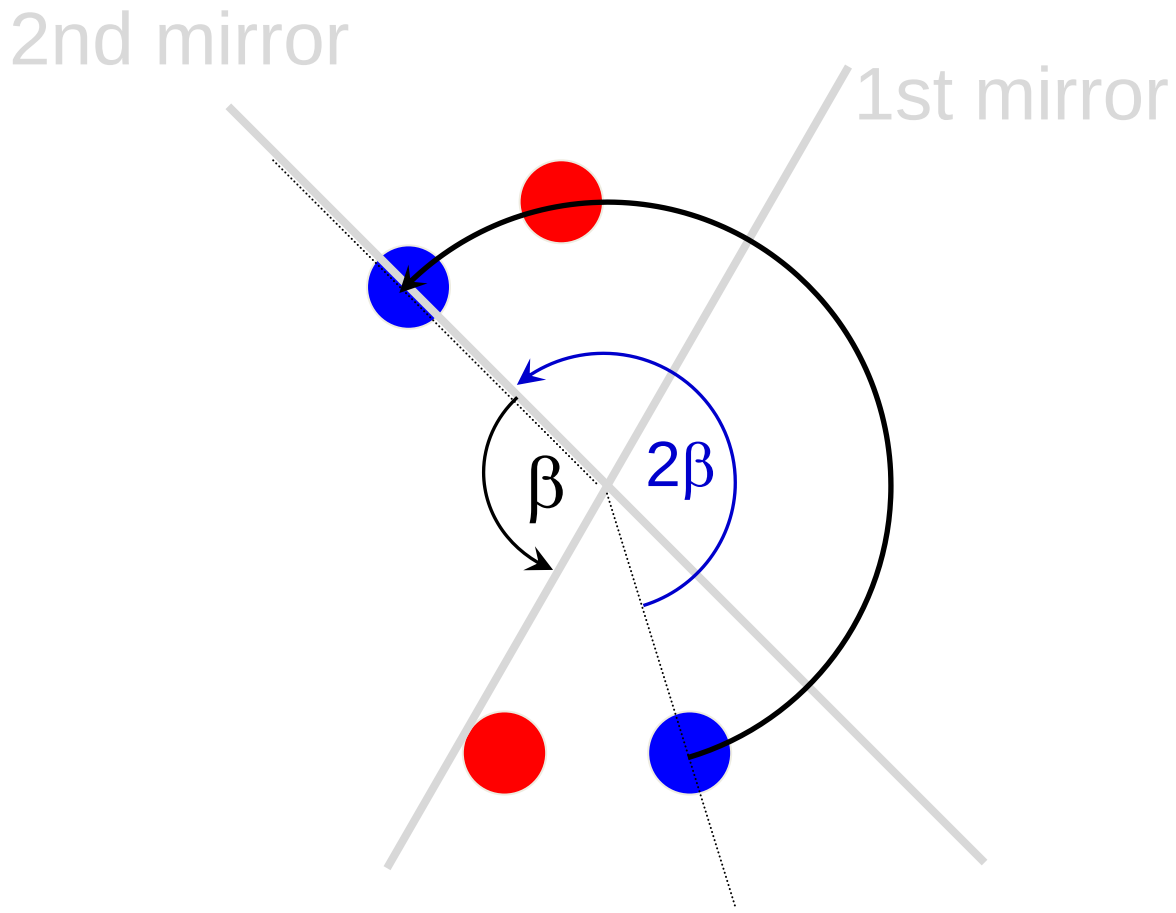


By the way: Reflections are not commutative.

Slides #7 to #10 show a reflection at the 1st mirror, followed by a second reflection at the 2nd mirror.

Slides #11 to #14 show, that a reflection at the 2nd mirror, followed by a reflection at the 1st mirror, will give a different result.

Double reflection at two mirrors #14



By the way: Reflections are not commutative.

As the angle between the 2nd mirror and the 1st mirror equals $\beta = 90° - \alpha$ (measured in positive, anticlockwise direction), the rotation angle now is $2\beta = 180° - 2\alpha$.

