A Fundamental Path to Quantum Physics
- The Space Paradox -

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Abstract

Quantum physics is a successful field of science with many relevant applications in everyday life. An exciting question is still, what is the fundamental reason for the postulates about quanta?
An answer would provide a deeper insight into nature. Such an insight can sometimes be achieved with help of a paradox. How can we derive the quantum postulates with help of a paradox?
In present-day science, space is usually modelled as a single entity. However, special relativity shows that this view is paradoxical. Instead, space consists of several volume portions that move at the velocity of light. According to the observed isotropy of space at a universal scale, space is a statistical average of rapidly moving volume portions. This solves the paradox. Therefrom, the deeper insight is achieved as follows:
The dynamics of these volume portions is derived. Therefrom, the Schrödinger equation and the quantum postulates are derived. In this manner, the dynamics of space provides the fundamental reason for the postulates about quanta.
In the learning process, the students achieve an intensive awareness, as they experience a cognitive conflict, the paradox. With it, they realize that the basic concept of a single entity of space is insufficient. They realize that the new concept requires rapidly moving volume portions and provides the quantum postulates. A concept for teaching is presented and experiences are reported.

1. Introduction

Why is this topic interesting? Students are interested in space and time, see Muckenfuß (1995). Space is basically described by General Relativity, GR, see Einstein (1915), whereby separate celestial reference systems are needed and proposed by the International Astronomical Union, IAU, see Soffel et al. (2003), Ashby and Patla (2024), Carmesin (2025). Quantum physics, QP, is essential in science and technology. However, Einstein et al. (1935) identified an incompatibility of QP, see Heisenberg (1925), Hilbert (1928), with GR - so the questions arise: How are space and QP related? How is QP founded? A derivation of the quantum postulates and a concept for teaching are provided.

1.1. Method

The hypothetic deductive method is used in the presented learning process and in the underlying scientific investigation, see Kircher et al. (2001), Popper (1935, 2002), Niiniluoto et al. (2004). Thereby, the used hypotheses are

(1) the observed universal isotropy, see Planck-Collaboration (2020),

(2) the energy momentum relation in special relativity, SR, see Einstein (1905), Hobson et al. (2006),

(3) space has no rest mass, see Workman et al (2022, p. 1142),

(4) space has an energy density, see Perlmutter et al. (1998), Riess et al. (2000), Smoot (2007), Carmesin (2024a-b, 2025).

The deduction part of the method provides the space paradox, the volume dynamics, VD, and the quantum postulates. These hypotheses are very founded. Therefore, there is little risk of failure.

2. Space paradox

The energy density of space is sometimes called u_Λ, corresponding to the cosmological constant Λ, see Einstein (1917). Sometimes, that energy density is called dark energy, u_{DE}, see Huterer and Turner (1999). It describes the energy ΔE in a volume portion (VP) ΔV of nature:

$$u_{DE} = \frac{\Delta E}{\Delta V}. \qquad \{1\}$$

Each VP ΔV has zero rest mass, $\Delta m_0 = 0$, and zero rest energy $\Delta E_0 = \Delta m_0 c^2 = 0$. Moreover, each VP at a possible velocity v obeys the energy momentum relation $\Delta E^2 = \frac{\Delta E_0^2}{1-v^2/c^2}$. As the energy density is nonzero, the energy ΔE is nonzero, and the above relation implies the following form of the energy momentum relation

$$1-\frac{v^2}{c^2}=\frac{\Delta E_0^2}{\Delta E^2}. \qquad \{2\}$$

The zero rest energy $\Delta E_0 = 0$ implies, that the right hand side in Eq. {2} is zero. As a consequence, the velocity of the VP is the velocity c of light in vacuum.

2.1. The paradox

If the volume would be a single entity, then that whole volume would move parallel to some unit direction vector $\vec{e}$ and with the velocity of light, $\vec{v} = c \cdot \vec{e}$. However, that velocity would break the isotropy of space observed at a universal scale. This is a contradiction. In general, a paradox is an apparent contradiction, the solution of which provides a deeper insight. As nature does not contradict itself, this contradiction is apparent, it is a paradox, called the space paradox.

