Hochschule für
Wirtschaft und Recht Berlin
Berlin School of Economics and Law

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Mathematics for Business and Economics

– LV-Nr. 200691.01 –

Modern Linear Algebra

(A Geometric Algebra crash course, Part III: The direct product & solving higher-dimensional systems of linear equations)

Stand: 28. Jan. 2015

Teaching & learning contents according to the modular description of LV 200 691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Most of this will be discussed in the standard language of the rather old-fashioned linear algebra or matrix algebra found in most textbooks of business mathematics or mathematical economics.

But as it might be helpful to get an impression of some more interesting new approaches, we will talk about solving systems of linear equations with more than three variables with the help of Geometric Algebra and the direct product in this third special GA lesson (2 x 45 min up to slide 34).

Repetition: Basics of Geometric Algebra in three-dimensional space

$1 + 3 + 3 + 1 = 2^3 = 8$ different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: $\sigma_x, \sigma_y, \sigma_z$

Three base bivectors: $\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x$
(sometimes called pseudovectors)

One base trivector: $\sigma_x\sigma_y\sigma_z$
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

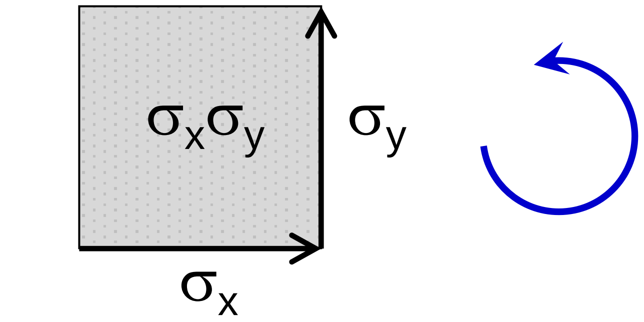
$$1^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

Base bivectors and base trivector square to minus one:

$$(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1$$

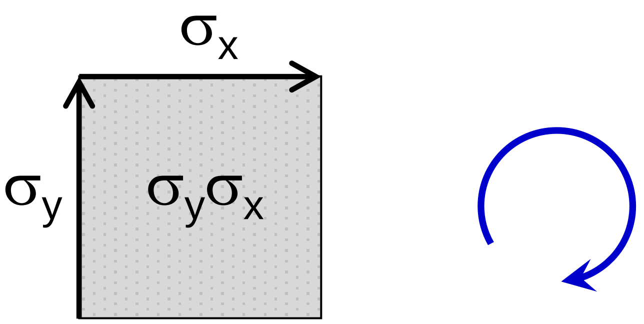
Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the re-sulting area elements.



right-handed orientation
(anticlockwise orientation)

positive orientation in a right-handed coordinate system



left-handed orientation
(clockwise orientation)

negative orientation in a right-handed coordinate system

Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

$$\sigma_x \sigma_y = -\sigma_y \sigma_x$$

$$\sigma_y \sigma_z = -\sigma_z \sigma_y$$

$$\sigma_z \sigma_x = -\sigma_x \sigma_z$$

Scalars

Scalars are geometric entities without direction. They can be expressed as a multiple of the base scalar:

$$k = k\,1$$

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

$$r = x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

$$A = A_{xy}\,\sigma_x\sigma_y + A_{yz}\,\sigma_y\sigma_z + A_{zx}\,\sigma_z\sigma_x$$

Trivectors

Trivectors are oriented volume elements. They can be expressed as a multiple of the base trivector:

$$V = V_{xyz}\,\sigma_x\sigma_y\sigma_z$$

Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$a\,b = a \bullet b + a \wedge b$$

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

$$a \bullet b = b \bullet a = \frac{1}{2}(a\,b + b\,a)$$

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

$$a \wedge b = -\,b \wedge a = \frac{1}{2}(a\,b - b\,a)$$

Geometric Multiplication of Vectors and Bivectors

The product of a vector r and a bivectors A consists of a vector term and a trivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

$$r\,A = r \bullet A + r \wedge A$$

or

$$A\,r = A \bullet r + A \wedge r$$

In this case, the inner products $r \bullet A$ or $A \bullet r$ are anti-commutative, while the outer products $r \wedge A$ or $A \wedge r$ are commutative:

$$r \bullet A = -\,A \bullet r = \frac{1}{2}\,(r\,A - A\,r)$$

$$r \wedge A = \;\; A \wedge r = \frac{1}{2}\,(r\,A + A\,r)$$

Associativity

The geometric product of three vectors

$$(a\ b)\ c = a\ (b\ c) = a\ b\ c$$

and the outer product of three vectors

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b \wedge c$$

are associative.

By the way: The pure inner product of three vectors is zero (and therefore again associative) as spaces with dimensions smaller than zero are not supposed to exist:

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) = 0$$

Systems of Two Linear Equations

$$\begin{aligned} a_1\, x + b_1\, y &= d_1 \\ a_2\, x + b_2\, y &= d_2 \end{aligned} \quad \Rightarrow \quad a\, x + b\, y = d$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$(a_1\, \sigma_x + a_2\, \sigma_y)\, x + (b_1\, \sigma_x + b_2\, \sigma_y)\, y = d_1\, \sigma_x + d_2\, \sigma_y$$

Solutions:

$$x = \frac{1}{a \wedge b}(d \wedge b) = (a \wedge b)^{-1}\,(d \wedge b)$$

$$y = \frac{1}{a \wedge b}\,(a \wedge d) = (a \wedge b)^{-1}\,(a \wedge d)$$

Systems of Three Linear Equations

$$\begin{aligned} a_1\, x + b_1\, y + c_1\, z &= d_1 \\ a_2\, x + b_2\, y + c_2\, z &= d_2 \quad \Rightarrow \quad a\, x + b\, y + c\, z = d \\ a_3\, x + b_3\, y + c_3\, z &= d_3 \end{aligned}$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Modern Geometric Algebra picture:

$$\begin{aligned} &(a_1\, \sigma_x + a_2 \sigma_y + a_3\, \sigma_z)\, x + (b_1\, \sigma_x + b_2\, \sigma_y + b_3\, \sigma_z)\, y \\ &\quad + (c_1\, \sigma_x + c_2\, \sigma_y + c_3\, \sigma_z)\, z = d_1\, \sigma_x + d_2\, \sigma_y + d_3\, \sigma_z \end{aligned}$$

Solutions:

$$x = (a \wedge b \wedge c)^{-1}\, (b \wedge c \wedge d)$$

$$y = (a \wedge b \wedge c)^{-1}\, (c \wedge a \wedge d)$$

$$z = (a \wedge b \wedge c)^{-1}\, (a \wedge b \wedge d)$$

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of the first and second parts.

Systems of Four Linear Equations

To avoid confusion, the coordinates now will be called x_1, x_2, x_3, and x_4 (instead of x, y, z, ...).

$$a_1\, x_1 + b_1\, x_2 + c_1\, x_3 + d_1\, x_4 = t_1$$
$$a_2\, x_1 + b_2\, x_2 + c_2\, x_3 + d_2\, x_4 = t_2$$
$$a_3\, x_1 + b_3\, x_2 + c_3\, x_3 + d_3\, x_4 = t_3$$
$$a_4\, x_1 + b_4\, x_2 + c_4\, x_3 + d_4\, x_4 = t_4$$

$$\Rightarrow \quad a\, x_1 + b\, x_2 + c\, x_3 + d\, x_4 = t$$

Old column vector picture:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \quad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \quad t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$$

Modern Geometric Algebra picture: ?

Problem: Pauli Algebra only has three base vectors σ_x, σ_y, σ_z. But four base vectors are required to construct the coefficient vectors a, b, c, d, & t.

A Closer Analysis of the Problem

Pauli matrices are 2 x 2 matrices. Therefore Pauli matrices or products of Pauli matrices have four elements, which can be complex.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_x\sigma_y = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \sigma_y\sigma_z = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad \sigma_z\sigma_x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_x\sigma_y\sigma_z = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

$\Rightarrow$ Only 4 x 2 = 8 linear independent 2 x 2 matrices exist.

number of elements

two alternatives: real or imaginary

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(1 + \sigma_z) \qquad \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma_x\sigma_y\sigma_z + \sigma_x\sigma_y)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma_x + \sigma_z\sigma_x) \qquad \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma_y\sigma_z - \sigma_y)$$

etc…

To construct the coefficient vectors a, b, c, d, and t of a system of four linear equations, higher-dimensional matrices are required.