Pauli Matrices

It is possible to find matrices which represent the three base vectors σ_x, $\sigma_{y,}$ and σ_z. These matrices are called Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The basic rules of Pauli algebra are rules, which describe three-dimensional vectors:

$$\sigma_x\sigma_y = \overbrace{\sigma_x\sigma_y\sigma_z}^{I}\sigma_z = I\,\sigma_z$$

$$\sigma_y\sigma_z = \sigma_x\sigma_y\sigma_z\sigma_x = I\,\sigma_x$$

$$\sigma_z\sigma_x = \sigma_x\sigma_y\sigma_z\sigma_y = I\,\sigma_y$$

Pauli Matrices

Base scalars, base bivectors, and base trivector can now be written as:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x\sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\sigma_y\sigma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\sigma_z\sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

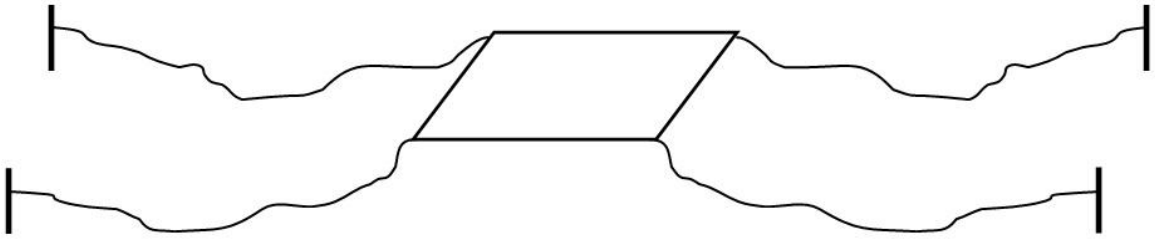
$$\sigma_x\sigma_y\sigma_z = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Dirac Belt trick

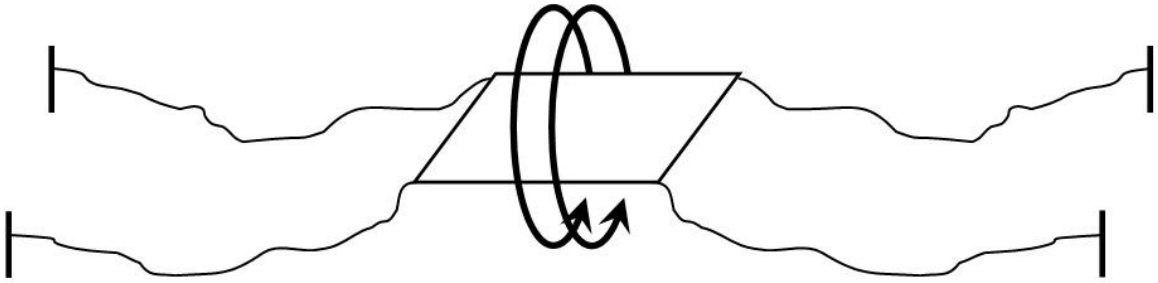
Originally Pauli matrices had been invented to describe the strange 4 π symmetry of the electron or some other elementary particles. But today we know that Pauli matrices represent base vectors of three-dimensional space. Or as Syngg puts it: "In recent years, it has become more widely recognized that objects larger than electrons also have 4 π periodicities." (John Snygg: Clifford Algebra. A Computational Tool for Physicists, Oxford University Press, New York, Oxford 1997)

Indeed every object (if attached to its surroundings) possesses such a symmetry as the Dirac belt trick shows. Therefore it seems sensible to assume that not only electrons but every object should be described with the help of Pauli matrices.

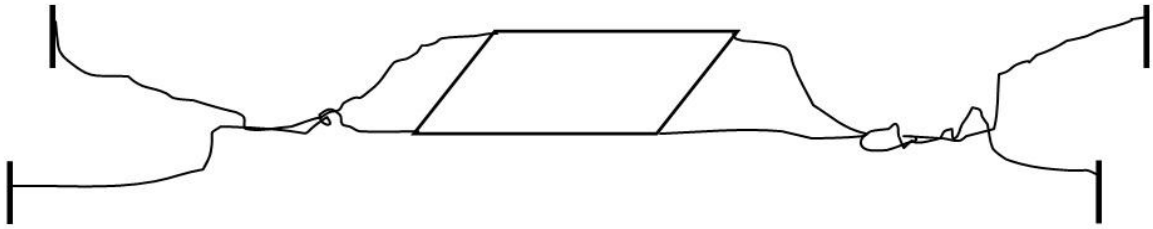
1. To reproduce the Dirac belt trick, fix an object with three or more strings.



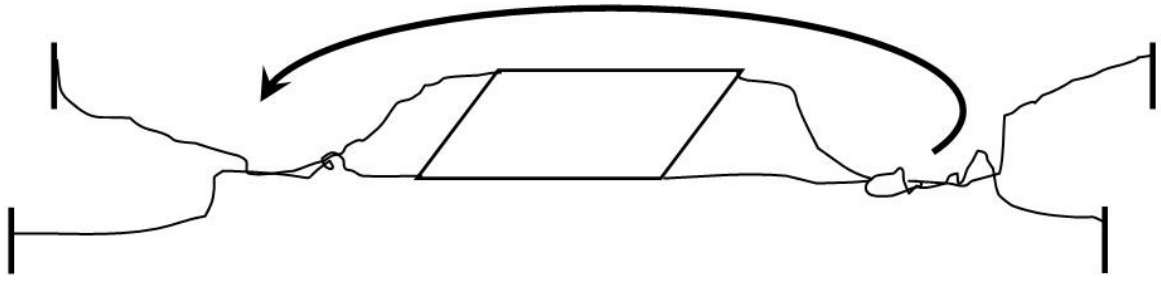
2. Now rotate the object about an angle of $4\pi = 720°$.