2.2. Solution

The four hypotheses in section 1.1 are very founded empirically. Consequently, the velocity $\vec{v} = c \cdot \vec{e}$ of each VP is very founded. Therefore, there must be several volume portions moving in all directions, so that their velocities average out at a universal scale, see Fig. (1). As a consequence, space is an average of volume portions. Moreover, at a universal scale, the velocities of these volume portions average out. Therefore, space is isotropic at a universal scale, as observed. In this manner, the space paradox is solved.

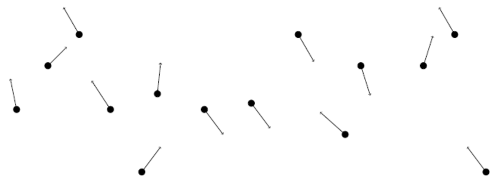

Fig. 1: Volume portions with velocities $\vec{v}_j$. The average of the velocities is zero, $average(\vec{v}_j) = \vec{0}$, own figure.

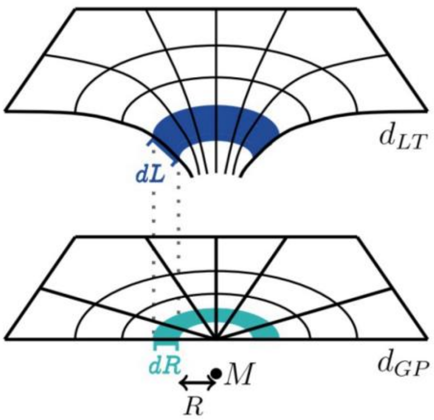

Fig. 2: In the vicinity of a mass M, the radial increment dL is increased with respect to the original increment dR that would occur in the limit M to zero. Note that the increment dL is measured as a light travel distance d_{LT}, and the increment dR is measured as a gravitational parallax distance d_{GP}, see text, own figure.

3. Dynamics of volume portions

A first insight of the paradox is that space is not a single entity. Instead, space consists of volume portions. In order to obtain a deeper insight, we analyse and derive properties of these volume portions. For it, we improve the usual description of curved space with help of metric tensors g_{ij}, see Hobson et al. (2006). For instance, in the vicinity of a mass M, the original radial distance dR is increased by the square root $\sqrt{g_{RR}}$ of the radial element g_{RR} of the metric tensor to a value dL, see Fig. (2),

$$dL = dR \cdot \sqrt{g_{RR}} \qquad \{3\}$$

Note that the value dL can be measured as a light travel distance d_{LT}, see Hobson et al. (2006), Note that similarly, the original value dR can be measured with help two hand leads, the distance is called gravitational parallax distance, see Carmesin (2025).

3.1. Additional volume

In order to describe curvature with help of volume portions, we realize that a mass M causes additional volume in its vicinity, see Fig. (2). For instance, in Fig. (2), flat space is illustrated in the lower part, and curved space is shown in the upper part of the figure. Thereby, the curved space is caused by the mass M, and space remains flat at zero mass $M = 0$. E. g. the shell in Fig. (2) with radius R and thickness dR in flat space has the volume $dV_R = 4\pi R^2 dR$. The corresponding shell in the curved space has the volume $dV_L = 4\pi R^2 dL.$ The difference of these two volumes is called additional volume δV,

$$\delta V = dV_L - dV_R = 4\pi R^2 \cdot (dL - dR). \qquad \{4\}$$

As another example, a mountain can be described with help of a metric tensor of its surface. Alternatively, the mountain can be described by its volume. It is a volume in addition to the volume of Earth below the mountain. So it is an additional volume. This example illustrates how a mountain can be described with help of a metric tensor or with help of an additional volume.

3.2. Relative additional volume

In general, in order to derive laws of physics, it is valuable to use the ratio of the additional volume $\delta V = dV_L - dV_R$ and of the volume dV_L of curved space in that difference. That ratio is called relative additional volume ε_L:

$$\varepsilon_L = \frac{\delta V}{dV_L}. \qquad \{5\}$$

In fact, inserting Eqs. {3,4} in the relative additional volume in Eq. {5} provides the relative additional volume as a function of the metric tensor element:

$$\varepsilon_L = \frac{dL - dR}{dL} = 1 - \frac{1}{\sqrt{g_{RR}}}. \qquad \{6\}$$

This confirms that the curvature can be described either by the metric tensor or by the additional volume. The description with the additional volume is compatible with the volume portions that become necessary for the solution of the space paradox. In contrast, the metric tensor does not include the concept of volume portions, and it does not provide the dynamics of the volume portions. As a consequence, the description with the volume portions provides the additional possibility to derive the dynamics of volume portions. Of course, the metric tensor provides the description of time dilations directly, while in the case of volume portions, we can apply the relation $g_{tt} = \frac{1}{g_{RR}}$ of the Schwarzschild (1916) metric. With help of the volume portions, we derive the dynamics of volume portions next:

3.3. **Dynamics of volume portions**

In this section, the propagation of a volume portion is described in terms of its relative additional volume ε_L, see Fig. (3).