Dirac has defined such 4 x 4 matrices. Consequently they are called Dirac matrices.

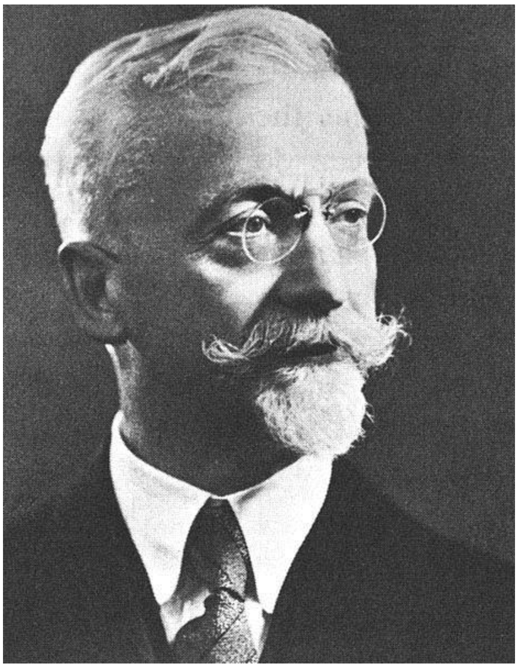
Wolfgang Pauli
(1900 – 1958)

P. A. M. Dirac
(1902 – 1984)

Dirac matrices are an important mathematical tool in physics. Usually they are symbolized by Greek gammas:

$$\gamma_x \qquad \gamma_y \qquad \gamma_z \qquad \gamma_t$$

(if spatial Dirac matrices square to $-\mathbf{1}$)



Élie Cartan
(1869 – 1951)

In this lesson we will first use another set of Dirac matrices originally constructed by Cartan.

→ **See his very important book:**

Élie Cartan: The Theory of Spinors. Unabridged republication of the complete English translation first published in 1966 (New York: Dover Publications, New York 1981.

Therefore we will call our 4 x 4 matrices

$$\sigma_1 \qquad \sigma_2 \qquad \sigma_3 \qquad \sigma_4$$

to indicate that they all square to one:

$$\sigma_1^{\,2} = \sigma_2^{\,2} = \sigma_3^{\,2} = \sigma_4^{\,2} = \mathbf{1}$$

The Direct Product

Dirac matrices are generalizations of Pauli matrices.

In a first step we will construct three 4 x 4 matrices σ_1, σ_2, σ_3 which behave mathematically totally identical to the three 2 x 2 Pauli matrices σ_x, σ_y, σ_z we have met in the first part of this GA lecture series.

This can be done with the direct product, Zehfuss invented in 1858 and Kronecker reinvented at around 1883.

→ **Literature:**

Harold V. Henderson, Friedrich Pukelsheim, Shayle R Searle: On the History of the Kronecker Product. Linear and Multilinear Algebra, Vol. 14, No. 2, 1983, pp. 113 – 120.

The direct product transforms lower-dimensional matrices into higher-dimensional matrices.

Definition of the Direct Product

Let A be an m x n matrix and let B be a p x q matrix. Then the direct product of A and B is the mp x nq matrix $A \otimes B$:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

Examples: $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\sigma_x \otimes \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \sigma_z = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\sigma_z \otimes \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \sigma_x = \begin{bmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Obviously, the direct product is not commutative.

Connection of Direct Multiplication and Matrix Multiplication

The matrix product of two direct products

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \text{ and } C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D \\ c_{21}D & c_{22}D \end{bmatrix}$$

can be found by straightforward calculation with the Falk scheme:

	$c_{11}D$ $c_{21}D$	$c_{12}D$ $c_{22}D$
$a_{11}B \quad a_{12}B$	$(a_{11}c_{11} + a_{12}c_{21})\,BD$	$(a_{11}c_{12} + a_{12}c_{22})\,BD$
$a_{21}B \quad a_{22}B$	$(a_{21}c_{11} + a_{22}c_{21})\,BD$	$(a_{21}c_{12} + a_{22}c_{22})\,BD$

As $$A\,C = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

it follows that

$$(A \otimes B)\,(C \otimes D) = (A\,C) \otimes (B\,D)$$

It can be shown that this rule is valid for higher-dimensional matrices too, if $A\,C$ and $B\,D$ exist.

Connection of Direct Multiplication and Matrix Multiplication

If $A\,C$ and $B\,D$ exist (i.e. the number of columns of the lead matrix equals to the number of rows of the lag matrix), the matrix product of two direct products can be simplified into:

$$(A \otimes B)\,(C \otimes D) = (A\,C) \otimes (B\,D)$$

The inverse of the direct product $A \otimes B$ can then be identified as

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

because

$$(A \otimes B)\,(A \otimes B)^{-1} = \overbrace{(A\,A^{-1})}^{1} \otimes \overbrace{(B\,B^{-1})}^{1} = \mathbf{1}$$

$$\updownarrow \qquad \updownarrow$$

$$(A \otimes B)\,(C \otimes D) = (A\,C) \otimes (B\,D)$$

→ **More about the mathematics of the direct product can be found in:**

Willi-Hans Steeb: Kronecker Product of Matrices and Applications. B. I. Wissenschaftsverlag, Mannheim, Wien, Zürich 1991.

4 x 4 Pauli Matrices

Cartan has chosen some 4 x 4 matrices as "matrices associated with the basis vectors". The first three of his matrices are:

$\sigma_1 = \sigma_z \otimes \sigma_x$

$\sigma_2 = -\sigma_z \otimes \sigma_y$

$\sigma_3 = \sigma_x \otimes 1$ (see p. 133 of his book)

All these matrices square to one:

$$\sigma_1^2 = (\sigma_z \otimes \sigma_x)^2 = (\sigma_z \otimes \sigma_x)(\sigma_z \otimes \sigma_x) = \sigma_z^2 \otimes \sigma_x^2 = 1 \otimes 1 = \mathbf{1}$$

$$\sigma_2^2 = (-\sigma_z \otimes \sigma_y)^2 = (-\sigma_z \otimes \sigma_y)(-\sigma_z \otimes \sigma_y) = \sigma_z^2 \otimes \sigma_y^2 = 1 \otimes 1 = \mathbf{1}$$

$$\sigma_3^2 = (\sigma_x \otimes 1)^2 = (\sigma_x \otimes 1)(\sigma_x \otimes 1) = \sigma_x^2 \otimes 1^2 = 1 \otimes 1 = \mathbf{1}$$

2 x 2 identity matrix

4 x 4 identity matrix

4 x 4 Pauli Matrices

And these three matrices anticommute:

$$\sigma_1\sigma_2 = (\sigma_z \otimes \sigma_x)\,(-\,\sigma_z \otimes \sigma_y)$$
$$= -\,\sigma_z^{\,2} \otimes (\sigma_x\sigma_y) = -\,1 \otimes (\sigma_x\sigma_y)$$

$$\sigma_2\sigma_1 = (-\,\sigma_z \otimes \sigma_y)\,(\sigma_z \otimes \sigma_x)$$
$$= -\,\sigma_z^{\,2} \otimes (\sigma_y\sigma_x) = 1 \otimes (\sigma_x\sigma_y)$$

$$\Rightarrow \qquad \sigma_1\sigma_2 = -\,\sigma_2\sigma_1$$

$$\sigma_2\sigma_3 = (-\,\sigma_z \otimes \sigma_y)\,(\sigma_x \otimes 1)$$
$$= (-\,\sigma_z\sigma_x) \otimes (\sigma_y\, 1) = -\,(\sigma_z\sigma_x) \otimes \sigma_y$$

$$\sigma_3\sigma_2 = (\sigma_x \otimes 1)\,(-\,\sigma_z \otimes \sigma_y)$$
$$= (-\,\sigma_x\sigma_z) \otimes (1\, \sigma_y) = (\sigma_z\sigma_x) \otimes \sigma_y$$

$$\Rightarrow \qquad \sigma_2\sigma_3 = -\,\sigma_3\sigma_2$$

And equivalently

$$\Rightarrow \qquad \sigma_3\sigma_1 = -\,\sigma_1\sigma_3$$

4 x 4 Pauli Matrices

The following three 4 x 4 Dirac matrices

$$\sigma_1 = \sigma_z \otimes \sigma_x = \begin{bmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sigma_2 = -\sigma_z \otimes \sigma_y = \begin{bmatrix} -\sigma_y & 0 \\ 0 & \sigma_y \end{bmatrix} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$\sigma_3 = \sigma_x \otimes 1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

obey Pauli algebra:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbf{1}$$

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 \qquad \sigma_2\sigma_3 = -\sigma_3\sigma_2 \qquad \sigma_3\sigma_1 = -\sigma_1\sigma_3$$

Therefore they can be considered as 4 x 4 matrix representations of Pauli matrices. Thus they represent three spatial base vectors.

