Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

Wintersemester 2014/2015

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Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part II: solving systems of linear equations)

Stand: 31. Dez. 2014

Teaching & learning contents according to the modular description of LV 200 691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Most of this will be discussed in the standard language of the rather old-fashioned linear algebra or matrix algebra found in most textbooks of business mathematics or mathematical economics.

But as it might be helpful to get an impression of some more interesting new approaches, we will talk about solving systems of linear equations in this second part (4 x 45 min.) of our short introduction to Geometric Algebra.

Repetition: Basics of Geometric Algebra

$1 + 3 + 3 + 1 = 2^3 = 8$ different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: $\sigma_x, \sigma_y, \sigma_z$

Three base bivectors: $\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

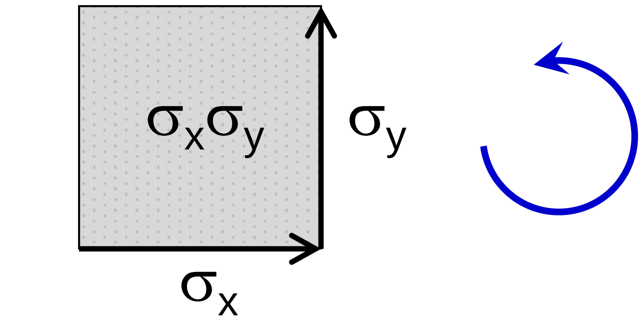
$$1^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

Base bivectors and base trivector square to minus one:

$$(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1$$

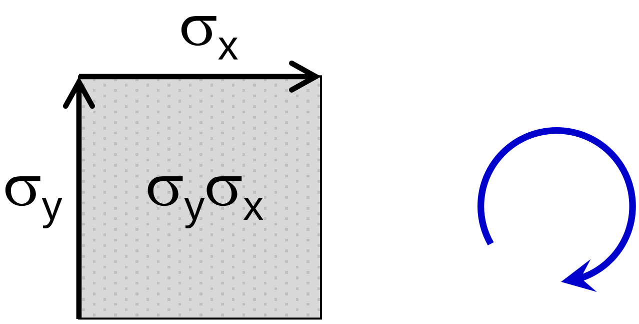
Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the resulting area elements.



right-handed orientation
(anticlockwise orientation)

positive orientation in a right-handed coordinate system



left-handed orientation
(clockwise orientation)

negative orientation in a right-handed coordinate system

Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

$$\sigma_x\sigma_y = -\,\sigma_y\sigma_x$$

$$\sigma_y\sigma_z = -\,\sigma_z\sigma_y$$

$$\sigma_z\sigma_x = -\,\sigma_x\sigma_z$$

Scalars

Scalars are geometric entities without direction. They can be expressed as a multiple of the base scalar:

$$k = k\ 1$$

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

$$r = x\ \sigma_x + y\ \sigma_y + z\ \sigma_z$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

$$A = A_{xy}\ \sigma_x\sigma_y + A_{yz}\ \sigma_y\sigma_z + A_{zx}\ \sigma_z\sigma_x$$

Trivectors

Trivectors are oriented volume elements. They can be expressed as a multiple of the base trivector:

$$V = V_{xyz}\ \sigma_x\sigma_y\sigma_z$$

Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$r_1 r_2 = r_1 \bullet r_2 + r_1 \wedge r_2$$

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

$$r_1 \bullet r_2 = r_2 \bullet r_1 = \frac{1}{2}(r_2 r_1 + r_1 r_2)$$

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

$$r_2 \wedge r_1 = -\, r_1 \wedge r_2 = \frac{1}{2}(r_2 r_1 - r_1 r_2)$$

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of the first part.

Systems of Two Linear Equations

Let's start with an example of a rather simple system of two linear equations:

$$\begin{aligned} 2x + y &= 3 \\ 2x + 4y &= 6 \end{aligned}$$

Of course this system of two linear equations can be solved algebraically:

$$\begin{aligned} 2x + y &= 3 \quad \Rightarrow \quad y = \underbrace{-2x + 3} \\ 2x + 4y &= 6 \end{aligned}$$

substitution

$$\begin{aligned} 2x + 4(-2x + 3) &= 6 \\ -6x + 12 &= 6 \\ x &= 1 \end{aligned}$$

substitution

$$y = -2x + 3 = 1$$

Check of the result:

$$\left.\begin{aligned} 2x + y &= 2 \cdot 1 + 1 = 3 \\ 2x + 4y &= 2 \cdot 1 + 4 \cdot 1 = 6 \end{aligned}\right\}$$

The result is correct.

Graphical Solutions of the System of Linear Equations

There are two different strategies to solve this system of linear equations graphically.

rows

$$\begin{aligned} 2x + y &= 3 \\ 2x + 4y &= 6 \end{aligned}$$

columns

- First strategy: Row picture

  The two rows $2x + y = 3$ and $2x + 4y = 6$

  are shown in a diagram.

- Second strategy: Column picture

  The columns $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ of the system of linear equations are shown in a diagram.

Row Picture

The two rows $2x + y = 3$

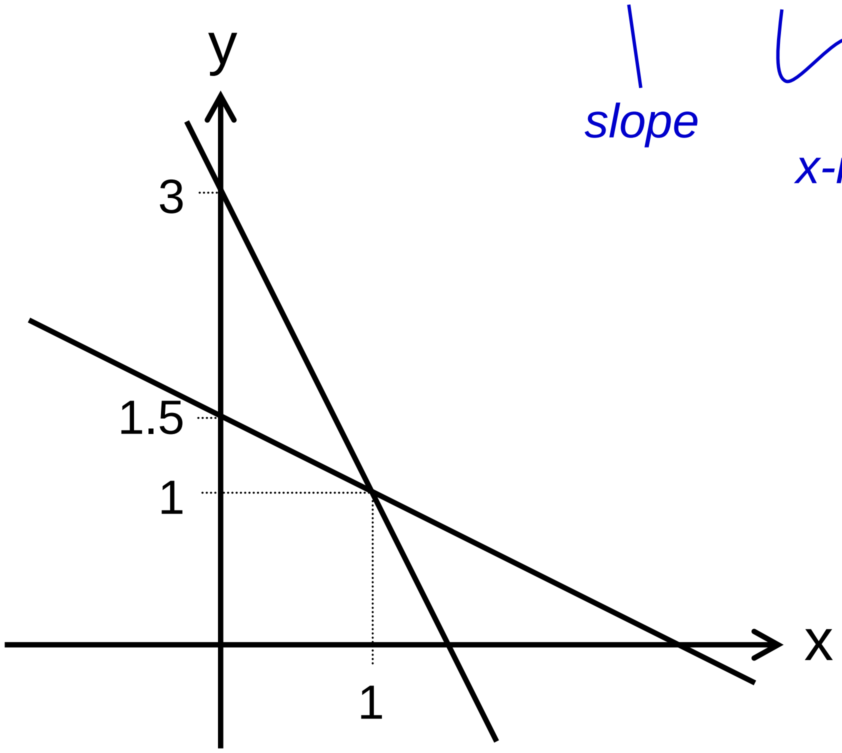
and $2x + 4y = 6$

are represented by the two straight lines

$$y = -2x + 3$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

slope *x-intercept*



The point of intersection (x, y) = (1, 1) of the two lines represents the solution x = 1 and y = 1 of the system of linear equations.

Column Picture

The system of linear equations $2x + y = 3$

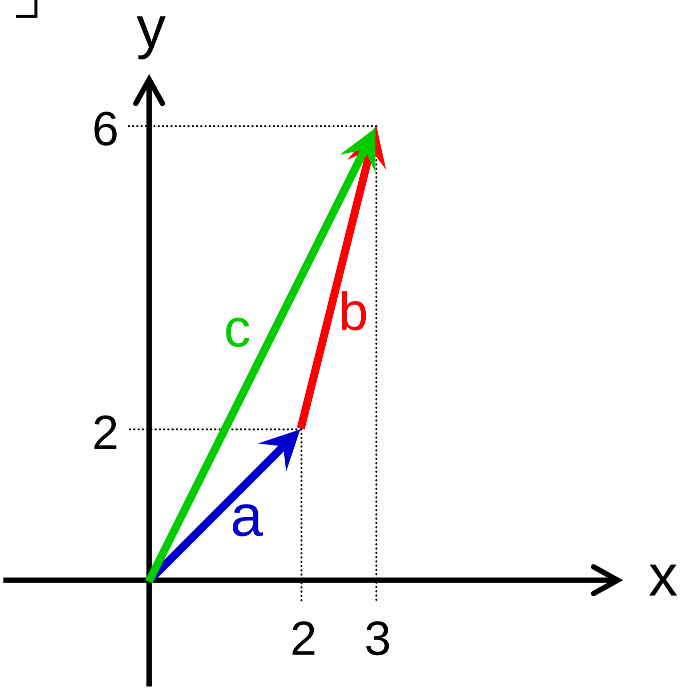
and $2x + 4y = 6$

is now written in vector form $\begin{bmatrix} 2 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \end{bmatrix} y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Solution strategy:

We are looking for a linear combination of vector $a = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ which gives us vector $c = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

Graphical solution:



As just one vector a and one vector b is needed to get vector c, the solution equals $x = 1$ and $y = 1$.