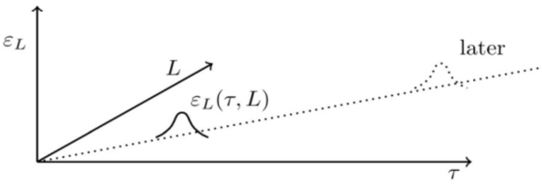

Fig. 3: The relative additional volume ε_L is shown as a function of location L and time τ. Thereby, the location L summarizes the three-dimensional vector $\vec{L}$ in a reduced manner, own figure.

The considered localized volume portion is represented by a relative additional volume $\varepsilon_L(\vec{L}, \tau)$ as a function of a three-dimensional vector $\vec{L}$ representing the location and of the time τ. That function $\varepsilon_L(\vec{L}, \tau)$ has a local maximum, as the VP is localized.

As a consequence, the total change or the total derivative is zero, $d\varepsilon_L(\vec{L}, \tau) = 0$. That is analysed in terms of the partial derivatives,

$$d\varepsilon_L(\vec{L}, \tau) = \frac{\partial \varepsilon_L}{\partial \tau} d\tau + \frac{\partial \varepsilon_L}{\partial \vec{L}} d\vec{L} = 0 \qquad \{7\}$$

The VP moves parallel to a corresponding direction unit vector $\vec{e}_v$. Therefore, during a time $d\tau$, the vector $\vec{L}$ changes by the amount $d\vec{L} = v \cdot d\tau \cdot \vec{e}_v$. With it, the total derivative in Eq. {7} is

$$d\varepsilon_L(\vec{L}, \tau) = \frac{\partial \varepsilon_L}{\partial \tau} d\tau + \frac{\partial \varepsilon_L}{\partial \vec{L}} \cdot v \cdot d\tau \cdot \vec{e}_v = 0. \{8\}$$

The above equation is solved for $\frac{\partial \varepsilon_L}{\partial \tau}$,

$$\frac{\partial \varepsilon_L}{\partial \tau} = -\frac{\partial \varepsilon_L}{\partial \vec{L}} \cdot v \cdot \vec{e}_v, \ \ with \ \ v = c. \qquad \{9\}$$

This is the differential equation, DEQ, of VPs or of volume dynamics, VD. A Lorentz invariant form of the DEQ is achieved as follows. The square is applied to Eq. {9}, and the right hand side is subtracted,

$$\left(\frac{\partial \varepsilon_L}{\partial \tau}\right)^2 - \left(\frac{\partial \varepsilon_L}{\partial \vec{L}}\right)^2 \cdot c^2 = 0. \qquad \{10\}$$

4. **Derivation of the Schrödinger equation**

In order to derive the Schrödinger (2026) equation, the DEQ of VPs in Eq. {9} is multiplied by (the operator ∂_τ marks $\partial/\partial\tau$, the time derivative is marked by a dot, $\dot{\varepsilon}_L$, the reduced Planck constant is $\hbar = \frac{h}{2\pi}$, the superscript cc marks the complex conjugate)

the operator $i\hbar\partial_\tau$. So, the DEQ of VPs has the form

$$i\hbar \frac{\partial}{\partial \tau} \dot{\varepsilon}_L = -i\hbar \frac{\partial}{\partial \vec{L}} \dot{\varepsilon}_L \cdot c \cdot \vec{e}_v. \qquad \{11\}$$

4.1. **Momentum operator**

The following plane wave is a solution to the DEQ of VPs in Eq. {11}

$$\varepsilon_L(\vec{L}, \tau) = \exp\left(-i\omega\tau + i \cdot \vec{e}_v \cdot \frac{E}{\hbar \cdot c} \cdot \vec{L}\right). \quad \{12\}$$