Defining a Fourth 4 x 4 Base Vector

To construct the coefficient vectors a, b, c, d, and t of the four-dimensional system of linear equations, a fourth 4 x 4 base vector is required. It can be defined as

$$\sigma_4 = \sigma_y \otimes 1$$

This vector again squares to one:

$$\sigma_4^{\,2} = (\sigma_y \otimes 1)^2 = (\sigma_y \otimes 1)\,(\sigma_y \otimes 1)$$
$$= \sigma_y^{\,2} \otimes 1^2 = 1 \otimes 1 = \mathbf{1}$$

And it anticommutes with the other three base vectors:

$$\sigma_4\sigma_1 = -\,\sigma_1\sigma_4$$

$$\sigma_4\sigma_2 = -\,\sigma_2\sigma_4$$

$$\sigma_4\sigma_3 = -\,\sigma_3\sigma_4$$

Consequently, the four 4 x 4 Dirac matrixes σ_1, σ_2, σ_3, and σ_4 are representations of base vectors of four-dimensional space.

Summary: Basics of Geometric Algebra in four-dimensional space

$1 + 4 + 6 + 4 + 1 = 2^4 = 16$ different base elements exist in four-dimensional space.

One base scalar: $\mathbf{1}$

Four base vectors: $\sigma_1, \sigma_2, \sigma_3, \sigma_4$

Six base bivectors: $\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_4, \sigma_4\sigma_1, \sigma_2\sigma_4, \sigma_4\sigma_3$
(or pseudobivectors if you like)

Four base trivectors: $\sigma_1\sigma_2\sigma_3, \sigma_2\sigma_3\sigma_4, \sigma_3\sigma_4\sigma_1, \sigma_4\sigma_1\sigma_2$
(sometimes called pseudovectors)

One base quadrovector: $\sigma_1\sigma_2\sigma_3\sigma_4$
(sometimes called pseudoscalar)

Base scalar, base vectors and base quadrovector square to one:

$$\mathbf{1}^2 = \sigma_1^{\ 2} = \sigma_2^{\ 2} = \sigma_3^{\ 2} = \sigma_4^{\ 2} = (\sigma_1\sigma_2\sigma_3\sigma_4)^2 = \mathbf{1}$$

Base bivectors and base trivectors square to minus one:

$$(\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_3\sigma_4)^2 = (\sigma_4\sigma_1)^2 = (\sigma_2\sigma_4)^2 = (\sigma_4\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3\sigma_4)^2 = (\sigma_3\sigma_4\sigma_1)^2 = (\sigma_4\sigma_1\sigma_2)^2 = -\mathbf{1}$$

Outer Products of Four Vectors

Remember former lessons (see repetition):

$$a\,b = a \bullet b + a \wedge b$$

Scalar: $$a \bullet b = \frac{1}{2}(a\,b + b\,a)$$

Bivector: $$a \wedge b = \frac{1}{2}(a\,b - b\,a)$$

$$(a \wedge b)\,c = (a \wedge b) \bullet c + (a \wedge b) \wedge c$$

Vector: $$(a \wedge b) \bullet c = \frac{1}{2}\big((a \wedge b)\,c - c\,(a \wedge b)\big)$$

Trivector: $$a \wedge b \wedge c = \frac{1}{2}\big((a \wedge b)\,c + c\,(a \wedge b)\big)$$

$$(a \wedge b \wedge c)\,d = (a \wedge b \wedge c) \bullet d + (a \wedge b \wedge c) \wedge d$$

Bivector:

$$(a \wedge b \wedge c) \bullet d = \frac{1}{2}\big((a \wedge b \wedge c)\,d + d\,(a \wedge b \wedge c)\big)$$

Quadrovector (or quadvector, grade-4 vector):

$$a \wedge b \wedge c \wedge d = \frac{1}{2}\big((a \wedge b \wedge c)\,d - d\,(a \wedge b \wedge c)\big)$$

The outer product of four vectors is an object of grade 4. And it is an associative product.

Systems of Four Linear Equations

Now we are able to translate the column vectors of a system of four linear equations

$$a_1\, x_1 + b_1\, x_2 + c_1\, x_3 + d_1\, x_4 = t_1$$

$$a_2\, x_1 + b_2\, x_2 + c_2\, x_3 + d_2\, x_4 = t_2$$

$$a_3\, x_1 + b_3\, x_2 + c_3\, x_3 + d_3\, x_4 = t_3$$

$$a_4\, x_1 + b_4\, x_2 + c_4\, x_3 + d_4\, x_4 = t_4$$

into the language of Geometric Algebra.

Coefficient vectors:

$$a = a_1\, \sigma_1 + a_2\, \sigma_2 + a_3\, \sigma_3 + a_4\, \sigma_4$$

$$b = b_1\, \sigma_1 + b_2\, \sigma_2 + b_3\, \sigma_3 + b_4\, \sigma_4$$

$$c = c_1\, \sigma_1 + c_2\, \sigma_2 + c_3\, \sigma_3 + c_4\, \sigma_4$$

$$d = d_1\, \sigma_1 + d_2\, \sigma_2 + d_3\, \sigma_3 + d_4\, \sigma_4$$

Vector of constant terms:

$$t = t_1\, \sigma_1 + t_2\, \sigma_2 + t_3\, \sigma_3 + t_4\, \sigma_4$$

$$\Rightarrow \quad a\, x_1 + b\, x_2 + c\, x_3 + d\, x_4 = t$$

Solving a System of Four Linear Equations in Geometric Algebra

$$a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4 = t$$

First Step:
Wedge product with vector b to get rid of $b\,x_2$:

$$(a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4) \wedge b = t \wedge b$$

$$(a \wedge b)\,x_1 + \underbrace{(b \wedge b)}\,x_2 + (c \wedge b)\,x_3 + (d \wedge b)\,x_4 = t \wedge b$$

$$(a \wedge b)\,x_1 + \quad 0 \quad + (c \wedge b)\,x_3 + (d \wedge b)\,x_4 = t \wedge b$$

Second Step:
Wedge product with vector c to get rid of $c\,x_3$:

$$\big((a \wedge b)\,x_1 + (c \wedge b)\,x_3 + (d \wedge b)\,x_4\big) \wedge c = t \wedge b \wedge c$$

$$(a \wedge b \wedge c)\,x_1 + \underbrace{(c \wedge b \wedge c)}\,x_3 + (d \wedge b \wedge c)\,x_4 = t \wedge b \wedge c$$

$$(a \wedge b \wedge c)\,x_1 + \quad 0 \quad + (d \wedge b \wedge c)\,x_4 = t \wedge b \wedge c$$

Third Step:
Wedge product with vector d to get rid of $d\,x_4$:

$$\big((a \wedge b \wedge c)\,x_1 + (d \wedge b \wedge c)\,x_4\big) \wedge d = t \wedge b \wedge c \wedge d$$

$$(a \wedge b \wedge c \wedge d)\,x_1 + \underbrace{(d \wedge b \wedge c \wedge d)}\,x_4 = t \wedge b \wedge c \wedge d$$

$$(a \wedge b \wedge c \wedge d)\,x_1 + \quad 0 \quad = t \wedge b \wedge c \wedge d$$