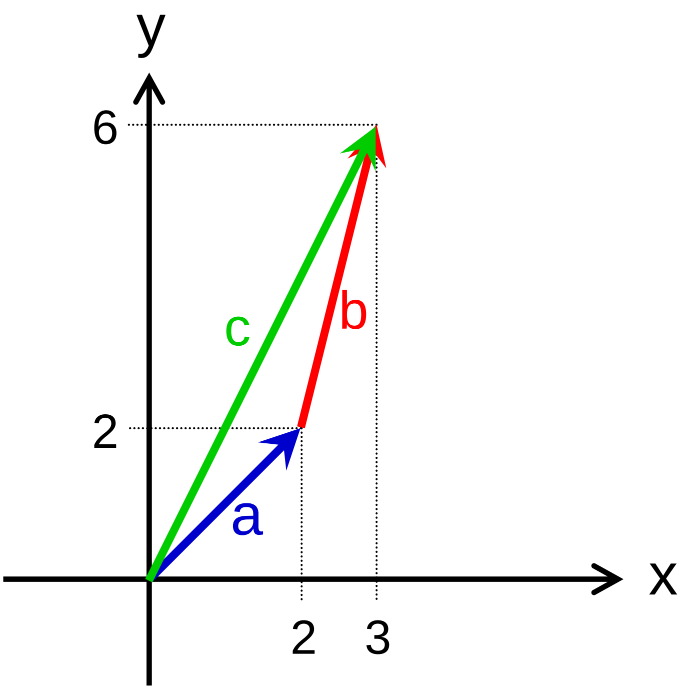
Translating the Column Picture into Geometric Algebra (Pauli Algebra)

$$\begin{aligned} 2x + y &= 3 \\ 2x + 4y &= 6 \end{aligned} \longrightarrow \begin{aligned} 2x\,\sigma_x + y\,\sigma_x &= 3\,\sigma_x \\ 2x\,\sigma_y + 4y\,\sigma_y &= 6\,\sigma_y \end{aligned}$$

$$a = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \longrightarrow a = 2\,\sigma_x + 2\,\sigma_y$$

$$b = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \longrightarrow b = \sigma_x + 4\,\sigma_y$$

$$c = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \longrightarrow c = 3\,\sigma_x + 6\,\sigma_y$$



Translating the Column Picture into Geometric Algebra (Pauli Algebra)

$$2\,x + \quad y = 3$$
$$2\,x + 4\,y = 6$$

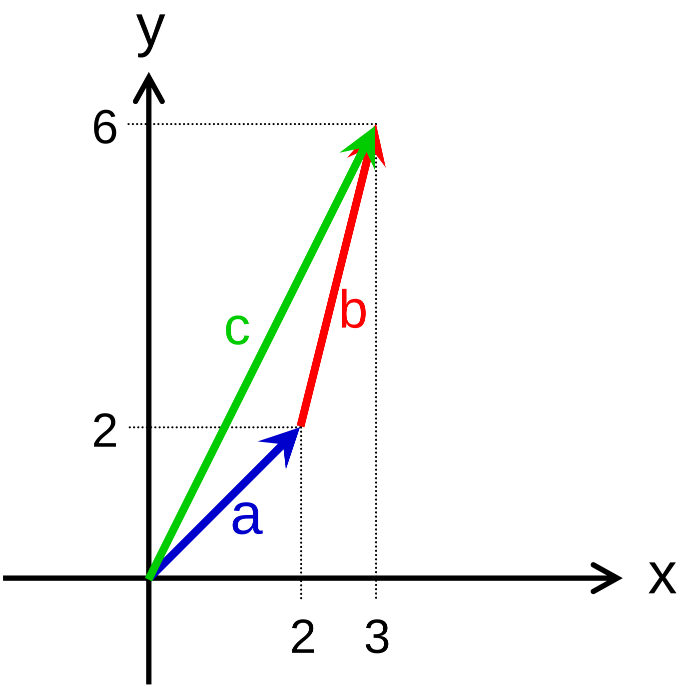
Two equations

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \end{bmatrix} y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$a\,x + b\,y = c$$

Only one equation

$$2\,x\,\sigma_x + \quad y\,\sigma_x = 3\,\sigma_x$$
$$2\,x\,\sigma_y + 4\,y\,\sigma_y = 6\,\sigma_y$$



$$a\,x + b\,y = c$$

Conceptional Core of Transforming Algebraic Problems into Geometric Situations

$$2x + y = 3$$
$$2x + 4y = 6$$

Two equations

↓

$$2x\sigma_x + y\sigma_x = 3\sigma_x$$
$$2x\sigma_y + 4y\sigma_y = 6\sigma_y$$

This is only one equation!

By adding directional information, we condense the two original equations into only one final equation:

$$2x\sigma_x + y\sigma_x + 2x\sigma_y + 4y\sigma_y = 3\sigma_x + 6\sigma_y$$

$$(2\sigma_x + 2\sigma_y)\,x + (\sigma_x + 4\sigma_y)\,y = 3\sigma_x + 6\sigma_y$$

$$a\,x + b\,y = c$$

See for example relativity: The four Maxwell equations can be written as one equation in Geometric Algebra.

$$(2\,\sigma_x + 2\,\sigma_y)\,x + (\sigma_x + 4\,\sigma_y)\,y = 3\,\sigma_x + 6\,\sigma_y$$

To recover the two original equations, just reflect the final equation at the x-axis (or at the y-axis) and add the results to the final equation:

$$\frac{1}{2}\left[\sigma_x\,(a\,x + b\,y)\,\sigma_x + (a\,x + b\,y)\right]$$

$$= \frac{1}{2}\left[(2\,\sigma_x - 2\,\sigma_y)\,x + (\sigma_x - 4\,\sigma_y)\,y + (2\,\sigma_x + 2\,\sigma_y)\,x + (\sigma_x + 4\,\sigma_y)\,y\right]$$

$$= 2\,\sigma_x\,x + \sigma_x\,y$$

$$\frac{1}{2}\left[\sigma_x\,c\,\sigma_x + c\right]$$

$$= \frac{1}{2}\left[3\,\sigma_x - 6\,\sigma_y + 3\,\sigma_x + 6\,\sigma_y\right]$$

$$= 3\,\sigma_x$$

$\Rightarrow$ First original equation:

$$2\,x\,\sigma_x + y\,\sigma_x = 3\,\sigma_x$$

But it is more interesting to look for a mathematical strategy to recover the columns (or vectors) as they will give us a simple solution of the system of linear equations.

To find such a strategy, remember the characteristic features of inner and outer product:

> If the product of two vectors equals the inner product (the bivector terms cancel), the two vectors are parallel.
>
> $$\Updownarrow$$
>
> If two parallel vectors are multiplied, the outer product will disappear and the product of the two parallel vectors will equal the inner product.

$\Rightarrow$ The outer product of a vector with itself equals zero:

$$a \wedge a = \frac{1}{2}(a^2 - a^2) = 0$$

To get rid of vector a, we only have to find the outer product of a linear equation with a.

Solving a System of Two Linear Equations in Geometric Algebra

The wedge product delivers a simple solution of a system of linear equations:

$$a\,x + b\,y = c$$

Getting rid of vector $a\,x$:

$$a \wedge (a\,x + b\,y) = a \wedge c$$

$$(a \wedge a)\,x + (a \wedge b)\,y = a \wedge c$$

$$(a \wedge b)\,y = a \wedge c$$

This gives the solution of variable y:

$$y = \frac{1}{a \wedge b}(a \wedge c)$$

Getting rid of vector $b\,y$:

$$(a\,x + b\,y) \wedge b = c \wedge b$$

$$(a \wedge b)\,x + (b \wedge b)\,y = c \wedge b$$

$$(a \wedge b)\,x = c \wedge b$$

This gives the solution of variable x:

$$x = \frac{1}{a \wedge b}(c \wedge b)$$

Solving a System of Two Linear Equations in Geometric Algebra

The wedge product delivers a simple solu-
tion of

In bivector multiplication the order of factors is important!

We are only allowed to write the solution in this way, if the planes (which are represented by the bivectors) are parallel and the order does not matter.

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$$(a \wedge b)\, y = a \wedge c$$

This gives the solution of variable y:

$$y = \frac{1}{a \wedge b}(a \wedge c) = \frac{a \wedge c}{a \wedge b}$$

Getting rid of vector b:

$$(a\,x + b\,y) \wedge b = c \wedge b$$

$$(a \wedge b)\, x + (b \wedge b)\, y = c \wedge b$$

$$(a \wedge b)\, x = c \wedge b$$

This gives the solution of variable x:

$$x = \frac{1}{a \wedge b}(c \wedge b) = \frac{c \wedge b}{a \wedge b}$$

Solving a System of Two Linear Equations in Geometric Algebra

Example:

$$2\,x + \,y = 3$$

$$2\,x + 4\,y = 6$$

$$c = 3\,\sigma_x + 6\,\sigma_y$$

$$a = 2\,\sigma_x + 2\,\sigma_y$$

$$b = \sigma_x + 4\,\sigma_y$$

$$a\,b = 10 + 6\,\sigma_x\sigma_y \quad \Rightarrow \quad a \wedge b = 6\,\sigma_x\sigma_y$$

$$a\,c = 18 + 6\,\sigma_x\sigma_y \quad \Rightarrow \quad a \wedge c = 6\,\sigma_x\sigma_y$$

$$b\,c = 27 - 6\,\sigma_x\sigma_y \quad \Rightarrow \quad b \wedge c = -\,6\,\sigma_x\sigma_y$$

$$c \wedge b = +\,6\,\sigma_x\sigma_y$$

Solution:

$$x = \frac{1}{a \wedge b}(c \wedge b) = (a \wedge b)^{-1}\,(c \wedge b)$$

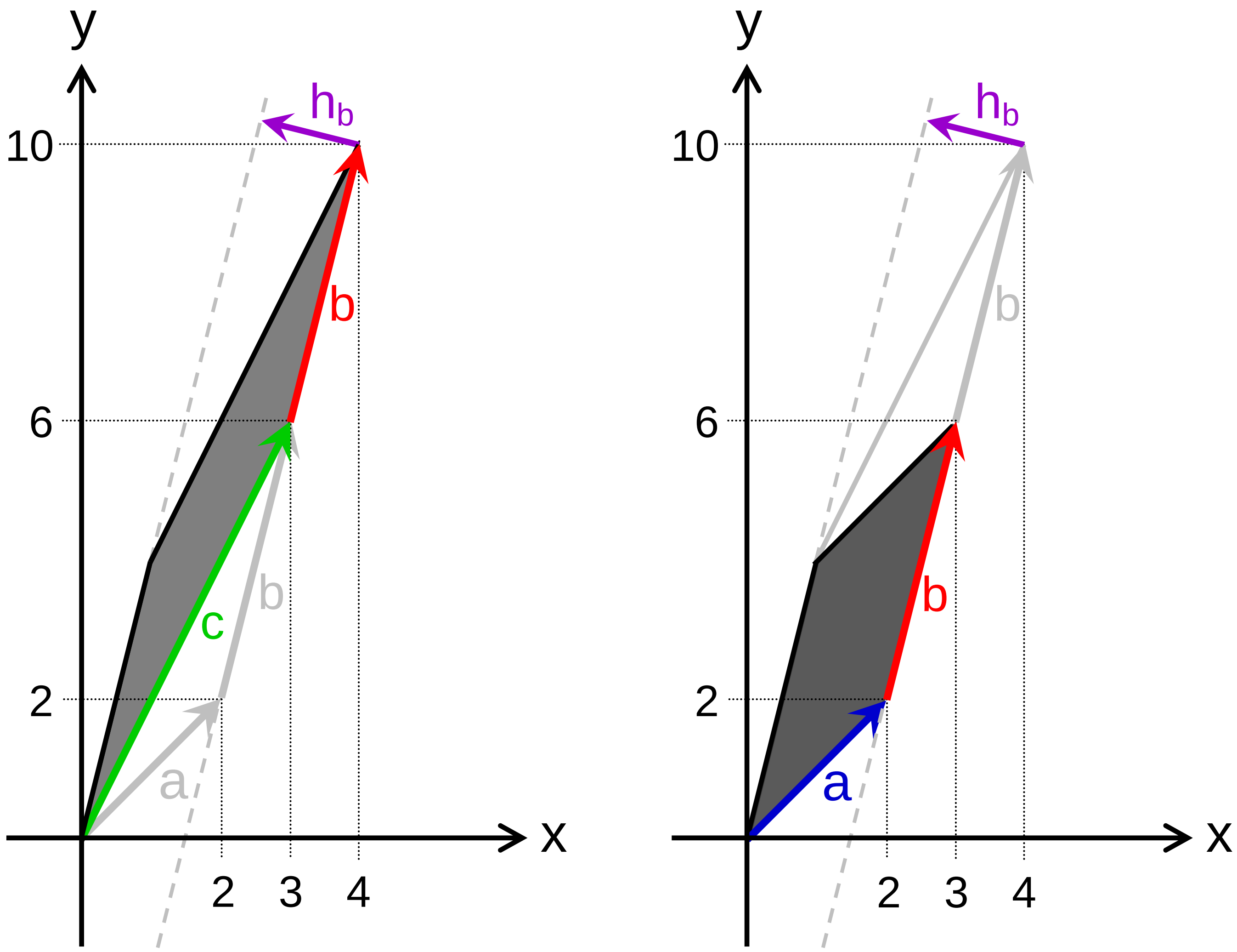
$$= -\frac{1}{36}\,6\,\sigma_x\sigma_y\,6\,\sigma_x\sigma_y = 1$$

$$y = \frac{1}{a \wedge b}\,(a \wedge c) = (a \wedge b)^{-1}\,(a \wedge c)$$

$$= -\frac{1}{36}\,6\,\sigma_x\sigma_y\,6\,\sigma_x\sigma_y = 1$$

Geometric Interpretation of the Result

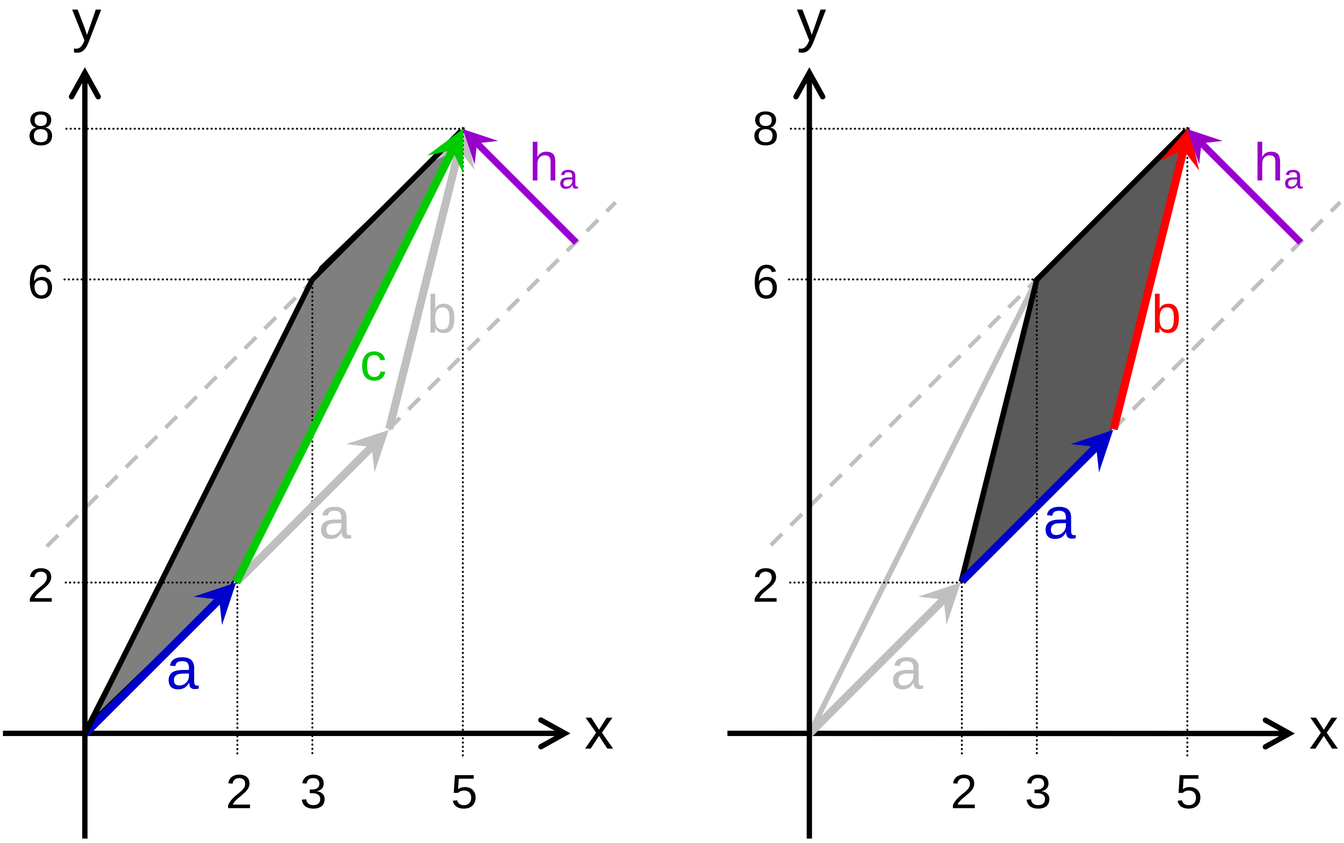
The equation $(a \wedge b)\, x = c \wedge b$ just says that we have to compare the areas of the parallelograms $a\,b$ and $c\,b$ to get x.



Both parallelograms have the same area. Therefore the scalar x should be 1.

Geometric Interpretation of the Result

The equation $(a \wedge b)\, y = a \wedge c$ just says that we have to compare the areas of the parallelograms $a\,b$ and $a\,c$ to get y.



Both parallelograms have the same area.
Therefore the scalar y should be 1.