This is confirmed by inserting. Hereby, the operator $-i\hbar\frac{\partial}{\partial \vec{L}}$ in the DEQ {11} of VPs generates the momentum $\vec{p} = p \cdot \vec{e}_v = \frac{E}{c} \cdot \vec{e}_v$. This is confirmed by inserting the solution in Eq. {12} into the DEQ {11} of VPs. Therefore, the operator $-i\hbar\frac{\partial}{\partial \vec{L}}$ is called momentum operator $\hat{\vec{p}}$,

$$\hat{\vec{p}} := -i\hbar \frac{\partial}{\partial \vec{L}} \dot{\varepsilon}_L. \qquad \{13\}$$

4.2. **Schrödinger equation**

In the DEQ {11} of VPs, the momentum operator is identified,

$$i\hbar \frac{\partial}{\partial \tau} \dot{\varepsilon}_L = \hat{\vec{p}} \cdot \vec{e}_v \cdot c \ \ \dot{\varepsilon}_L. \qquad \{14\}$$

The product of the direction unit vector $\vec{e}_v$ and of the momentum operator $\hat{\vec{p}}$ is the operator of the absolute value of the momentum, as the momentum and the unit vector are parallel, $\hat{\vec{p}} \cdot \vec{e}_v = \hat{p}$. Moreover, the energy momentum relation of special relativity implies that the product of the momentum and the velocity c of light is the energy. According to Bohr's (1920) correspondence principle, the product $c \cdot \hat{p}$ is equal to the energy operator $\hat{E}$. Thereby, the energy operator is usually called Hamilton operator $\hat{H}$. Moreover, the time derivative of the relative additional volume $\dot{\varepsilon}_L$ is normalized. For it, $\dot{\varepsilon}_L$ is multiplied by a normalisation factor t_n, so that the usual (Sakurai 1994, Kumar 2018) normalization holds,

$$\int (t_n \dot{\varepsilon}_L) \cdot (t_n \dot{\varepsilon}_L)^{cc} dL^3 = 1. \qquad \{15\}$$

As a consequence, the DEQ {14} of VPs implies the Schrödinger (1926) equation,

$$i\hbar \frac{\partial}{\partial \tau} t_n \dot{\varepsilon}_L = \hat{E} t_n \dot{\varepsilon}_L \ or \ i\hbar \frac{\partial}{\partial \tau} \psi = \hat{E}\psi. \qquad \{16\}$$

Hereby, we identify the wave function

$$\psi = t_n \cdot \dot{\varepsilon}_L. \qquad \{17\}$$

The Schrödinger equation in eq. {16} has the general form proposed by Schrödinger (1926). In this sense, the derived Schrödinger eq. {16} should hold in general in physics. This is confirmed additionally by the general applicability of that Schrödinger eq.: All relativistic wave equations in physics can be derived from the Schrödinger equation by using an appropriate particular Hamilton operator for the system under investigation, see Sakurai (1994). Moreover, also the fields in quantum field theory are solutions of the Schrödinger equation, see Carmesin (2024a).

4.3. **Schrödinger equation and masses**

According to the Higgs (1964) mechanism, mass forms by a phase transition from vacuum, which is represented by VPs more generally, according to the space paradox. As a consequence, the present derivation of the Schrödinger equation includes masses. In particular, the Schrödinger equation {16} describes all relativistic species described in Workman (2022, section 25.2.3). In this sense, the Schrödinger equation {16} is more general than the DEQ derived by Schrödinger (1926), which holds for nonrelativistic objects. Accordingly, we call the Schrödinger equation {16} a generalized Schrödinger equation, GSEQ. Of course, the generalization is a consequence of the Hamilton operator $c \cdot \hat{p} = \hat{E} = \hat{H}$.