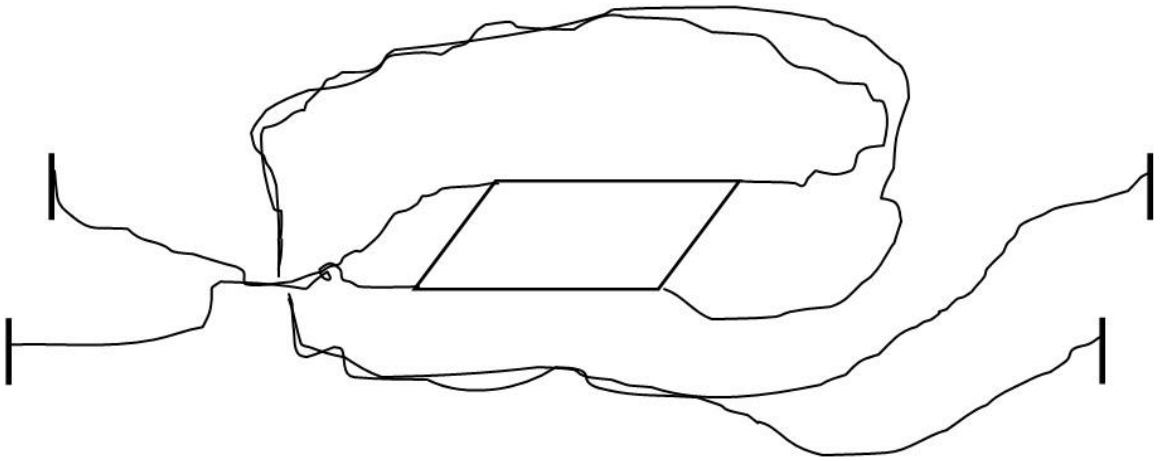
3. Obviously the strings are heavily entangled now.