Finding Inverse Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

To find the inverse A^{-1} of a (2 x 2)-matrix A, we split the Falk scheme of matrix multiplication into two parts.

		b_{11}	b_{12}
		b_{21}	b_{22}
a_{11}	a_{12}	1	0
a_{21}	a_{22}	0	1

The first (blue) part results in a system of two linear equations, which can be transformed into one Geometric Algebra equation:

$$a_{11}\, b_{11} + a_{12}\, b_{21} = 1$$

$$a_{21}\, b_{11} + a_{22}\, b_{21} = 0$$

$$\downarrow$$

$$(a_{11}\, \sigma_x + a_{21}\, \sigma_y)\, b_{11} + (a_{12}\, \sigma_x + a_{22}\, \sigma_y)\, b_{21} = \sigma_x$$

And the second (green) part results in yet another system of two linear equations:

$$a_{11}\, b_{12} + a_{12}\, b_{22} = 1$$

$$a_{21}\, b_{12} + a_{22}\, b_{22} = 0$$

$$\downarrow$$

$$\underbrace{(a_{11}\,\sigma_x + a_{21}\,\sigma_y)}_{r_1}\, b_{12} + \underbrace{(a_{12}\,\sigma_x + a_{22}\,\sigma_y)}_{r_2}\, b_{22} = \sigma_y$$

According to the solution strategy presented at slide # 15 *(get rid of disturbing vectors with the outer product!)*, we will get the four elements b_{ij} of the inverse matrix A^{-1}.

$$b_{11} = \frac{1}{r_1 \wedge r_2}\,(\sigma_x \wedge r_2) = (r_1 \wedge r_2)^{-1}\,(\sigma_x \wedge r_2)$$

$$b_{21} = \frac{1}{r_1 \wedge r_2}\,(r_1 \wedge \sigma_x) = (r_1 \wedge r_2)^{-1}\,(r_1 \wedge \sigma_x)$$

$$b_{12} = \frac{1}{r_1 \wedge r_2}\,(\sigma_y \wedge r_2) = (r_1 \wedge r_2)^{-1}\,(\sigma_y \wedge r_2)$$

$$b_{22} = \frac{1}{r_1 \wedge r_2}\,(r_1 \wedge \sigma_y) = (r_1 \wedge r_2)^{-1}\,(r_1 \wedge \sigma_y)$$

Checking the Inverse Matrix

		$\sigma_x \wedge r_2$ $-\sigma_x \wedge r_1$	$\sigma_y \wedge r_2$ $-\sigma_y \wedge r_1$
a_{11}	a_{12}	$r_1 \wedge r_2$	0
a_{21}	a_{22}	0	$r_1 \wedge r_2$

Calculations:

$$r_1 = a_{11}\,\sigma_x + a_{21}\,\sigma_y$$
$$r_2 = a_{12}\,\sigma_x + a_{22}\,\sigma_y$$

$$r_1 \wedge r_2 = a_{11}\,a_{22}\,\sigma_x\sigma_y - a_{12}\,a_{21}\,\sigma_x\sigma_y$$

$$\sigma_x \wedge r_1 = a_{21}\,\sigma_x\sigma_y \qquad \sigma_x \wedge r_2 = a_{22}\,\sigma_x\sigma_y$$
$$\sigma_y \wedge r_1 = -\,a_{11}\,\sigma_x\sigma_y \qquad \sigma_y \wedge r_2 = -\,a_{12}\,\sigma_x\sigma_y$$

$$e_{11} = a_{11}\,(\sigma_x \wedge r_2) + a_{12}\,(-\,\sigma_x \wedge r_1)$$
$$= a_{11}\,a_{22}\,\sigma_x\sigma_y - a_{12}\,a_{21}\,\sigma_x\sigma_y = r_1 \wedge r_2$$

$$e_{12} = a_{11}\,(\sigma_y \wedge r_2) + a_{12}\,(-\,\sigma_y \wedge r_1)$$
$$= -\,a_{11}\,a_{12}\,\sigma_x\sigma_y + a_{11}\,a_{12}\,\sigma_x\sigma_y = 0$$

$e_{12} = \dots \quad e_{22} = \dots$

Example for Finding the Inverse Matrix

$$A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = ?$$

$$\Rightarrow \quad r_1 = a_{11}\,\sigma_x + a_{21}\,\sigma_y = 8\,\sigma_x + 5\,\sigma_y$$

$$r_2 = a_{12}\,\sigma_x + a_{22}\,\sigma_y = 3\,\sigma_x + 2\,\sigma_y$$

$$\Rightarrow \quad r_1 \wedge r_2 = (16 - 15)\,\sigma_x\sigma_y = \sigma_x\sigma_y$$

$$(r_1 \wedge r_2)^{-1} = -\,\sigma_x\sigma_y$$

Elements of the inverse matrix:

$$b_{11} = (r_1 \wedge r_2)^{-1}\,(\sigma_x \wedge r_2) = -\,\sigma_x\sigma_y\,2\,\sigma_x\sigma_y = \;2$$

$$b_{12} = (r_1 \wedge r_2)^{-1}\,(\sigma_y \wedge r_2) = -\,\sigma_x\sigma_y\,3\,\sigma_y\sigma_x = -\,3$$

$$b_{21} = (r_1 \wedge r_2)^{-1}\,(r_1 \wedge \sigma_x) = -\,\sigma_x\sigma_y\,5\,\sigma_y\sigma_x = -\,5$$

$$b_{22} = (r_1 \wedge r_2)^{-1}\,(r_1 \wedge \sigma_y) = -\,\sigma_x\sigma_y\,8\,\sigma_x\sigma_y = \;8$$

Result:

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$$

Check of result:

$A\,A^{-1}$		2	−3
		−5	8
8	3	1	0
5	2	0	1

Quotation from Wikipedia

"When a linear system solution is introduced via the wedge product, Cramer's rule follows as a side-effect, and there is no need to lead up to the end results with definitions of minors, matrices, matrix invertibility, adjoints, cofactors, Laplace expansions, theorems on determinant multiplication and row column exchanges, and so forth …"

Wikipedia: Comparison of vector algebra and geometric algebra [23. Nov. 2014],

URL: http://en.wikipedia.org/wiki/Comparison_of_vector_algebra_and_geometric_algebra

$\Rightarrow$ And the comparison of the wedge product solution with vector algebra reveals another astonishing fact.

What are Determinants?

A comparison of the standard solution and the Geometric Algebra solution for constructing inverse matrixes reveals the geometric nature of determinants!

Standard solution:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} +|A_{11}| & -|A_{21}| \\ -|A_{12}| & +|A_{22}| \end{bmatrix}$$

determinant of
matrix A

determinants
of submatrices A_{ij}

Geometric Algebra solution:

$$A^{-1} = \frac{1}{r_1 \wedge r_2}\begin{bmatrix} \sigma_x \wedge r_2 & \sigma_y \wedge r_2 \\ -\sigma_x \wedge r_1 & -\sigma_y \wedge r_1 \end{bmatrix}$$

What are Determinants?

A comparison of the standard solution and the Geometric Algebra solution for constructing inverse matrixes reveals the geometric nature of determinants!

Standard solution:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} +|A_{11}| & -|A_{21}| \\ -|A_{12}| & +|A_{22}| \end{bmatrix}$$

determinant of matrix A

determinants of submatrices A_{ij}

Geometric Algebra solution:

$$A^{-1} = \frac{1}{r_1 \wedge r_2}\begin{bmatrix} \sigma_x \wedge r_2 & \sigma_y \wedge r_2 \\ -\sigma_x \wedge r_1 & -\sigma_y \wedge r_1 \end{bmatrix}$$

The determinant of a (2 x 2)-matrix is given by the area (i.e. the two-dimensional volume) of parallelogram $r_1\, r_2$.

Solving (2 x 2)-Matrix Equations

Matrix equation: $A\,B = C$

If matrices A and C are known, the unknown lag matrix B can be found by again splitting the Falk scheme into two parts:

		b_{11}	b_{12}
		b_{21}	b_{22}
a_{11}	a_{12}	c_{11}	c_{12}
a_{21}	a_{22}	c_{21}	c_{22}

The first (blue) part results in a system of two linear equations, which can be transformed into one Geometric Algebra equation:

$$a_{11}\,b_{11} + a_{12}\,b_{21} = c_{11}$$

$$a_{21}\,b_{11} + a_{22}\,b_{21} = c_{21}$$

$$\downarrow$$

$$(a_{11}\,\sigma_x + a_{21}\,\sigma_y)\,b_{11} + (a_{12}\,\sigma_x + a_{22}\,\sigma_y)\,b_{21} = \overbrace{c_{11}\,\sigma_x + c_{21}\,\sigma_y}^{c_{blue}}$$

And the second (green) part results in yet another system of two linear equations:

$$a_{11}\,b_{12} + a_{12}\,b_{22} = c_{12}$$
$$a_{21}\,b_{12} + a_{22}\,b_{22} = c_{22}$$
$$\downarrow$$

$$\underbrace{(a_{11}\,\sigma_x + a_{21}\,\sigma_y)}_{r_1}\,b_{12} + \underbrace{(a_{12}\,\sigma_x + a_{22}\,\sigma_y)}_{r_2}\,b_{22} = \underbrace{c_{11}\,\sigma_x + c_{21}\,\sigma_y}_{c_{green}}$$

According to the solution strategy presented at slide # 15 *(get rid of disturbing vectors with the outer product!)*, we can find the four elements b_{ij} of the unknown matrix B.

$$b_{11} = \frac{1}{r_1 \wedge r_2}(c_{blue} \wedge r_2) = (r_1 \wedge r_2)^{-1}(c_{blue} \wedge r_2)$$

$$b_{21} = \frac{1}{r_1 \wedge r_2}(r_1 \wedge c_{blue}) = (r_1 \wedge r_2)^{-1}(r_1 \wedge c_{blue})$$

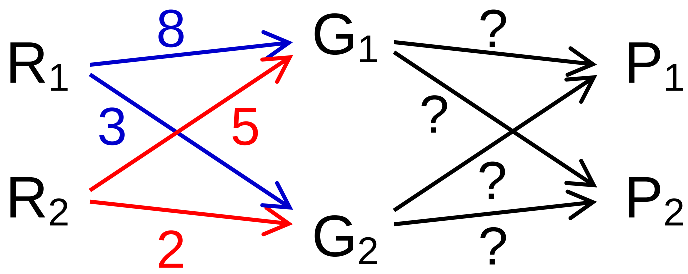
$$b_{12} = \frac{1}{r_1 \wedge r_2}(c_{green} \wedge r_2) = (r_1 \wedge r_2)^{-1}(c_{green} \wedge r_2)$$

$$b_{22} = \frac{1}{r_1 \wedge r_2}(r_1 \wedge c_{green}) = (r_1 \wedge r_2)^{-1}(r_1 \wedge c_{green})$$

Example (Problem)

A firm manufactures two different types of final products P_1 and P_2. To produce these products two intermediate goods G_1 and G_2 are required.