Solving a System of Four Linear Equations in Geometric Algebra

To solve the system of four linear equations

$$a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4 = t$$

we finally get the four wedge product equations

$$(a \wedge b \wedge c \wedge d)\,x_1 = t \wedge b \wedge c \wedge d$$

$$(a \wedge b \wedge c \wedge d)\,x_2 = a \wedge t \wedge c \wedge d$$

$$(a \wedge b \wedge c \wedge d)\,x_3 = a \wedge b \wedge t \wedge d$$

$$(a \wedge b \wedge c \wedge d)\,x_4 = a \wedge b \wedge c \wedge t$$

Thus the solutions are:

$$x_1 = (a \wedge b \wedge c \wedge d)^{-1}\,(t \wedge b \wedge c \wedge d)$$

$$x_2 = (a \wedge b \wedge c \wedge d)^{-1}\,(a \wedge t \wedge c \wedge d)$$

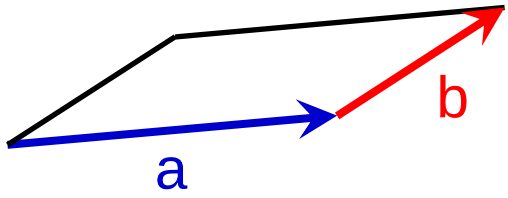
$$x_3 = (a \wedge b \wedge c \wedge d)^{-1}\,(a \wedge b \wedge t \wedge d)$$

$$x_4 = (a \wedge b \wedge c \wedge d)^{-1}\,(a \wedge b \wedge c \wedge t)$$

Geometric Interpretation of the Result

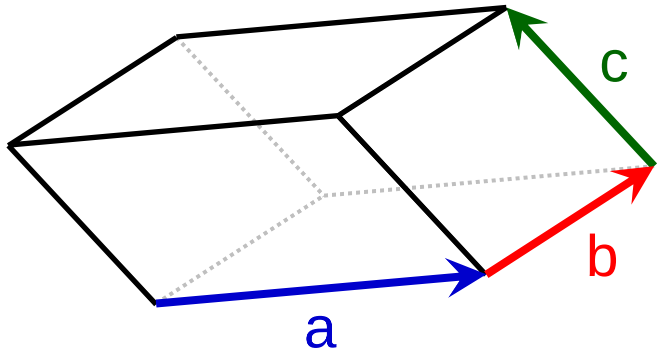
- The product R of two vectors a and b represents an oriented parallelogram.

  R = a b



- The product P of three vectors a, b, c represents an oriented parallelepiped.

  P = a b c

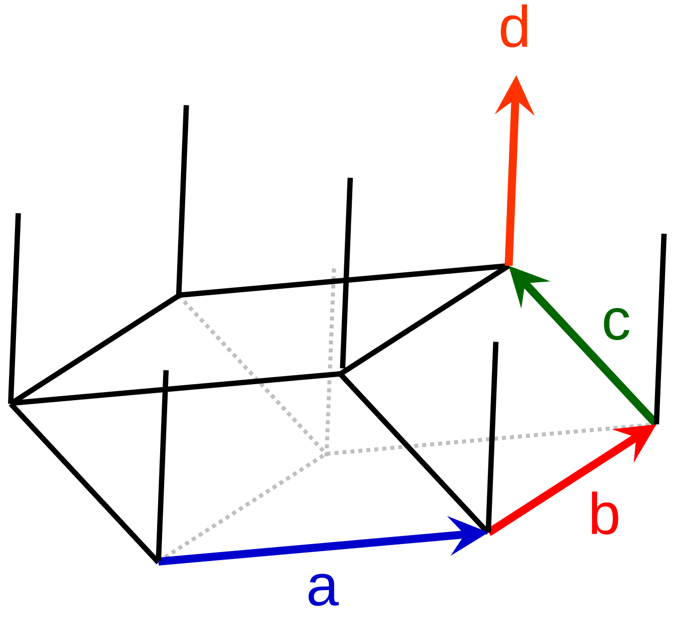


- The product Q of four vectors a, b, c, d represents an oriented, four-dimensional hyper-parallelepiped, which can be visualized best in four-dimensional space.

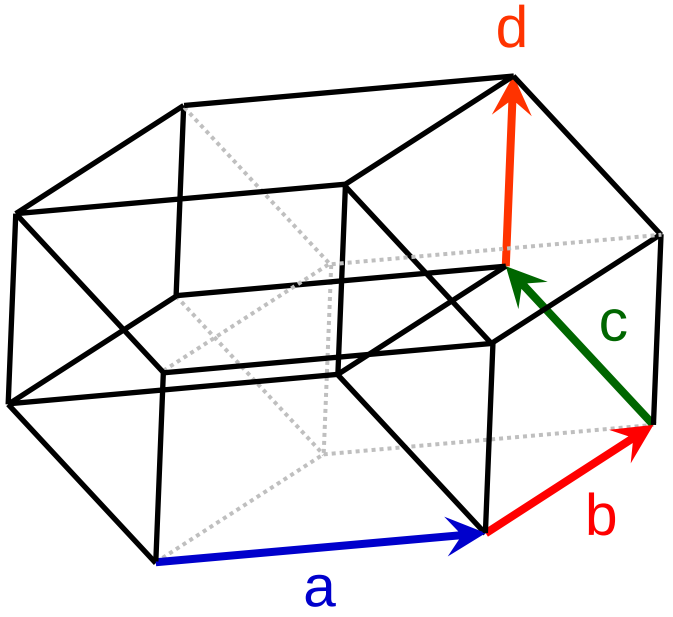
  Q = a b c d

Four-dimensional Hyper-Parallelepipeds

Simply move the eight corners of the three-dimensional parallelepiped along a vector which points into the direction of the forth dimension …



… to get an hyper-parallelepiped with now 8 x 2 = 16 corners.



Four-dimensional Hyper-Parallelepipeds

	Corners	Edges	Faces	Solids	Hyper-volumes
Points	1	0	0	0	0
Line Segments	2	1	0	0	0
Parallelograms	4	4	1	0	0
Parallelepipeds	8	12	6	1	0
Hyper-Parallelepipeds	16	32	24	8	1

→ **More about visualizing higher-dimensional figures can be found in the well-written, popular scientific book:**

Clifford A. Pickover: Surfing Through Hyperspace. Understanding Higher Universes in Six Easy Lessons. Oxford University Press, Oxford, New York 1999.

Geometric Interpretation of the Result

- The product of two vectors a and b represents an oriented parallelogram.
  The outer product of these two vectors $a \wedge b$ represents the area (i.e. the two-dimensional "volume") of this parallelogram.

- The product of three vectors a, b, and c represents an oriented parallelepiped.
  The outer product of these three vectors $a \wedge b \wedge c$ represents the volume of this parallelepiped.

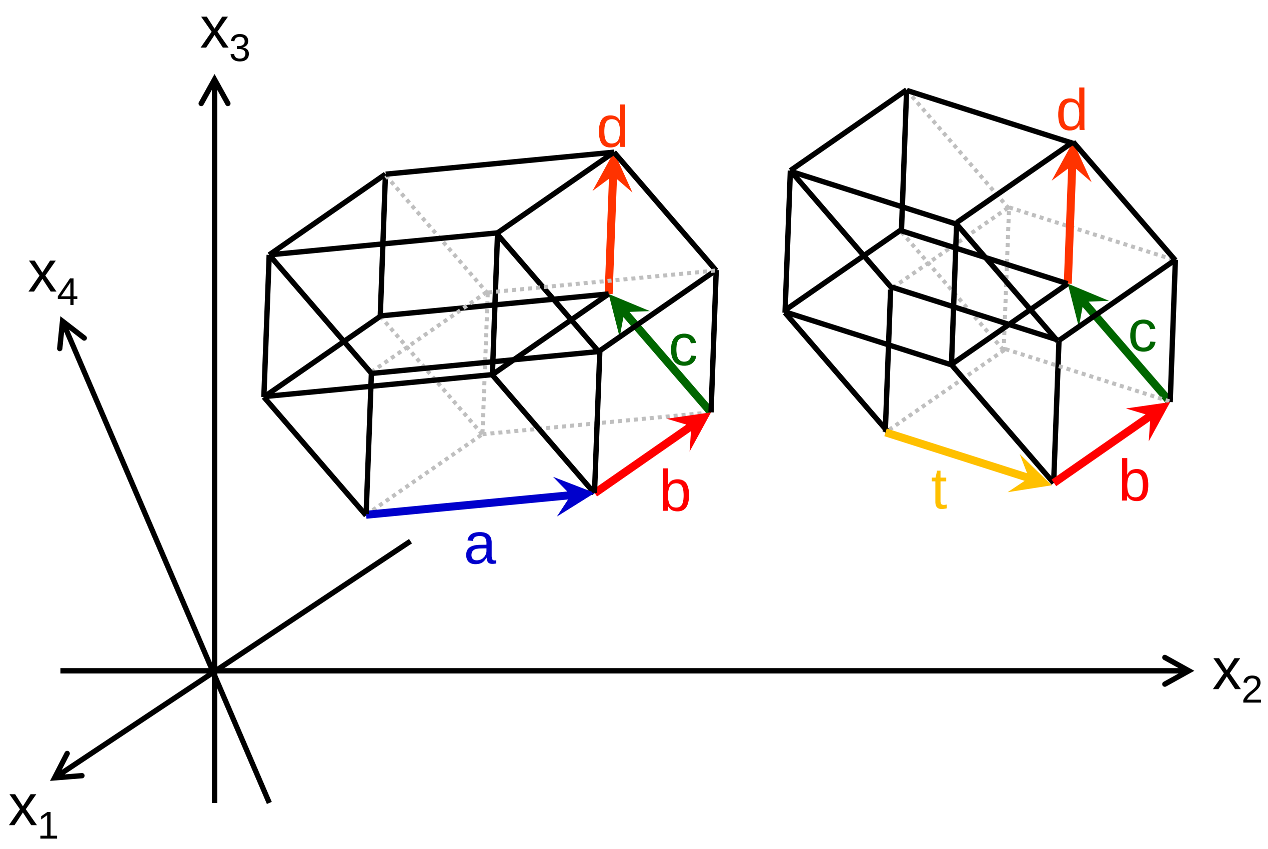
- The product of four vectors a, b, c, and d represents an oriented, four-dimensional hyper-parallelepiped.
  The outer product of these four vectors $a \wedge b \wedge c \wedge d$ represents the four-dimensional hyper-volume of this hyper-parallelepiped.