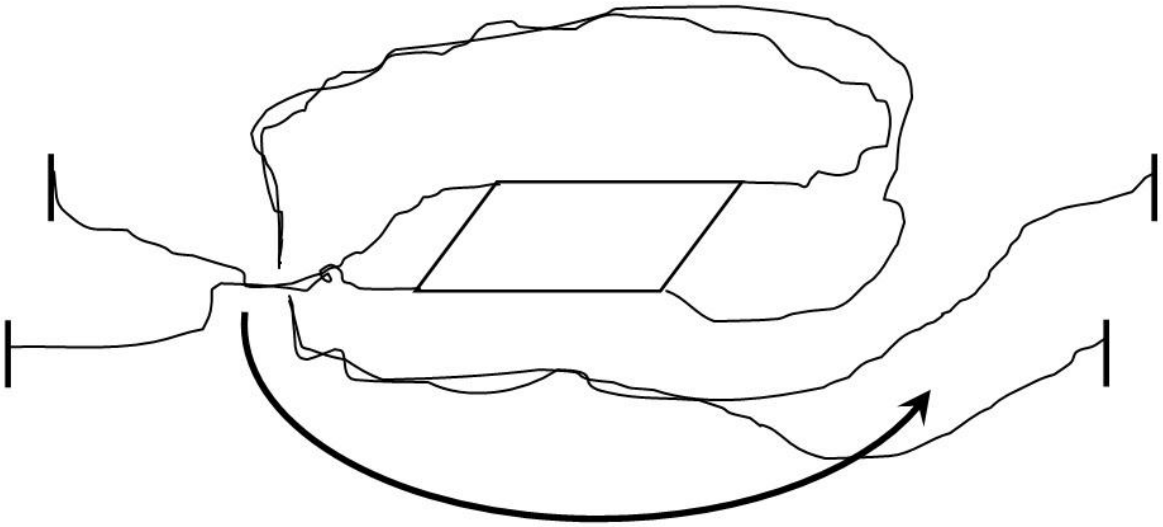


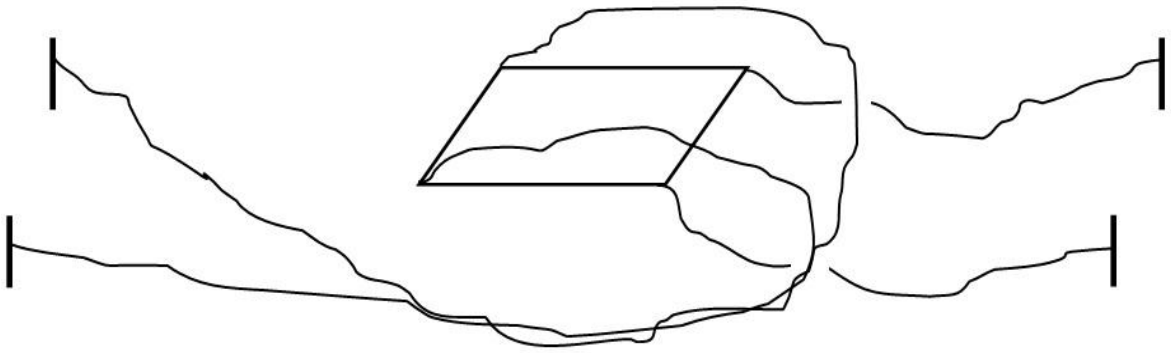
4. But it is possible to disentangle them again without moving or rotating the object. Just follow the procedure indicated in the following pictures.

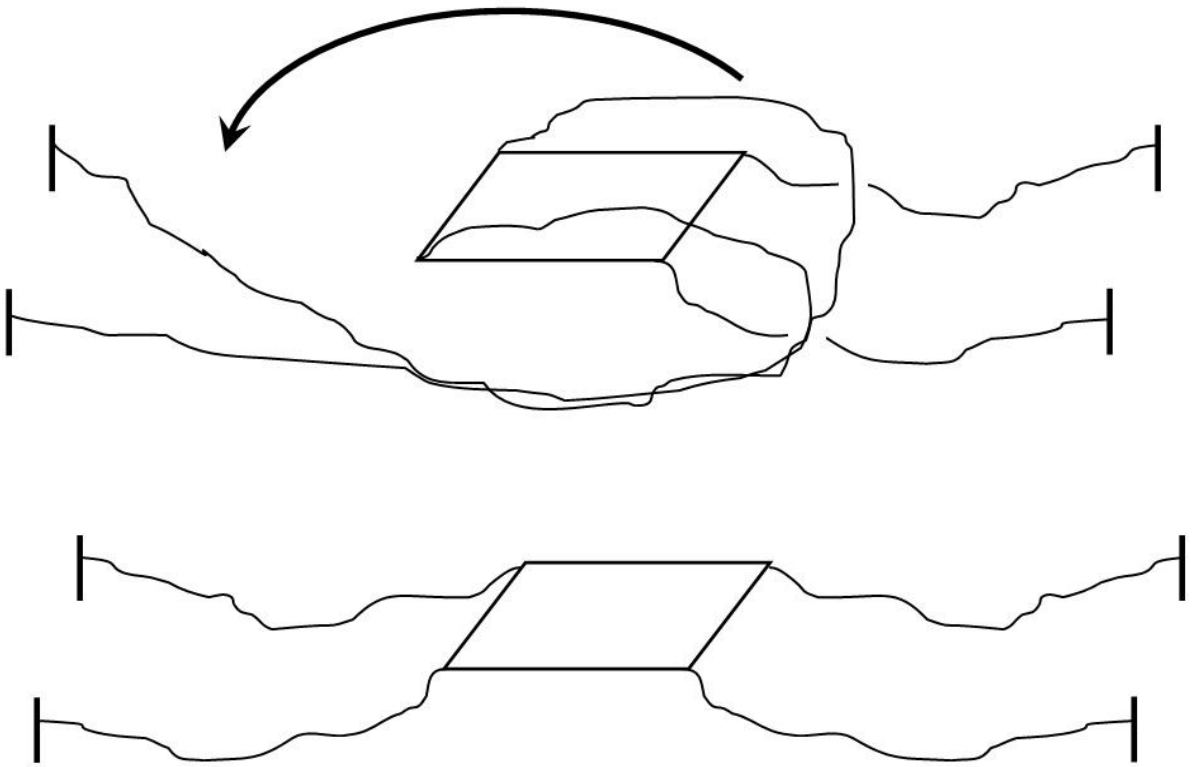


Only the strings are moved. The object remains in its position.









5. As the situation now is totally equivalent to the original situation, we should draw the conclusion, that the object after a rotation of 720° is in the same topological state as in the original position before the rotation.

6. The Dirac belt trick does not work after a rotation of only 2 π = 360°. Thus the topological states are different.

Outlook

The Dirac belt trick is named after P. A. M. Dirac, who invented matrices which represent base vectors in higher-dimensional spaces (or spacetimes).