4.4. **Nonrelativistic Schrödinger equation**

The generalized Schrödinger equation {16} implies the nonrelativistic form for the case of relatively slow objects with a rest mass m_0, a momentum p and a small ratio

$$\frac{p^2}{m_0^2 c^2} \ll 1. \qquad \{18\}$$

With it, the energy momentum relation of special relativity $E = \sqrt{p^2c^2 + m_0^2c^4}$ can be approximated in linear order in the fraction in Eq. {18},

$$E \doteq E_0 \cdot \left(1 + \frac{p^2}{2m_0^2c^2}\right) = E_0 + \frac{p^2}{2m_0}. \qquad \{19\}$$

In that approximation, the Schrödinger equation, SEQ, {16} has the form

$$i\hbar \frac{\partial}{\partial \tau} \psi_{E_0} = \left(E_0 + \frac{\hat{p}^2}{2m_0}\right) \psi_{E_0}. \qquad \{20\}$$

The wave function ψ_{E_0} includes the description of the rest energy E_0. Accordingly, we apply a factorization Ansatz,

$$\psi_{E_0} = \exp\left(\frac{E_0 \tau}{i\hbar}\right) \psi. \qquad \{21\}$$

Inserting the wave function {21} into the SEQ {20} yields for $i\hbar \frac{\partial}{\partial \tau} \psi_{E_0}$ the equation

$$E_0 \psi_{E_0} + e^{\frac{E_0 \tau}{i\hbar}} i\hbar \partial_\tau \psi = \left(E_0 + \frac{\hat{p}^2}{2m_0}\right) \psi_{E_0}. \qquad \{22\}$$

Subtracting $E_0 \psi_{E_0}$ and multiplication of $\exp\left(\frac{-E_0 \tau}{i\hbar}\right)$ implies the nonrelativistic SEQ

$$i\hbar \partial_\tau \psi = \frac{\hat{p}^2}{2m_0} \psi. \qquad \{23\}$$

More generally, a potential V can be added,

$$i\hbar \partial_\tau \psi = \frac{\hat{p}^2}{2m_0} \psi + V\psi = \hat{H}\psi. \qquad \{24\}$$

4.5. **Postulate about the Schrödinger equation**

This time-dependent Schrödinger equation represents one of the quantum postulates. Here, we use the formulation of the postulates provided by Kumar (2018, p 168-170). So, postulate 5 is: 'The time evolution of the state vector is governed by the time-dependent Schrödinger equation {24}, where $\hat{H}$ is the Hamilton operator corresponding to the total energy of the system.' As we derived this postulate, we explained its origin and its foundation. Therefore, that postulate becomes a derived rule of QP.

4.6. **Interpretation of the Schrödinger equation**

As an interpretation, we realize that the Schrödinger equation describes the most general propagation of a localized VP in nature, see Fig. 3. This includes the formation of orbits, such as in a hydrogen atom. These orbits represent the chemical elements in the periodic system of elements, see Mayer-Kuckuk (1980). The GSEQ and the SEQ hold also for masses, as these are formed from VPs via a phase transition, see Higgs (1964). Note that more generally, the algebraic structure of the wave equation of electromagnetic waves can be derived in the framework of the DEQ of VPs, when more general tensors are derived additionally, so that electromagnetic waves are included in a tensor version of the present framework, see Carmesin (2024a).

5. **Derivation of the postulate about states**

The first quantum postulate in Kumar (2018, p 168-170) is as follows: 'The state of a quantum mechanical system, at a given instant of time, is described by a vector $|\psi(\tau)\rangle$, in the abstract Hilbert space H of the system.' This postulate is derived as follows: The states of a quantum mechanical system are the solutions of the Schrödinger equation, see section (4). As the SEQ is a linear DEQ, its solutions form a Hilbert space. With the scalar product in Eq. {15}, the solution space is a Hilbert space H. This implies the above first quantum postulate, q. e. d.

6. **Derivation of the postulate about observables**

The second quantum postulate in Kumar (2018, p 168-170) is as follows: 'A measurable physical quantity A (called an observable or dynamical physical quantity), is represented by a linear and hermitian (or self-adjoint, see Sakurai 1994) operator $\hat{A}$ acting in the Hilbert space of the state vectors.' This postulate is derived as follows: In general, each measurable quantity is obtained by a measurement, and each measurement is applied to a state of the system. That state is a vector $\psi(\tau)$ in H, see section (5). As a consequence, the measurement represents a function f of $\psi(\tau)$,

$$measurement(state(\psi)) \triangleq f(\psi). \qquad \{25\}$$

In general, a state vector $\psi(\tau)$ in H is a linear superposition $\psi(\tau) = \psi_1(\tau) + \psi_2(\tau)$. The above postulate states that the measurement should not cause any nonlinear effects, as these would decrease the quality of the measurement. Accordingly, the above postulates require an ideal completely linear measurement. Therefore, an ideal and postulated measurement is linear,