The production of the intermediate goods requires the raw materials R_1 and R_2 (see diagram).



Altogether 65 units of R_1 and 41 units of R_2 are required to produce one unit of the first final product P_1.
And 64 units of R_1 and 41 units of R_2 are required to produce one unit of the second final product P_2.

$\Rightarrow$ Find the demand matrix which shows the demand of intermediate goods to produce one unit of each final product.

Example (Answer)

Demand of raw materials to produce one unit of the intermediate goods: $A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$

Demand of intermediate goods to produce one unit of the final products: $B = ?$

Demand of raw materials to produce one unit of the final products: $D = \begin{bmatrix} 65 & 64 \\ 41 & 41 \end{bmatrix}$

Matrix equation: $A\,B = D$

$$\Rightarrow \quad r_1 = a_{11}\,\sigma_x + a_{21}\,\sigma_y = 8\,\sigma_x + 5\,\sigma_y$$

$$r_2 = a_{12}\,\sigma_x + a_{22}\,\sigma_y = 3\,\sigma_x + 2\,\sigma_y$$

$$\Rightarrow \quad r_1 \wedge r_2 = (16 - 15)\,\sigma_x\sigma_y = \sigma_x\sigma_y$$

$$(r_1 \wedge r_2)^{-1} = -\,\sigma_x\sigma_y$$

see slide # 23

$$\Rightarrow \quad d_1 = d_{11}\,\sigma_x + d_{21}\,\sigma_y = 65\,\sigma_x + 41\,\sigma_y$$

$$d_2 = d_{12}\,\sigma_x + d_{22}\,\sigma_y = 64\,\sigma_x + 41\,\sigma_y$$

$$\Rightarrow \quad d_1 \wedge r_1 = (325 - 328)\, \sigma_x\sigma_y = -\, 3\, \sigma_x\sigma_y$$

$$d_1 \wedge r_2 = (130 - 123)\, \sigma_x\sigma_y = 7\, \sigma_x\sigma_y$$

$$d_2 \wedge r_1 = (320 - 328)\, \sigma_x\sigma_y = -\, 8\, \sigma_x\sigma_y$$

$$d_2 \wedge r_2 = (128 - 123)\, \sigma_x\sigma_y = 5\, \sigma_x\sigma_y$$

$$b_{11} = (r_1 \wedge r_2)^{-1}\, (d_1 \wedge r_2) = -\, \sigma_x\sigma_y\, (7\, \sigma_x\sigma_y) = 7$$

$$b_{21} = (r_1 \wedge r_2)^{-1}\, (r_1 \wedge d_1) = -\, \sigma_x\sigma_y\, (+\, 3\, \sigma_x\sigma_y) = 3$$

7 units of G_1 and 3 units of G_2 are required to produce one unit of final product P_1.

$$b_{12} = (r_1 \wedge r_2)^{-1}\, (d_2 \wedge r_2) = -\, \sigma_x\sigma_y\, (5\, \sigma_x\sigma_y) = 5$$

$$b_{22} = (r_1 \wedge r_2)^{-1}\, (r_1 \wedge d_2) = -\, \sigma_x\sigma_y\, (+\, 8\, \sigma_x\sigma_y) = 8$$

5 units of G_1 and 8 units of G_2 are required to produce one unit of final product P_2.

Demand of intermediate goods to produce one unit of each final product:

$$B = \begin{bmatrix} 7 & 5 \\ 3 & 8 \end{bmatrix}$$

Check of result:

A B		7	5
		3	8
8	3	65	64
5	2	41	41

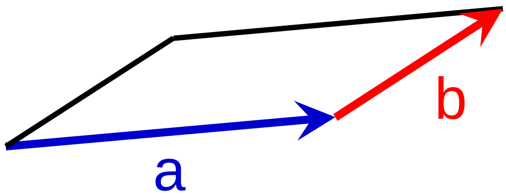
The Product of Three Vectors

To understand systems of three linear equations, we should understand geometric products and wedge products of three vectors.

The product R of two vectors a, b

$$R = a\,b$$

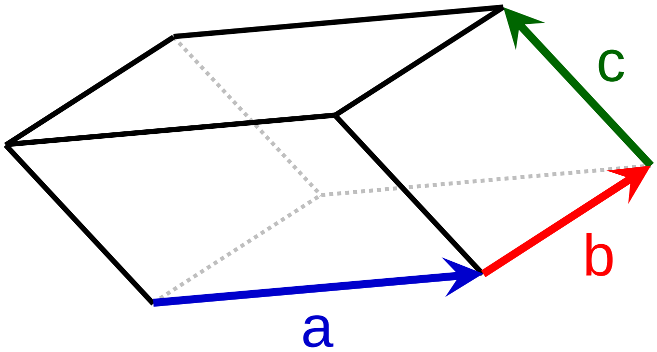
can be interpreted geometrically as an oriented parallelogram.



The product P of three vectors a, b, c

$$P = a\,b\,c$$

extends this geometric structure into yet another direction, and an oriented parallelepiped is formed.



The Product of Three Vectors

The product of two vectors usually consists of a scalar part k and a bivector part A:

$$R = a\,b = \underbrace{a \bullet b}_{\text{scalar}} + \underbrace{a \wedge c}_{\text{bivector}} = k + A$$

If R = a b is again multiplied by a vector, the scalar part transforms into a vector part and the bivector part transforms partially into a scalar part and partially into a trivector part:

$$P = R\,c = k\,c + A\,c$$

$$= \underbrace{k\,c + A \bullet c}_{\text{vector}} + \underbrace{A \wedge c}_{\text{trivector}} = r + V$$

Thus a product of three vectors is …

… an oriented parallelepiped
(geometric viewpoint)

… a linear combination of a vector and a trivector.
(algebraic viewpoint)

Volume of an Oriented Parallelepiped

The trivector part V of an oriented parallel-epiped characterizes the volume of this parallelepiped. It is a scalar V_{xyz} multiplied by the unit trivector (or pseudoscalar) $\sigma_x\sigma_y\sigma_z$:

$$V = \langle P \rangle_{trivector} = \langle a\ b\ c \rangle_{trivector} = a \wedge b \wedge c$$

$$= V_{xyz}\ \sigma_x\sigma_y\sigma_z$$

Associativity

As the product of vectors is associative

$$(a\ b)\ c = a\ (b\ c) = a\ b\ c$$

it follows that the wedge product is an associative product too:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b \wedge c$$

Commutativity & Anti-Commutativity

We have found that the outer product (wedge product) of two vectors is anti-commutative:

$$a \wedge b = -\, b \wedge a$$

Surprisingly, the outer product of a bivector $A = a \wedge b$ with a vector c is commutative

$$\begin{aligned} A \wedge c &= a \wedge b \wedge c \\ &= -\, a \wedge c \wedge b \\ &= \quad c \wedge a \wedge b \\ &= c \wedge A \end{aligned}$$

and the inner product $A \bullet c$ is anti-commutative instead. Therefore the definition of outer and inner products are:

$$A \wedge c = \frac{1}{2}\,(A\,c + c\,A)$$

$$A \bullet c = \frac{1}{2}\,(A\,c - c\,A)$$

Systems of Three Linear Equations

Systems of three linear equations

$$a_{11}\,x + a_{12}\,y + a_{13}\,z = d_1$$
$$a_{21}\,x + a_{22}\,y + a_{23}\,z = d_2$$
$$a_{31}\,x + a_{32}\,y + a_{33}\,z = d_3$$

can be transformed into an old-fashioned vector equation with four vectors a, b, c, and d:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} x + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} y + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} z = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

In modern form of Geometric Algebra the same system of linear equations reads:

$$a_{11}\,x\,\sigma_x + a_{12}\,y\,\sigma_x + a_{13}\,z\,\sigma_x = d_1\,\sigma_x$$
$$a_{21}\,x\,\sigma_y + a_{22}\,y\,\sigma_y + a_{23}\,z\,\sigma_y = d_2\,\sigma_y$$
$$a_{31}\,x\,\sigma_z + a_{32}\,y\,\sigma_z + a_{33}\,z\,\sigma_z = d_3\,\sigma_z$$

Again, we do not see three equations here, but only one equation which is composed of four Pauli vectors a, b, c, and d.

That's our starting point: One equation

$$(a_{11}\,\sigma_x + a_{21}\,\sigma_y + a_{31}\,\sigma_z)\,x$$
$$+\,(a_{12}\,\sigma_x + a_{22}\,\sigma_y + a_{32}\,\sigma_z)\,y$$
$$+\,(a_{13}\,\sigma_x + a_{23}\,\sigma_y + a_{33}\,\sigma_z)\,z$$
$$= d_1\,\sigma_x + d_2\,\sigma_y + d_3\,\sigma_z$$

containing the four Pauli vectors

$$a = a_{11}\,\sigma_x + a_{21}\,\sigma_y + a_{31}\,\sigma_z$$

$$b = a_{12}\,\sigma_x + a_{22}\,\sigma_y + a_{32}\,\sigma_y$$

$$c = a_{13}\,\sigma_x + a_{23}\,\sigma_y + a_{33}\,\sigma_z$$

and

$$d = d_1\,\sigma_x + d_2\,\sigma_y + d_3\,\sigma_z$$

$$\boxed{a\,x + b\,y + c\,z = d}$$

To solve this equation for x, y, and z we again try to recover the vectors (or former columns) by getting rid of other vectors with the wedge product.