Geometric Interpretation of the Result

The equation

$$(a \wedge b \wedge c \wedge d)\, x_1 = t \wedge b \wedge c \wedge d$$

just says that we have to compare the hyper-volumes of the hyper-parallelepipeds a b c d, which represents the determinant of the co-efficient matrix, with the hyper-volume of the hyper-parallelepiped t b c d to find the value of x_1.



Please mind the four orthogonal axes!

Finding the Inverse of 4 x 4 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

To find the inverse A^{-1} of a 4 x 4 matrix A, the Falk scheme of matrix multiplication can now be split into four parts.

$$\begin{array}{cccc|cccc}
 & & & & b_{11} & b_{12} & b_{13} & b_{14} \\
 & & & & b_{21} & b_{22} & b_{23} & b_{24} \\
 & & & & b_{31} & b_{32} & b_{33} & b_{34} \\
 & & & & b_{41} & b_{42} & b_{43} & b_{44} \\
\hline
a_{11} & a_{12} & a_{13} & a_{14} & 1 & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & a_{24} & 0 & 1 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 1 & 0 \\
a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 1 & 0
\end{array}$$

Thus we get four systems of four linear equations.

The first (blue) part results in the following system of linear equations given:

$$a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} + a_{14}\, b_{41} = 1$$

$$a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31} + a_{24}\, b_{41} = 0$$

$$a_{31}\, b_{11} + a_{32}\, b_{21} + a_{33}\, b_{31} + a_{34}\, b_{41} = 0$$

$$a_{41}\, b_{11} + a_{42}\, b_{21} + a_{43}\, b_{31} + a_{44}\, b_{41} = 0$$

As usual, this results in the following coefficient vectors:

$$a = a_1\, \sigma_1 + a_2\, \sigma_2 + a_3\, \sigma_3 + a_4\, \sigma_4$$

$$b = b_1\, \sigma_1 + b_2\, \sigma_2 + b_3\, \sigma_3 + b_4\, \sigma_4$$

$$c = c_1\, \sigma_1 + c_2\, \sigma_2 + c_3\, \sigma_3 + c_4\, \sigma_4$$

$$d = d_1\, \sigma_1 + d_2\, \sigma_2 + d_3\, \sigma_3 + d_4\, \sigma_4$$

And now we get four different vectors of constant terms:

$$t_{blue} = \sigma_1$$
$$t_{green} = \sigma_2$$
$$t_{red} = \sigma_3$$
$$t_{orange} = \sigma_4$$

Finding the Inverse of 4 x 4 Matrices

$\Rightarrow$ Elements of the inverse matrix A^{-1}:

$$b_{11} = (a \wedge b \wedge c \wedge d)^{-1} (\sigma_1 \wedge b \wedge c \wedge d)$$
$$b_{21} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge \sigma_1 \wedge c \wedge d)$$
$$b_{31} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge b \wedge \sigma_1 \wedge d)$$
$$b_{41} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge b \wedge c \wedge \sigma_1)$$

$$b_{12} = (a \wedge b \wedge c \wedge d)^{-1} (\sigma_2 \wedge b \wedge c \wedge d)$$
$$b_{22} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge \sigma_2 \wedge c \wedge d)$$
$$b_{32} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge b \wedge \sigma_2 \wedge d)$$
$$b_{42} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge b \wedge c \wedge \sigma_2)$$

$$b_{13} = (a \wedge b \wedge c \wedge d)^{-1} (\sigma_3 \wedge b \wedge c \wedge d)$$
$$b_{23} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge \sigma_3 \wedge c \wedge d)$$
$$b_{33} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge b \wedge \sigma_3 \wedge d)$$
$$b_{43} = (a \wedge b \wedge c \wedge d)^{-1} (a \wedge b \wedge c \wedge \sigma_3)$$

$b_{14} = \ldots$ etc.

Systems of Five Linear Equations

In a straightforward way all this can be generalized for systems of five linear equations:

$$a_1\, x_1 + b_1\, x_2 + c_1\, x_3 + d_1\, x_4 + e_1\, x_5 = t_1$$

$$a_2\, x_1 + b_2\, x_2 + c_2\, x_3 + d_2\, x_4 + e_2\, x_5 = t_2$$

$$a_3\, x_1 + b_3\, x_2 + c_3\, x_3 + d_3\, x_4 + e_3\, x_5 = t_3$$

$$a_4\, x_1 + b_4\, x_2 + c_4\, x_3 + d_4\, x_4 + e_4\, x_5 = t_4$$

$$a_5\, x_1 + b_5\, x_2 + c_5\, x_3 + d_5\, x_4 + e_5\, x_5 = t_5$$

We only have to find a fifth base vector σ_5 to get the required five coefficient vectors

$$a = a_1\, \sigma_1 + a_2\, \sigma_2 + a_3\, \sigma_3 + a_4\, \sigma_4 + a_5\, \sigma_5$$

$$b = b_1\, \sigma_1 + b_2\, \sigma_2 + b_3\, \sigma_3 + b_4\, \sigma_4 + b_5\, \sigma_5$$

$$c = c_1\, \sigma_1 + c_2\, \sigma_2 + c_3\, \sigma_3 + c_4\, \sigma_4 + c_5\, \sigma_5$$

$$d = d_1\, \sigma_1 + d_2\, \sigma_2 + d_3\, \sigma_3 + d_4\, \sigma_4 + d_5\, \sigma_5$$

$$e = e_1\, \sigma_1 + e_2\, \sigma_2 + e_3\, \sigma_3 + e_4\, \sigma_4 + e_5\, \sigma_5$$

and the vector of constant terms

$$t = t_1\, \sigma_1 + t_2\, \sigma_2 + t_3\, \sigma_3 + t_4\, \sigma_4 + t_5\, \sigma_5$$

Solving a System of Five Linear Equations in Geometric Algebra

$$a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4 + e\,x_5 = t$$

First Step:
Wedge product with vector b to get rid of $b\,x_2$:

$$(a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4 + e\,x_5) \wedge b = t \wedge b$$

$$(a \wedge b)\,x_1 + (c \wedge b)\,x_3 + (d \wedge b)\,x_4 + (e \wedge b)\,x_5 = t \wedge b$$

Second Step:
Wedge product with vector c to get rid of $c\,x_3$:

$$\big((a \wedge b)\,x_1 + (c \wedge b)\,x_3 + (d \wedge b)\,x_4 + (e \wedge b)\,x_5\big) \wedge c = t \wedge b \wedge c$$

$$(a \wedge b \wedge c)\,x_1 + (d \wedge b \wedge c)\,x_4 + (e \wedge b \wedge c)\,x_5 = t \wedge b \wedge c$$

Third Step:
Wedge product with vector d to get rid of $d\,x_4$:

$$\big((a \wedge b \wedge c)\,x_1 + (d \wedge b \wedge c)\,x_4 + (e \wedge b \wedge c)\,x_5\big) \wedge d = t \wedge b \wedge c \wedge d$$

$$(a \wedge b \wedge c \wedge d)\,x_1 + (e \wedge b \wedge c \wedge d)\,x_5 = t \wedge b \wedge c \wedge d$$