$$f(\psi) = f(\psi_1 + \psi_2) = f(\psi_1) + f(\psi_2). \quad \{26\}$$

Consequently, the function represents a linear operator $\hat{A}$. Moreover, the observables have real values, so the operators are self-adjoint,

$$measurement(state(\psi)) \triangleq \hat{A}\,\psi. \quad \{27\}$$

Altogether, we realized that the postulate requires ideal linear measurement devices. And in that framework, we derived the postulate, q. e. d.

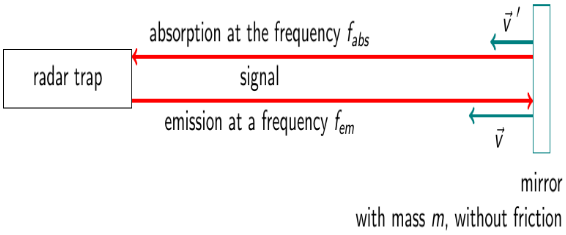

Fig. 4: A radar trap emits an electromagnetic signal to a car that drives with a velocity v. When the signal is reflected, energy and momentum are conserved, so that the velocity of the car reduces to a value v'. The car has a mass m and acts as a mirror, own figure.

7. Universal quantization

In this section, we show that the quantization constant is the same for all frequencies of electromagnetic signals. For it, we analyse a radar trap, see Fig. (4). The trap emits a signal with an energy E_{em}, frequency f_{em} and action $S_{em} = \frac{E_{em}}{f_{em}}$. The reflected signal with the respective values $S_{abs} = \frac{E_{abs}}{f_{abs}}$ is absorbed by the trap. The action of the signal is invariant in that process, $S_{em} = S_{abs}$. It has been shown in Carmesin (2023a, 2024a), that the action of a falling object is also invariant when a gravitational redshift occurs. The minimal action S_{min} is the Planck constant, and the value has to be measured.

7.1. Radar trap

At the radar trap, the frequency is increased by the Doppler effect as follows, see Burisch et al. (2025, p. 485) or Carmesin (2025), with $\beta = \frac{v}{c}$,

$$f_{abs} = f_{em} \cdot \frac{1+\beta}{1-\beta}. \quad \{28\}$$

When the signal is reflected at the car, energy and momentum are conserved. Hereby, the ratio u of the momentum p_{em} of the emitted signal and of the momentum $p_m = m \cdot v$ of the car is $u = \frac{p_{em}}{p_m}$. As a consequence, see Carmesin (2025), the energies of the emitted and of the absorbed signal are as follows,

$$E_{abs} = E_{em} \cdot \frac{1+\beta}{1-\beta-u\beta}. \quad \{29\}$$

The mass of the car is chosen large, so that the ratio u can be omitted. As a consequence, the ratio of the actions is one,

$$\frac{S_{abs}}{S_{em}} = \frac{E_{abs}}{f_{abs}} \cdot \frac{f_{em}}{E_{em}} = \frac{1+\beta}{1-\beta} \cdot \frac{1-\beta}{1+\beta} = 1. \quad \{30\}$$

Thus, the action is an invariant of the signal in this process. The minimal value S_{min} of this action is positive and represents to minimal invariant signal. Its value must be measured, it is the Planck constant, see Workman (2022).

8. Derivation of the postulate about eigenvalues

The third quantum postulate in Kumar (2018, p 168-170) is as follows: 'The measurement of an observable A in a given state may be represented formally by the action of an operator $\hat{A}$ on the state vector $\psi(\tau)$. The only possible outcome of such a measurement is one of the eigenvalues.' This postulate is derived as follows: The first sentence of the postulate has already been derived in section (6). The second sentence is founded as follows. In general, a wave function ψ is a linear combination $\psi = \Sigma_j\, c_j\, \psi_j$ of eigenvectors ψ_j with eigenvalues a_j of the considered operator $\hat{A}$. The application of $\hat{A}$ to ψ yields the following result

$$\hat{A}\psi = \Sigma_j\, c_j\, a_j\, \psi_j. \quad \{31\}$$

As a consequence, such an application of the operator changes the linear combination of the eigenvectors. But an ideal measurement should not remix the linear combination of the eigenvectors. An ideal measurement should leave the state unchanged, this is possible in the case of a single eigenvector,

$$\hat{A}\psi = \hat{A}\psi_j = a_j\, \psi_j, \quad \{32\}$$

or it should project to an eigenvector, this is possible in this case

$$\hat{A}\psi = \hat{A}\, \Sigma_j\, c_j\, \psi_j = a_j \psi_j. \quad \{33\}$$

In this sense, the third postulate again defines an ideal type of measurement and excludes conceivable other measurement devices. Of course, additionally, the postulate is in accordance with empirical findings.