Solving a System of Three Linear Equations in Geometric Algebra

$$a\,x + b\,y + c\,z = d$$

Wedge product with vector a to get rid of a x:

$$a \wedge (a\,x + b\,y + c\,z) = a \wedge d$$

$$(a \wedge a)\,x + (a \wedge b)\,y + (a \wedge c)\,z = a \wedge d$$

$$(a \wedge b)\,y + (a \wedge c)\,z = a \wedge d$$

Wedge product with vector b to get rid of b y:

$$b \wedge \big((a \wedge b)\,y + (a \wedge c)\,z\big) = b \wedge (a \wedge d)$$

$$(b \wedge a \wedge b)\,y + (b \wedge a \wedge c)\,z = b \wedge a \wedge d$$

$$-(a \wedge \underbrace{b \wedge b)}_{0}\,y - (a \wedge b \wedge c)\,z = -\,a \wedge b \wedge d$$

$$(a \wedge b \wedge c)\,z = a \wedge b \wedge d$$

This gives the solution of variable z:

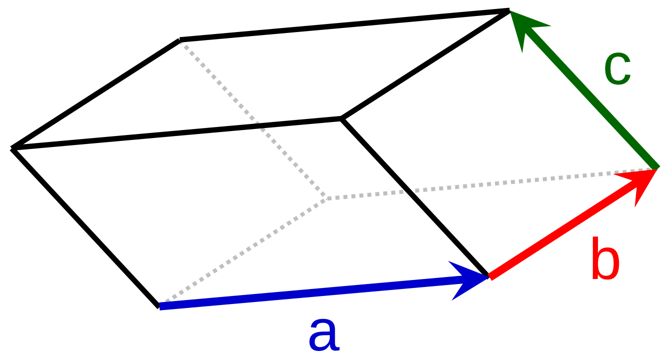
$$z = \frac{1}{a \wedge b \wedge c}(a \wedge b \wedge d)$$

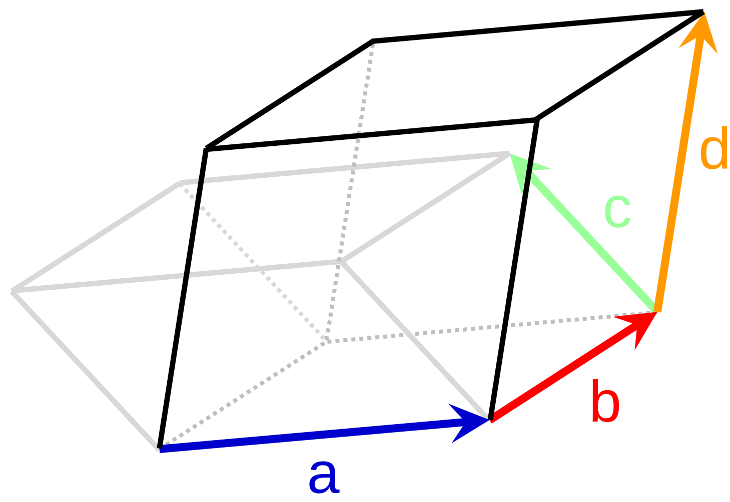
In a similar way, we get:

$$x = \frac{1}{a \wedge b \wedge c}(b \wedge c \wedge d) \qquad y = \frac{1}{a \wedge b \wedge c}(c \wedge a \wedge d)$$

Geometric Interpretation of the Result

The equation $(a \wedge b \wedge c)\, z = a \wedge b \wedge d$ just says that we have to compare the volumes of the parallelepipeds a b c and a b d to get the value of z.





And again the volume of the parallelepiped a b c can be identified with the determinant of the (3 x 3)-matrix:

$$\det(a_{ij}) = a \wedge b \wedge c = \langle a\ b\ c \rangle_{\text{trivector}}$$

Finding the Inverse of (3 x 3)-Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

To find the inverse A^{-1} of a (3 x 3)-matrix A, we now split the Falk scheme of matrix multiplication into three parts, which give us three systems of three linear equations.

			b_{11}	b_{12}	b_{13}
			b_{21}	b_{22}	b_{23}
			b_{31}	b_{32}	b_{33}
a_{11}	a_{12}	a_{13}	1	0	0
a_{21}	a_{22}	a_{23}	0	1	0
a_{31}	a_{32}	a_{33}	0	0	1

The first (blue) part results in the system of linear equations given on the following slide.

$$a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} = 1$$
$$a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31} = 0$$
$$a_{31}\, b_{11} + a_{32}\, b_{21} + a_{33}\, b_{31} = 0$$

The translation into Geometric Algebra results in one equation:

$$(a_{11}\,\sigma_x + a_{21}\,\sigma_y + a_{31}\,\sigma_z)\, b_{11} + (a_{12}\,\sigma_x + a_{22}\,\sigma_y + a_{32}\,\sigma_z)\, b_{21} + (a_{13}\,\sigma_x + a_{23}\,\sigma_y + a_{33}\,\sigma_z)\, b_{31} = \sigma_x$$

$$r_1\, b_{11} + r_2\, b_{21} + r_3\, b_{31} = \sigma_x$$

According to the solution strategy of slide # 38 *(get rid of disturbing vectors by outer multiplication)*, we will get the first three elements b_{ij} of the inverse matrix A^{-1}.

$$b_{11} = \frac{1}{r_1 \wedge r_2 \wedge r_3}\,(r_2 \wedge r_3 \wedge \sigma_x)$$

$$b_{21} = \frac{1}{r_1 \wedge r_2 \wedge r_3}\,(r_3 \wedge r_1 \wedge \sigma_x)$$

$$b_{31} = \frac{1}{r_1 \wedge r_2 \wedge r_3}\,(r_1 \wedge r_2 \wedge \sigma_x)$$

Summary: The Inverse of (3 x 3)-Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11}\,\sigma_x + a_{21}\,\sigma_y + a_{31}\,\sigma_z = r_1$$

$$a_{12}\,\sigma_x + a_{22}\,\sigma_y + a_{32}\,\sigma_z = r_2$$

$$a_{13}\,\sigma_x + a_{23}\,\sigma_y + a_{33}\,\sigma_z = r_3$$

$$A^{-1} = \frac{1}{r_1 \wedge r_2 \wedge r_3} \begin{bmatrix} r_2 \wedge r_3 \wedge \sigma_x & r_2 \wedge r_3 \wedge \sigma_y & r_2 \wedge r_3 \wedge \sigma_z \\ r_3 \wedge r_1 \wedge \sigma_x & r_3 \wedge r_1 \wedge \sigma_y & r_3 \wedge r_1 \wedge \sigma_z \\ r_1 \wedge r_2 \wedge \sigma_x & r_1 \wedge r_2 \wedge \sigma_y & r_1 \wedge r_2 \wedge \sigma_z \end{bmatrix}$$

$\det(a_{ij}) = a \wedge b \wedge c$ is an oriented volume.

If you prefer the determinant to be a scalar, simply identify it as coefficient of the unit tri-vector $\sigma_x\sigma_y\sigma_z$:

$$|\det(a_{ij})| = -(a \wedge b \wedge c)\,\sigma_x\sigma_y\sigma_z$$

It can now be seen as a volume scale factor.

Checking the Inverse Matrix

			$r_2 \wedge r_3 \wedge \sigma_x$	$r_2 \wedge r_3 \wedge \sigma_y$	$r_2 \wedge r_3 \wedge \sigma_z$
			$r_3 \wedge r_1 \wedge \sigma_x$	$r_3 \wedge r_1 \wedge \sigma_y$	$r_3 \wedge r_1 \wedge \sigma_z$
			$r_1 \wedge r_2 \wedge \sigma_x$	$r_1 \wedge r_2 \wedge \sigma_y$	$r_1 \wedge r_2 \wedge \sigma_z$
a_{11}	a_{12}	a_{13}	$r_1 \wedge r_2 \wedge r_3$	0	0
a_{21}	a_{22}	a_{23}	0	$r_1 \wedge r_2 \wedge r_3$	0
a_{31}	a_{32}	a_{33}	0	0	$r_1 \wedge r_2 \wedge r_3$

Calculations:

$$r_1 \wedge r_2 \wedge \sigma_x = (a_{21}\, a_{32} - a_{31}\, a_{22})\, \sigma_x\sigma_y\sigma_z$$

$$r_2 \wedge r_3 \wedge \sigma_x = (a_{22}\, a_{33} - a_{32}\, a_{23})\, \sigma_x\sigma_y\sigma_z$$

$$r_3 \wedge r_1 \wedge \sigma_x = (a_{23}\, a_{31} - a_{33}\, a_{21})\, \sigma_x\sigma_y\sigma_z$$

$$r_1 \wedge r_2 \wedge r_3 = (a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23} - a_{11} a_{32} a_{23} - a_{21} a_{12} a_{33} - a_{31} a_{22} a_{13})\, \sigma_x\sigma_y\sigma_z$$