Fourth Step:
Wedge product with vector e to get rid of $e\,x_5$:

$$\big((a \wedge b \wedge c \wedge d)\,x_1 + (e \wedge b \wedge c \wedge d)\,x_5\big) \wedge e = t \wedge b \wedge c \wedge d \wedge e$$

$$(a \wedge b \wedge c \wedge d \wedge e)\,x_1 = t \wedge b \wedge c \wedge d \wedge e$$

Solving a System of Four Linear Equations in Geometric Algebra

To solve the system of five linear equations

$$a\, x_1 + b\, x_2 + c\, x_3 + d\, x_4 + e\, x_5 = t$$

we finally get five wedge product equations:

$$(a \wedge b \wedge c \wedge d \wedge e)\, x_1 = t \wedge b \wedge c \wedge d \wedge e$$

$$(a \wedge b \wedge c \wedge d \wedge e)\, x_2 = a \wedge t \wedge c \wedge d \wedge e$$

$$(a \wedge b \wedge c \wedge d \wedge e)\, x_3 = a \wedge b \wedge t \wedge d \wedge e$$

$$(a \wedge b \wedge c \wedge d \wedge e)\, x_4 = a \wedge b \wedge c \wedge t \wedge e$$

$$(a \wedge b \wedge c \wedge d \wedge e)\, x_5 = a \wedge b \wedge c \wedge d \wedge t$$

Thus the solutions are:

$$x_1 = (a \wedge b \wedge c \wedge d \wedge e)^{-1}\, (t \wedge b \wedge c \wedge d \wedge e)$$

$$x_2 = (a \wedge b \wedge c \wedge d \wedge e)^{-1}\, (a \wedge t \wedge c \wedge d \wedge e)$$

$$x_3 = (a \wedge b \wedge c \wedge d \wedge e)^{-1}\, (a \wedge b \wedge t \wedge d \wedge e)$$

$$x_4 = (a \wedge b \wedge c \wedge d \wedge e)^{-1}\, (a \wedge b \wedge c \wedge t \wedge e)$$

$$x_5 = (a \wedge b \wedge c \wedge d \wedge e)^{-1}\, (a \wedge b \wedge c \wedge d \wedge t)$$

Systems of Five (or More) Linear Equations: Clifford Algebra

We then have to work with five (or more) base vectors σ_i to get the coefficient vectors and the vector of constant terms:

$$a\, x_1 + b\, x_2 + c\, x_3 + \ldots + g\, x_n = t$$

It is always possible, to construct higher-dimensional square matrices (e.g. 8 x 8 or 16 x 16 matrices, etc.) with the direct product, which represent base vectors:

$$\sigma_i^2 = 1$$

σ_i are unit vectors.

$$\sigma_i\, \sigma_j = -\, \sigma_j\, \sigma_i$$

Different unit vectors anticommute.

These algebraic rules and the geometric interpretation of these rules are important.

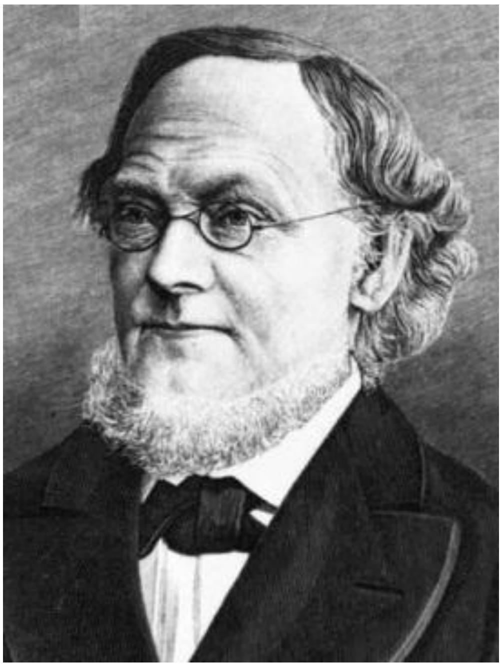
They form the mathematical core of Clifford Algebra, which was invented by Hermann Grassmann and William Kingdon Clifford.

(Positive) Clifford Algebra

$\sigma_i^2 = 1$ — σ_i are unit vectors.

$\sigma_i \, \sigma_j = - \sigma_j \, \sigma_j$ — Different unit vectors anticommute.

These algebraic rules form the mathematical core of Clifford Algebra, which was invented by Hermann Grassmann and William Kingdon Clifford.

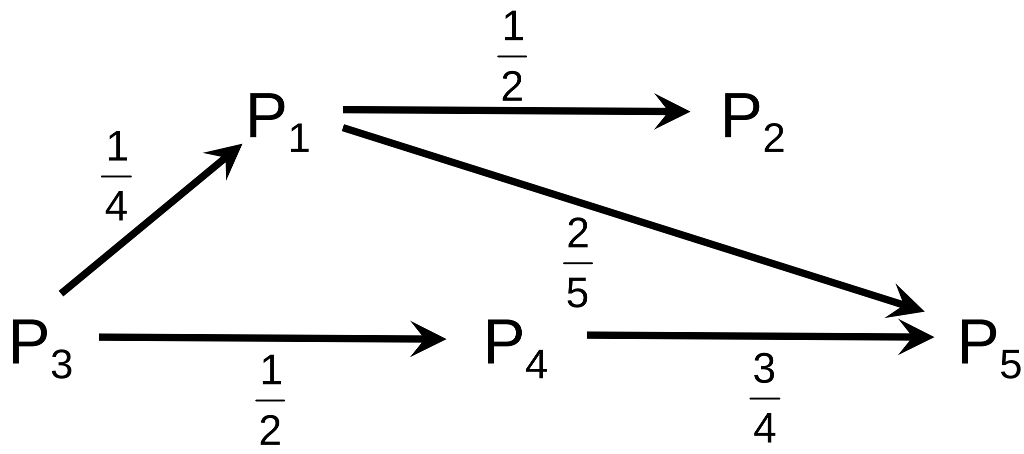


Hermann Günther Graßmann
(1809 – 1877)

William Kingdon Clifford
(1845 – 1879)

Example: Input-Output Analysis

The following flow diagram (gozintograph) shows the demand of products of industries P_1, P_2, P_3, P_4, and P_5 (input measured in units of money) which are required to produce goods of industry P_i exactly worth one unit of money (output).



Find the interindustry transaction demand table, if final demand is worth 1000 units of money for industry 1, 1200 units of money for industry 2, 1400 units of money for industry 3, 1600 units of money for industry 4, and 1800 units of money for industry 5.

Explain, why the value added of industry 5 should be negative.

Step 1: Input-Output Analysis

Finding the matrix of technical coefficients:

Goods worth 0.25 units of money of industry 3 are required to produce goods worth exactly one unit of money of industry 1. $\Rightarrow$ $a_{31} = 0.25$

Goods worth 0.50 units of money of industry 1 are required to produce goods worth exactly one unit of money of industry 2. $\Rightarrow$ $a_{12} = 0.50$

Goods worth 0.50 units of money of industry 4 are required to produce goods worth exactly one unit of money of industry 4. $\Rightarrow$ $a_{34} = 0.50$

Goods worth 0.40 units of money of industry 1 and 0.75 units of money of industry 4 are required to produce goods worth exactly one unit of money of industry 5. $\Rightarrow$ $a_{15} = 0.40$

$\Rightarrow$ $a_{45} = 0.75$

The matrix of technical coefficient equals

$$A = \begin{pmatrix} 0 & 0.50 & 0 & 0 & 0.40 \\ 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2: Input-Output Analysis

Finding the Leontief matrix:

$$\mathrm{I} - A = \begin{pmatrix} 1 & -0.50 & 0 & 0 & -0.40 \\ 0 & 1 & 0 & 0 & 0 \\ -0.25 & 0 & 1 & -0.50 & 0 \\ 0 & 0 & 0 & 1 & -0.75 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Finding the coefficient vectors

$a = \sigma_1 - 0.25\,\sigma_3$

$b = -\,0.50\,\sigma_1 + \sigma_2$

$c = \sigma_3$

$d = -\,0.50\,\sigma_3 + \sigma_4$

$e = -\,0.40\,\sigma_1 - 0.75\,\sigma_4 + \sigma_5$

and the vector of constant terms

$$t = 1000\,\sigma_1 + 1200\,\sigma_2 + 1400\,\sigma_3 + 1600\,\sigma_4 + 1800\,\sigma_5$$

Step 3: Input-Output Analysis

Solving the input-output equation

$$(\mathrm{I} - a)\, x = d$$

for total demand x.