9. Derivation of the postulate on probabilities

The fourth quantum postulate in Kumar (2018, p 168-170) is as follows: 'If a measurement of an observable A is made in a state $\psi(\tau)$, of the quantum mechanical system, the following holds:

(1) The probability of obtaining one of the non-degenerate discrete eigenvalues $\boldsymbol{a_j}$ of the corresponding operator $\hat{A}$ is given by $P(a_j) = \frac{|\langle\phi_j|\psi\rangle|^2}{\langle\psi|\psi\rangle}$, where $|\phi_j\rangle$ is the eigenfunction of $\hat{A}$ with the eigenvalue a_j. If the state vector is normalized to unity, $P(a_j) = |\langle\phi_j|\psi\rangle|^2$.

(2) If the eigenvalue $\boldsymbol{a_j}$ is m-fold degenerate, this probability is given by $P(a_j) = \Sigma_{j=1}^{m} \frac{|\langle \phi_j | \psi \rangle|^2}{\langle \psi | \psi \rangle}$.

(3) If the operator $\hat{A}$ possesses a continuous eigenspectrum $\{ a \}$, the probability that the result of a measurement will yield a value between a and $a + da$ is given by $P(a) = \frac{|\langle \phi(a) | \psi \rangle|^2}{\langle \psi | \psi \rangle} da = \frac{|\langle \phi(a) | \psi \rangle|^2}{\int |\psi(a')|^2 da'} da$.

The postulate has been derived in Carmesin (2021a-d, 2022a-d, 2023a-d, 2024a-g, 2025). Here, we outline the proof of part (1). According to the central limit theorem, the relative frequency of a_j is proportional to the energy density $u_{kin} = \frac{\dot{\varepsilon}_L^2}{8\pi G}$ of the state $\psi = t_n \cdot \dot{\varepsilon}_L$. This result is insightful, as it shows how the physics of the VPs provides the probabilities that are essential for QP. This result has been derived in Carmesin (2018a-b, 2019, 2020, 2021a-d, 2022a-d, 2023a-d, 2024a-g, 2025). Moreover, this result shows that the probability density $p(L)$ of the state $\psi(L)$ is proportional to the energy density $u_{kin} = \frac{\dot{\varepsilon}_L^2}{8\pi G}$. As a consequence, the probability density $p(L)$ of the state $\psi(L)$ is proportional to the absolute square of the wave function, $p(L) \propto |\psi|^2(L)$. This is a direct and empirically tested expression for the probabilistic outcomes of measurements. The result in part (1) can be derived with help of a transformation of these results: We use the bracket notation and orthonormal eigenfunctions $|\phi_j\rangle$. We represent the state $|\psi\rangle$ as a linear superposition of eigenfunctions

$$|\psi\rangle = \Sigma_j |\phi_j\rangle\langle\phi_j|\psi\rangle = \sum_j |\psi_j\rangle. \qquad \{34\}$$

The factor $\Sigma_j |\phi_j\rangle\langle\phi_j|$ is identified with the identity matrix $\hat{1}$, as it keeps $|\psi\rangle$ identical,

$$\hat{1} = \Sigma_j |\phi_j\rangle\langle\phi_j|. \qquad \{35\}$$

As shown above, the probability $P(a_j)$ of the state $|\psi\rangle$ is proportional to the respective energy density u_{kin}, which is proportional to the square $\langle\psi_j|\psi_j\rangle$. That proportionality is an equality, as the sum $\Sigma_j P(a_j)$ is one,

$$P(a_j) = \langle\psi_j|\psi_j\rangle. \qquad \{36\}$$

Thereby, $\Sigma_j P(a_j) = 1$ is derived as follows: Eq. {34} implies

$$\langle\psi_j|\psi_j\rangle = \langle\psi|\phi_j\rangle\langle\phi_j|\phi_j\rangle\langle\phi_j|\psi\rangle = |\langle\psi|\phi_j\rangle|^2. \{37\}$$