Checking the Inverse Matrix

$$e_{11} = a_{11} (r_2 \wedge r_3 \wedge \sigma_x) + a_{12} (r_3 \wedge r_1 \wedge \sigma_x) + a_{13} (r_1 \wedge r_2 \wedge \sigma_x)$$

$$= (a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{32} a_{23} - a_{12} a_{33} a_{21} - a_{13} a_{31} a_{22}) \sigma_x \sigma_y \sigma_z$$

$$= r_1 \wedge r_2 \wedge r_3$$

$$e_{21} = a_{21} (r_2 \wedge r_3 \wedge \sigma_x) + a_{22} (r_3 \wedge r_1 \wedge \sigma_x) + a_{23} (r_1 \wedge r_2 \wedge \sigma_x)$$

$$= (a_{21} a_{22} a_{33} - a_{21} a_{32} a_{23} + a_{22} a_{23} a_{31} - a_{22} a_{33} a_{21} + a_{23} a_{21} a_{32} - a_{23} a_{31} a_{22}) \sigma_x \sigma_y \sigma_z$$

$$= 0$$

$e_{31} = \ldots$

$e_{12} = \ldots$ etc …

Solving (3 x 3)-Matrix Equations

Matrix equation: $A\,B = C$

If matrices A and C are known, the unknown lag matrix B can be found by again splitting the Falk scheme into three parts:

			b_{11}	b_{12}	b_{13}
			b_{21}	b_{22}	b_{23}
			b_{31}	b_{32}	b_{33}
a_{11}	a_{12}	a_{13}	c_{11}	c_{12}	c_{13}
a_{21}	a_{22}	a_{23}	c_{21}	c_{22}	c_{23}
a_{31}	a_{32}	a_{33}	c_{31}	c_{32}	c_{33}

The first (blue) part results in the following system of three linear equations:

$$a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31} = c_{11}$$
$$a_{21}\,b_{11} + a_{22}\,b_{21} + a_{23}\,b_{31} = c_{21}$$
$$a_{31}\,b_{11} + a_{32}\,b_{21} + a_{33}\,b_{31} = c_{31}$$

The translation into Geometric Algebra results in one equation:

$$(a_{11}\,\sigma_x + a_{21}\,\sigma_y + a_{31}\,\sigma_z)\, b_{11} + (a_{12}\,\sigma_x + a_{22}\,\sigma_y + a_{32}\,\sigma_z)\, b_{21} + (a_{13}\,\sigma_x + a_{23}\,\sigma_y + a_{33}\,\sigma_z)\, b_{31} = c_{11}\,\sigma_x + c_{21}\,\sigma_y + c_{31}\,\sigma_z$$

$$r_1\, b_{11} + r_2\, b_{21} + r_3\, b_{31} = c_1$$

Similar equations can be formulated by the second (green) and third (red) parts.

Applying again the solution strategy of slide # 38 *(get rid of disturbing vectors by outer multiplication)*, matrix B can be found:

$$B = \frac{1}{r_1 \wedge r_2 \wedge r_3} \begin{bmatrix} r_2 \wedge r_3 \wedge c_1 & r_2 \wedge r_3 \wedge c_2 & r_2 \wedge r_3 \wedge c_3 \\ r_3 \wedge r_1 \wedge c_1 & r_3 \wedge r_1 \wedge c_2 & r_3 \wedge r_1 \wedge c_3 \\ r_1 \wedge r_2 \wedge c_1 & r_1 \wedge r_2 \wedge c_2 & r_1 \wedge r_2 \wedge c_3 \end{bmatrix}$$

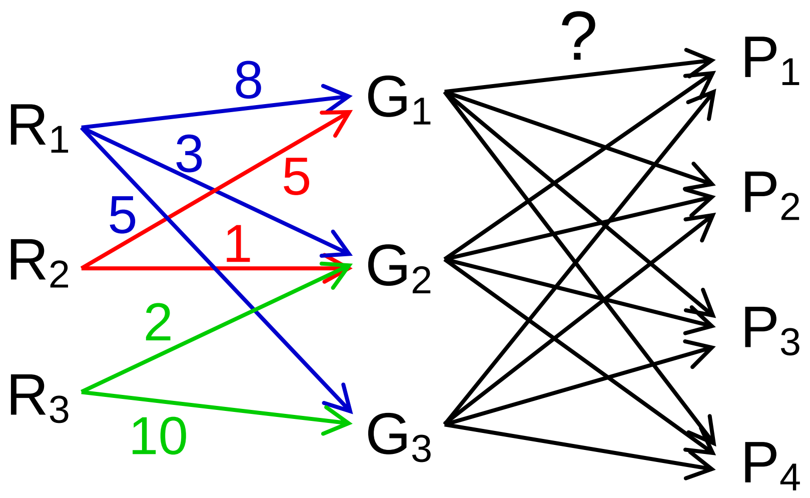
Alternatively a pre-multiplication of D by A^{-1} *(see slide 42)* gives the same result:

$$B = A^{-1}\, D$$

Example (Problem)

A firm manufactures four different types of final products P_1, P_2, P_3, and P_4. To produce these products three intermediate goods G_1, G_2, and G_3 are required.

The production of the intermediate goods requires the raw materials R_1, R_2, and R_3 (see diagram).



Total demand of raw materials:

100 units of R_1, 70 units of R_2, and 40 units of R_3 are required to produce one unit of the first final product P_1.

60 units of R_1, 40 units of R_2, and 20 units of R_3 are required to produce one unit of the second final product P_2.

40 units of R_1, 10 units of R_2, and 40 units of R_3 are required to produce one unit of the third final product P_3.

60 units of R_1, 28 units of R_2, and 44 units of R_3 are required to produce one unit of the fourth final product P_4.

⇒ **Find the demand matrix which shows the demand of intermediate goods to produce one unit of each final product.**

⇒ **Please use two different solution strategies.**

Example (Answer)

First production step:

Demand of raw materials to produce one unit of the intermediate goods:

$$A = \begin{bmatrix} 8 & 3 & 5 \\ 7 & 1 & 0 \\ 0 & 2 & 10 \end{bmatrix}$$

Second production step:

Demand of intermediate goods to produce one unit of the final products:

$$B = ?$$

Combination of both production steps:

Demand of raw materials to produce one unit of the final products:

$$D = \begin{bmatrix} 100 & 60 & 40 & 60 \\ 70 & 40 & 10 & 28 \\ 40 & 20 & 40 & 44 \end{bmatrix}$$

Matrix equation: $A\,B = D$

First solution strategy:

- Direct calculation of B.

Second solution strategy:

- Find the inverse A^{-1} and pre-multiply D by A^{-1}.

$$B = A^{-1} D$$

Example (Answer)

Associated vectors of the system of linear equations:

- Demand of raw materials A of first production step:

$$r_1 = 8\,\sigma_x + 7\,\sigma_y$$

$$r_2 = 3\,\sigma_x + \sigma_y + 2\,\sigma_z$$

$$r_3 = 5\,\sigma_x + 10\,\sigma_z$$

- Total demand of raw materials D of both production steps:

$$d_1 = 100\,\sigma_x + 70\,\sigma_y + 40\,\sigma_z$$

$$d_2 = 60\,\sigma_x + 40\,\sigma_y + 20\,\sigma_z$$

$$d_3 = 40\,\sigma_x + 10\,\sigma_y + 40\,\sigma_z$$

$$d_4 = 60\,\sigma_x + 28\,\sigma_y + 44\,\sigma_z$$

Calculation of determinant:
(volume of oriented parallelepiped)

$$\begin{aligned} r_1 \wedge r_2 \wedge r_3 &= (80 + 0 + 70 - 0 - 210 - 0)\,\sigma_x\sigma_y\sigma_z \\ &= -60\,\sigma_x\sigma_y\sigma_z \end{aligned}$$

Example (First Strategy)

Calculation of outer products:

Determinants of mixed matrices $\rightarrow$ First column of B (volume of mixed oriented parallelepipeds):

$$r_1 \wedge r_2 \wedge d_1$$
$$= (320 + 1400 + 0 - 1120 - 840 - 0)\, \sigma_x\sigma_y\sigma_z$$
$$= -240\, \sigma_x\sigma_y\sigma_z$$

$$r_2 \wedge r_3 \wedge d_1$$
$$= (0 + 1000 + 700 - 2100 - 200 - 0)\, \sigma_x\sigma_y\sigma_z$$
$$= -600\, \sigma_x\sigma_y\sigma_z$$

$$r_3 \wedge r_1 \wedge d_1$$
$$= (1400 + 0 + 5600 - 0 - 0 - 7000)\, \sigma_x\sigma_y\sigma_z$$
$$= 0\, \sigma_x\sigma_y\sigma_z$$

Calculation of the first column of matrix B:

$$b_{11} = \frac{1}{-60\, \sigma_x\sigma_y\sigma_z} (-600\, \sigma_x\sigma_y\sigma_z) = 10$$

$$b_{21} = \frac{1}{-60\, \sigma_x\sigma_y\sigma_z} (0\, \sigma_x\sigma_y\sigma_z) = 0$$

$$b_{31} = \frac{1}{-60\, \sigma_x\sigma_y\sigma_z} (-240\, \sigma_x\sigma_y\sigma_z) = 4$$

Example (First Strategy)

Calculation of outer products:

Determinants of mixed matrices → Second column of B (volume of mixed oriented parallelepipeds):

$r_1 \wedge r_2 \wedge d_2$

$= (160 + 840 + 0 - 640 - 420 - 0)\ \sigma_x\sigma_y\sigma_z$

$= -\ 60\ \sigma_x\sigma_y\sigma_z$

$r_2 \wedge r_3 \wedge d_2$

$= (0 + 600 + 400 - 1200 - 100 - 0)\ \sigma_x\sigma_y\sigma_z$

$= -\ 300\ \sigma_x\sigma_y\sigma_z$

$r_3 \wedge r_1 \wedge d_2$

$= (700 + 0 + 3200 - 0 - 0 - 4200)\ \sigma_x\sigma_y\sigma_z$

$= -\ 300\ \sigma_x\sigma_y\sigma_z$

Calculation of the second column of matrix B:

$$b_{21} = \frac{1}{-60\,\sigma_x\sigma_y\sigma_z}\ (-\ 300\ \sigma_x\sigma_y\sigma_z) = 5$$

$$b_{22} = \frac{1}{-60\,\sigma_x\sigma_y\sigma_z}\ (-\ 300\ \sigma_x\sigma_y\sigma_z) = 5$$

$$b_{23} = \frac{1}{-60\,\sigma_x\sigma_y\sigma_z}\ (-\quad 60\ \sigma_x\sigma_y\sigma_z) = 1$$