Wedge products:

$$a \wedge b \wedge c \wedge d \wedge e = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

$$\Rightarrow\ (a \wedge b \wedge c \wedge d \wedge e)^{-1} = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

$$t \wedge b \wedge c \wedge d \wedge e = 2320\ \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

$$a \wedge t \wedge c \wedge d \wedge e = 1200\ \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

$$a \wedge b \wedge t \wedge d \wedge e = 3455\ \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

$$a \wedge b \wedge c \wedge t \wedge e = 2950\ \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

$$a \wedge b \wedge c \wedge d \wedge t = 1800\ \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$$

Step 3: Input-Output Analysis

Solving the input-output equation

$$(\mathrm{I} - a)\, x = d$$

for total demand x.

Total demand:

$$x_1 = (a \wedge b \wedge c \wedge d \wedge e)^{-1} (t \wedge b \wedge c \wedge d \wedge e)$$
$$= 2320$$

$$x_2 = (a \wedge b \wedge c \wedge d \wedge e)^{-1} (a \wedge t \wedge c \wedge d \wedge e)$$
$$= 1200$$

$$x_3 = (a \wedge b \wedge c \wedge d \wedge e)^{-1} (a \wedge b \wedge t \wedge d \wedge e)$$
$$= 3455$$

$$x_4 = (a \wedge b \wedge c \wedge d \wedge e)^{-1} (a \wedge b \wedge c \wedge t \wedge e)$$
$$= 2950$$

$$x_5 = (a \wedge b \wedge c \wedge d \wedge e)^{-1} (a \wedge b \wedge c \wedge d \wedge t)$$
$$= 1800$$

Step 4: Input-Output Analysis

Finding the transaction vectors:

$t_1 = x_1\ a_1 = 2320\ (0.25\ \sigma_3) = 580\ \sigma_3$

$t_2 = x_2\ a_2 = 1200\ (0.50\ \sigma_1) = 600\ \sigma_1$

$t_3 = x_3\ a_3 = 0$

$t_4 = x_4\ a_4 = 2950\ (0.50\ \sigma_3) = 1475\ \sigma_3$

$t_5 = x_5\ a_5 = 1800\ (0.40\ \sigma_1 + 0.75\ \sigma_4)$

$\qquad\qquad = 720\ \sigma_1 + 1350\ \sigma_4$

Finding the values added:

$v_1 = x_1 - t_1 \bullet (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5) = 1740$

$v_2 = x_2 - t_2 \bullet (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5) = 600$

$v_3 = x_3 - t_3 \bullet (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5) = 0$

$v_4 = x_4 - t_4 \bullet (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5) = 1475$

$v_5 = x_5 - t_5 \bullet (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_4) = -270$

Step 5: Input-Output Analysis

Interindustry transaction demand table:

Sector of Origin	Sector of Destination 1	2	3	4	5	Final Demand	Total Demand
1	0	600	0	0	720	1000	2320
2	0	0	0	0	0	1200	1200
3	580	0	0	1475	0	1400	3455
4	0	0	0	0	1350	1600	2950
5	0	0	0	0	0	1800	1800
Value Added	1740	600	0	1475	–270		
Gross Production	2320	1200	3455	2950	1800		

Goods worth 0.40 + 0.75 = 1.15 units of money are required to produce goods worth exactly one unit of money of industry 5. Thus industry 5 is not a profitable industry and the value added is negative.

Outlook: Some Physics & Philosophy

Mathematics ⟷ Physics

science of logical systems (constructed by the human mind)

science of nature (given by the world which exists outside of us)

Mathematics is part of the humanities.

(German: Geisteswissenschaft)

Physics is a natural science.

(Naturwissenschaft)

The Relation between Mathematics and Physics

Dirac: "One may describe the situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, …"

Mathematical laws seem to be inventions…

Physical laws seem to be discoveries…

The Relation between Mathematics and Physics

Dirac: "One may describe the situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen."

Mathematical laws seem to be inventions, but whatever we invent in mathematics (provided it is beautiful) will be found in physics as a law of nature one day.

$\rightarrow$ **Literature:**

Paul Adrien Maurice Dirac: The Relation between Mathematics and Physics. Lecture delivered on presentation of the James Scott prize, February 6, 1939. Published in: Proceedings of the Royal Society (Edinburgh), Vol. 59, 1938 – 39, Part II, pp. 122 – 129.

The Relation between Mathematics and Physics

Dirac: “Possibly, the two subjects (physics and mathematics) will ultimately unify, every branch of pure mathematics then having its physical application, its importance in physics being proportional to its interest in mathematics.”

To solve systems of five linear equations, we need a fifth base vector. As said, this base vector can be constructed in a very simple way as an ugly 8 x 8 matrix by using the direct product.

But there exists a very interesting fifth base vector, which can be identified with a 4 x 4 matrix, which is mathematically much more beautiful than a 8 x 8 matrix.

And this new 4 x 4 base vector matrix is of tremendous importance in physics.

Searching for a Fifth Base Vector

$1 + 4 + 6 + 4 + 1 = 2^4 = 16$ different base elements exist in four-dimensional space:

- One base scalar
- Four base vectors
- Six base bivectors
- Four base trivectors
- One base quadrovector

But Dirac matrices are 4 x 4 matrices! Thus $4 \times 4 \times 2 = 32 = 2^5$ linear independent matrices must exist:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \cdots$$

$$\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \cdots$$

Therefore it should be possible, to construct all $1 + 5 + 10 + 10 + 5 + 1 = 2^5 = 32$ different base elements which should exist in five-dimensional space.

Searching for a Fifth Base Vector

$1 + 5 + 10 + 10 + 5 + 1 = 2^5 = 32$ different base elements should exist in five-dimensional space:

$\binom{5}{0}$ One base scalar: $\mathbf{1}$

$\binom{5}{1}$ Five base vectors: $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$

$\binom{5}{2}$ Ten base bivectors: $\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_4,$ etc…
(or pseudotrivectors if you like)

$\binom{5}{3}$ Ten base trivectors: $\sigma_1\sigma_2\sigma_3,\ \sigma_2\sigma_3\sigma_4,$ etc…
(sometimes called pseudobivectors)

$\binom{5}{4}$ Five base quadrovectors: $\sigma_1\sigma_2\sigma_3\sigma_4,$ etc…
(sometimes called pseudovectors)

$\binom{5}{5}$ One base pentavector: $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$
(sometimes called pseudoscalar)

vectors of grade 4

vector of grade 5

To find a fifth base vector, it makes sense to generalize the pseudoscalar of three-dimensional space:

$$\sigma_x\sigma_y\sigma_z = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \quad \rightarrow \quad \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

Searching for a Fifth Base Vector

$$\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} = (\sigma_x\sigma_y\sigma_z) \otimes 1 = i\,(1 \otimes 1)$$

Together with the four base vectors

$$\sigma_1 = \sigma_z \otimes \sigma_x$$
$$\sigma_2 = -\,\sigma_z \otimes \sigma_y$$
$$\sigma_3 = \sigma_x \otimes 1$$
$$\sigma_4 = \sigma_y \otimes 1$$

$$\Rightarrow \quad \sigma_1\sigma_2\sigma_3\sigma_4 = (-\,\sigma_z\,\sigma_z\,\sigma_x\,\sigma_y) \otimes (\sigma_x\,\sigma_y)$$
$$= (-\,i\,\sigma_z) \otimes (i\,\sigma_z) = \sigma_z \otimes \sigma_z$$

$$\Rightarrow \quad (\sigma_1\sigma_2\sigma_3\sigma_4)^{-1} = \sigma_z \otimes \sigma_z$$

a new fifth base vector can be identified as:

$$\sigma_5^{*} = (\sigma_1\sigma_2\sigma_3\sigma_4)^{-1}\,(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)$$
$$= i\,(\sigma_z \otimes \sigma_z)\,(1 \otimes 1)$$
$$= i\,(\sigma_z \otimes \sigma_z)$$

An asterisk is added to indicate, that
this base vector is of different quality.

Searching for a Fifth Base Vector

$$\sigma_5^* = i\,(\sigma_z \otimes \sigma_z) = \begin{bmatrix} i\sigma_z & 0 \\ 0 & -i\sigma_z \end{bmatrix} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

$$\sigma_5^{*2} = -\,(1 \otimes 1) = -\,\mathbf{1}$$

Please compare with

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = +\,\mathbf{1}$$

This new base vector σ_5^* is of totally different quality compared with the other four base vectors σ_1, σ_2, σ_3, σ_4. It squares to minus one.

This is of tremendous importance in physics:

→ Base vectors, which square to minus one, are called spacelike base vectors.

→ Base vectors, which square to one, are called timelike base vectors.

These different signs of base vectors allow us to describe space and time. We do not live in space, we live in spacetime instead!

Summary

One base scalar

$\mathbf{1} = 1 \otimes 1$

Five base vectors

$\sigma_1 = \sigma_z \otimes \sigma_x$

$\sigma_2 = -\,\sigma_z \otimes \sigma_y$

$\sigma_3 = \sigma_x \otimes 1$

$\sigma_4 = \sigma_y \otimes 1$

$\sigma_5^* = i\,(\sigma_z \otimes \sigma_z)$

Ten base bivectors

$\sigma_1\sigma_2 = -\,1 \otimes \sigma_x\sigma_y \quad = -\,i\,(1 \otimes \sigma_z)$

$\sigma_2\sigma_3 = -\,\sigma_z\sigma_x \otimes \sigma_y \quad = -\,i\,(\sigma_y \otimes \sigma_y)$

$\sigma_3\sigma_4 = \sigma_x\sigma_y \otimes 1 \quad = \quad i\,(\sigma_z \otimes 1)$ *see comment*

$\sigma_1\sigma_4 = -\,\sigma_y\sigma_z \otimes \sigma_x \quad = -\,i\,(\sigma_x \otimes \sigma_x)$

$\sigma_4\sigma_2 = -\,\sigma_y\sigma_z \otimes \sigma_y \quad = -\,i\,(\sigma_x \otimes \sigma_y)$

$\sigma_3\sigma_1 = -\,\sigma_z\sigma_x \otimes \sigma_x \quad = -\,i\,(\sigma_y \otimes \sigma_x)$

$\sigma_4\sigma_5^* = i\,(\sigma_y\sigma_z \otimes \sigma_z) = -\;(\sigma_x \otimes \sigma_z)$

$\sigma_5^*\sigma_1 = i\,(1 \otimes \sigma_z\sigma_x) \quad = -\;(1 \otimes \sigma_y)$

$\sigma_2\sigma_5^* = -\,i\,(1 \otimes \sigma_y\sigma_z) = \quad (1 \otimes \sigma_x)$ *see comment*

$\sigma_5^*\sigma_3 = i\,(\sigma_z\sigma_x \otimes \sigma_z) = -\;(\sigma_y \otimes \sigma_z)$

Obviously there are some minus signs missing. It seems that some strange symmetry (or breaking of symmetry) is hiding behind this.
Heisenberg: “Beauty is the proper conformity of the parts to one another and to the whole.”
Bacon: “There is no excellent beauty that hath not some strangeness in the proportion.”
(see: S. Chandrasekhar: Truth and Beauty. Aesthetics and Motivations in Science. Paperback edition, University of Chicago Press, Chicago & London 1990, p. 70). These definitions of beauty may apply to base vector products. There seems to be some strangeness about base vector products with an index sum of 7 in case of base bivectors and an index sum of 8 in case of base trivectors.

Ten base trivectors

$\sigma_3\sigma_4\sigma_5^* = -\ (1 \otimes \sigma_z) \quad \Leftrightarrow \quad \sigma_3\sigma_4\sigma_5^* = -\,i\ \sigma_1\sigma_2$

$\sigma_4\sigma_5^*\sigma_1 = -\ (\sigma_y \otimes \sigma_y) \quad \Leftrightarrow \quad \sigma_4\sigma_5^*\sigma_1 = -\,i\ \sigma_2\sigma_3$

$\sigma_5^*\sigma_1\sigma_2 = \ (\sigma_z \otimes 1) \quad \Leftrightarrow \quad \sigma_5^*\sigma_1\sigma_2 = -\,i\ \sigma_3\sigma_4$

$\sigma_2\sigma_3\sigma_5^* = -\ (\sigma_x \otimes \sigma_x) \quad \Leftrightarrow \quad \sigma_2\sigma_3\sigma_5^* = -\,i\ \sigma_1\sigma_4$

$\sigma_5^*\sigma_1\sigma_3 = -\ (\sigma_x \otimes \sigma_y) \quad \Leftrightarrow \quad \sigma_5^*\sigma_1\sigma_3 = -\,i\ \sigma_4\sigma_2$

$\sigma_4\sigma_5^*\sigma_2 = -\ (\sigma_y \otimes \sigma_x) \quad \Leftrightarrow \quad \sigma_4\sigma_5^*\sigma_2 = -\,i\ \sigma_3\sigma_1$

$\sigma_1\sigma_2\sigma_3 = -\,i\ (\sigma_x \otimes \sigma_z) \quad \Leftrightarrow \quad \sigma_1\sigma_2\sigma_3 = i\ \sigma_4\sigma_5^*$

$\sigma_2\sigma_3\sigma_4 = -\,i\ (1 \otimes \sigma_y) \quad \Leftrightarrow \quad \sigma_2\sigma_3\sigma_4 = i\ \sigma_5^*\sigma_1$

$\sigma_3\sigma_4\sigma_1 = \ i\ (1 \otimes \sigma_x) \quad \Leftrightarrow \quad \sigma_3\sigma_4\sigma_1 = i\ \sigma_2\sigma_5^*$

$\sigma_1\sigma_2\sigma_4 = -\,i\ (\sigma_y \otimes \sigma_z) \quad \Leftrightarrow \quad \sigma_1\sigma_2\sigma_4 = i\ \sigma_5^*\sigma_3$

Five base quadrovectors

$\sigma_1\sigma_2\sigma_3\sigma_4 = -\,i\ \sigma_5^* = \ \sigma_z \otimes \sigma_z$

$\sigma_2\sigma_3\sigma_4\sigma_5^* = \ i\ \sigma_1 = \ i\ (\sigma_z \otimes \sigma_x)$

$\sigma_3\sigma_4\sigma_5^*\sigma_1 = \ i\ \sigma_2 = -\,i\ (\sigma_z \otimes \sigma_y)$

$\sigma_4\sigma_5^*\sigma_1\sigma_2 = \ i\ \sigma_3 = \ i\ (\sigma_x \otimes 1)$

$\sigma_5^*\sigma_1\sigma_2\sigma_3 = \ i\ \sigma_4 = \ i\ (\sigma_y \otimes 1)$

One base pentavector

$\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5^* = i\ (1 \otimes 1)$

Renaming Base Vectors

To show, that this five-dimensional system of base vectors does not consist of Pauli vectors any more, the base vectors had been renamed. They are now called Dirac matrices.

In standard textbooks usually the following definitions are used:
(e.g. see Doran, Lasenby 2003, p. 278)

$$\gamma_t = \sigma_z \otimes 1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_5{}^* \sigma_1 \sigma_2$$

$$\gamma_x = -\,i\,(\sigma_y \otimes \sigma_x) = \begin{bmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{bmatrix} = \sigma_3 \sigma_1$$

$$\gamma_y = -\,i\,(\sigma_y \otimes \sigma_y) = \begin{bmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{bmatrix} = \sigma_2 \sigma_3$$

$$\gamma_z = -\,i\,(\sigma_y \otimes \sigma_z) = \begin{bmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{bmatrix} = -\,\sigma_3 \sigma_4 \sigma_5{}^* \sigma_1$$

$$\gamma_v = \sigma_x \otimes 1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_3$$

Outlook I: The Relation between Mathematics and Physics

Dirac: “One may describe the situation by saying that the mathematician plays a game in which he himself invent the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen.”

Mathematical laws seem to be inventions, but whatever we invent in mathematics (provided it is beautiful) will be found in physics as a law of nature one day.

After having invented 4 x 4 matrices, we are able to identify spacelike and timelike matrices. Thus as philosophers of science we should perhaps conclude that time must exist (which some of us might have known before by experience).

Outlook II: The Relation between Mathematics and Economics

Mathematical laws seem to be inventions, but whatever we invent in mathematics (provided it is beautiful) will perhaps be found in business and economics as a helpful tool to describe relevant laws one day?

Open Research Questions

In business mathematics and mathematical economics we mainly discuss variables of Euclidean geometry. But shouldn't it be possible to identify variables of pseudo-Euclidean geometry, building a model world with some base vectors squaring to plus one and other base vectors squaring to minus one?