Example (First Strategy)

Calculation of outer products:

Determinants of mixed matrices $\rightarrow$ Third column of B (volume of mixed oriented parallelepipeds):

$$r_1 \wedge r_2 \wedge d_3$$
$$= (320 + 560 + 0 - 160 - 840 - 0)\, \sigma_x\sigma_y\sigma_z$$
$$= -120\, \sigma_x\sigma_y\sigma_z$$

$$r_2 \wedge r_3 \wedge d_3$$
$$= (0 + 400 + 100 - 300 - 200 - 0)\, \sigma_x\sigma_y\sigma_z$$
$$= 0\, \sigma_x\sigma_y\sigma_z$$

$$r_3 \wedge r_1 \wedge d_3$$
$$= (1400 + 0 + 800 - 0 - 0 - 2800)\, \sigma_x\sigma_y\sigma_z$$
$$= -600\, \sigma_x\sigma_y\sigma_z$$

Calculation of the third column of matrix B:

$$b_{31} = \frac{1}{-60\, \sigma_x\sigma_y\sigma_z} (\quad 0\, \sigma_x\sigma_y\sigma_z) = 0$$

$$b_{32} = \frac{1}{-60\, \sigma_x\sigma_y\sigma_z} (-600\, \sigma_x\sigma_y\sigma_z) = 10$$

$$b_{33} = \frac{1}{-60\, \sigma_x\sigma_y\sigma_z} (-120\, \sigma_x\sigma_y\sigma_z) = 2$$

Example (First Strategy)

Calculation of outer products:

Determinants of mixed matrices $\rightarrow$ Forth column of B (volume of mixed oriented parallelepipeds):

$$r_1 \wedge r_2 \wedge d_4$$
$$= (352 + 840 + 0 - 448 - 924 - 0)\,\sigma_x\sigma_y\sigma_z$$
$$= -\,180\;\sigma_x\sigma_y\sigma_z$$

$$r_2 \wedge r_3 \wedge d_4$$
$$= (0 + 600 + 280 - 840 - 220 - 0)\,\sigma_x\sigma_y\sigma_z$$
$$= -\,180\;\sigma_x\sigma_y\sigma_z$$

$$r_3 \wedge r_1 \wedge d_4$$
$$= (1540 + 0 + 2240 - 0 - 0 - 4200)\,\sigma_x\sigma_y\sigma_z$$
$$= -\,420\;\sigma_x\sigma_y\sigma_z$$

Calculation of the forth column of matrix B:

$$b_{41} = \frac{1}{-60\,\sigma_x\sigma_y\sigma_z}\,(-\,180\;\sigma_x\sigma_y\sigma_z) = 3$$

$$b_{42} = \frac{1}{-60\,\sigma_x\sigma_y\sigma_z}\,(-\,420\;\sigma_x\sigma_y\sigma_z) = 7$$

$$b_{43} = \frac{1}{-60\,\sigma_x\sigma_y\sigma_z}\,(-\,180\;\sigma_x\sigma_y\sigma_z) = 3$$

Example (Result)

Demand matrix B which shows the demand of intermediate goods to produce one unit of each final product is given by

$$B = \begin{bmatrix} 10 & 5 & 0 & 3 \\ 0 & 5 & 10 & 7 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Check of result:

A B = D						
			10	5	0	3
			0	5	10	7
			4	1	2	3
8	3	5	100	60	40	60
7	1	0	70	40	10	28
0	2	10	40	20	40	44

Example (Result)

Interpretation of result:

10 units of G_1 and 4 units of G_3 are required to produce one unit of the final product P_1.

5 units of G_1, 5 units of G_2, and 1 unit of G_3 are required to produce one unit of the final product P_2.

10 units of G_2 and 2 units of G_3 are required to produce one unit of the final product P_3.

3 units of G_1, 7 units of G_2, and 3 units of G_3 are required to produce one unit of the final product P_3.

Please compare now with the second solution strategy on the following slides.

Example (Second Strategy)

Calculation of outer products:

- Determinant (volume of oriented parallelepiped):

$$r_1 \wedge r_2 \wedge r_3 = (80 + 0 + 70 - 0 - 210 - 0)\, \sigma_x\sigma_y\sigma_z$$
$$= -60\, \sigma_x\sigma_y\sigma_z$$

- Determinants of submatrices
  (volume of associated oriented parallelepipeds):

$$r_1 \wedge r_2 \wedge \sigma_x = (14 - 0)\, \sigma_x\sigma_y\sigma_z = 14\, \sigma_x\sigma_y\sigma_z$$
$$r_2 \wedge r_3 \wedge \sigma_x = (10 - 0)\, \sigma_x\sigma_y\sigma_z = 10\, \sigma_x\sigma_y\sigma_z$$
$$r_3 \wedge r_1 \wedge \sigma_x = (0 - 70)\, \sigma_x\sigma_y\sigma_z = -70\, \sigma_x\sigma_y\sigma_z$$

$$r_1 \wedge r_2 \wedge \sigma_y = (0 - 16)\, \sigma_x\sigma_y\sigma_z = -16\, \sigma_x\sigma_y\sigma_z$$
$$r_2 \wedge r_3 \wedge \sigma_y = (10 - 30)\, \sigma_x\sigma_y\sigma_z = -20\, \sigma_x\sigma_y\sigma_z$$
$$r_3 \wedge r_1 \wedge \sigma_y = (80 - 0)\, \sigma_x\sigma_y\sigma_z = 80\, \sigma_x\sigma_y\sigma_z$$

$$r_1 \wedge r_2 \wedge \sigma_z = (8 - 21)\, \sigma_x\sigma_y\sigma_z = -13\, \sigma_x\sigma_y\sigma_z$$
$$r_2 \wedge r_3 \wedge \sigma_z = (0 - 5)\, \sigma_x\sigma_y\sigma_z = -5\, \sigma_x\sigma_y\sigma_z$$
$$r_3 \wedge r_1 \wedge \sigma_z = (35 - 0)\, \sigma_x\sigma_y\sigma_z = 35\, \sigma_x\sigma_y\sigma_z$$

Example (Second Strategy)

Calculation of the inverse A^{-1}:

$$A^{-1} = \frac{1}{r_1 \wedge r_2 \wedge r_3} \begin{bmatrix} r_2 \wedge r_3 \wedge \sigma_x & r_2 \wedge r_3 \wedge \sigma_y & r_2 \wedge r_3 \wedge \sigma_z \\ r_3 \wedge r_1 \wedge \sigma_x & r_3 \wedge r_1 \wedge \sigma_y & r_3 \wedge r_1 \wedge \sigma_z \\ r_1 \wedge r_2 \wedge \sigma_x & r_1 \wedge r_2 \wedge \sigma_y & r_1 \wedge r_2 \wedge \sigma_z \end{bmatrix}$$

$$= \frac{1}{-60\,\sigma_x\sigma_y\sigma_z} \begin{bmatrix} 10\,\sigma_x\sigma_y\sigma_z & -20\,\sigma_x\sigma_y\sigma_z & -5\,\sigma_x\sigma_y\sigma_z \\ -70\,\sigma_x\sigma_y\sigma_z & 80\,\sigma_x\sigma_y\sigma_z & 35\,\sigma_x\sigma_y\sigma_z \\ 14\,\sigma_x\sigma_y\sigma_z & -16\,\sigma_x\sigma_y\sigma_z & -13\,\sigma_x\sigma_y\sigma_z \end{bmatrix}$$

$$= \frac{1}{60} \begin{bmatrix} -10 & 20 & 5 \\ 70 & -80 & -35 \\ -14 & 16 & 13 \end{bmatrix}$$

Check of result:

	$60\,A^{-1}A$		8	3	5
			7	1	0
			0	2	10
–10	20	5	60	0	0
70	–80	–35	0	60	0
–14	16	13	0	0	60

Example (Second Strategy)

Calculation of demand matrix B:

$A^{-1}D$	$\begin{matrix} 100 & 60 & 40 & 60 \\ 70 & 40 & 10 & 28 \\ 40 & 20 & 40 & 44 \end{matrix}$
$\begin{matrix} -\frac{10}{60} & \frac{20}{60} & \frac{5}{60} \\ \frac{70}{60} & -\frac{80}{60} & -\frac{35}{60} \\ -\frac{14}{60} & \frac{16}{60} & \frac{13}{60} \end{matrix}$	$\underbrace{\begin{matrix} 10 & 5 & 0 & 3 \\ 0 & 5 & 10 & 7 \\ 4 & 1 & 2 & 3 \end{matrix}}$

This result is identical to the result of slide # 54.

Outlook

Systems of more than three linear equations can be solved in a similar way. To do this, we need vectors of higher-dimensional spaces.

P. A. M. Dirac invented some matrices which represent base vectors in four- or five-dimensional spaces (or spacetimes). Thus rather simple solutions of systems of four or five linear equations are possible if the outer product is applied in a straightforward manner.

Systems of more than five linear equations can be solved in the same way, if base vectors of these higher-dimensional spaces are constructed using the direct product of Zehfuss and Kronecker. It is really no big deal to do that after having understood how things work in two- or three-dimensional spaces.

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Merry Christmas and a Happy New Year!