Hereby, we used $\langle\phi_j|\phi_j\rangle = 1$. The sum is

$$\Sigma_j\langle\psi_j|\psi_j\rangle = \langle\psi|\phi_j\rangle\langle\phi_j|\psi\rangle. \qquad \{38\}$$

The identity matrix provides

$$\Sigma_j\langle\psi_j|\psi_j\rangle = \Sigma_j\langle\psi|\hat{1}|\psi\rangle = 1. \qquad \{39\}$$

We apply Eq. {37} to Eq. {36},

$$P(a_j) = \langle\psi_j|\psi_j\rangle = |\langle\psi|\phi_j\rangle|^2. \qquad \{40\}$$

This shows part (1) for the case of normalized functions considered here. q. e. d. The other cases are derived analogously.

10. Proposed learning process

In this section, I present a learning process that I used in two learning groups. I utilized the concept in in a research club with students in classes ranging from 8 to 13. Moreover, I used the learning process in general studies courses at the university. In both cases, the results are documented, for instance with photos of the blackboard and with slides of a presentation. In both cases, exercises and discussions are applied.

(1) As a preliminary step, Newton's mechanics and gravity is summarized, see Carmesin et al. (2018c). This takes about 90 min.

(2) In a second preliminary step, basics of special relativity is summarized, see Burisch et al. (2025). This takes about 90 min.

(3) In a preliminary step three, basics of general relativity are summarized, see Burisch et al. (2025). This takes about 90 min.

(4) As first main step, the space paradox is derived. This takes about 45 min.

(5) In a second main step, the concepts of additional volume and relative additional volume are introduced. This takes about 45 min.

(6) At a third main step, the volume dynamics is derived. This takes about 45 min.

(7) In a fourth preliminary step, basics of quantum physics are summarized, see Carmesin et al. (2020). This takes about 90 min.

(8) As a fourth main step, the Schrödinger equation is derived from the VD and utilized. This takes about 90 min.

(9) At a next main step, the postulate about states is derived and used. This takes about 45 min.

(10) In a main step six, the postulate about observables is derived and used. This takes about 45 min.

(11) In a next main step, the postulate about eigenvalues is derived and used. This takes about 45 min.

(12) As a main step eight, universal quantization is derived and discussed. This takes about 45 min.

(13) In a next main step, gravity, curvature and the energy density of gravitational fields are derived, applied and discussed. This is an essential preparation for the derivation of probabilistic outcomes. This takes about 135 min.

(14) In a completing main step, the postulate about probabilistic outcomes is derived, discussed and utilized. This takes about 135 min.

Altogether, the course provides a founded understanding of quantum physics and its connection to gravity and space physics. Preliminary introductions require 360 min (1-3,7). The derivation of volume dynamics needs 135 min (4-6). The Schrödinger equation is treated in 90 minutes (8). The three postulates about the algebraic structure of QP are derived and used in 135 min (9-11). Universal quantization is special, as it shows that an object exhibits a universal invariant positive action, and its minimal value provides quantization - of course the value of the minimal action must be measured, this topic takes 45 min (12). The derivation of gravity and curvature from the VD needs 135 min (13). This prepares the derivation of the postulate about probabilistic outcomes. The derivation and application of that postulate takes 135 min (14). In summary, the course needs 360 min preparation, 270 min for the VD and the implication of gravity and energy density, and 405 min for the derivation of the postulates and universal quantization. The needed 1035 min correspond to 23 lectures of 45 min each.

11. Experience from teaching

All students chose the course on a voluntary basis. Correspondingly, they were highly motivated. The students received a script, so they could concentrate on the lecture, discussion and exercises. For comparison, in a usual curricular course about quantum physics, experiments are used regularly. This course focusses on theoretical concepts. In general, experimental and theoretical courses have complementary purposes and are not intended to replace each other. In this course, the large introductory part has the purpose to enable everybody to participate in all coming exercises and derivations. In fact, everybody was able to do the exercises and to contribute to the discussions. Moreover, many students participated in small research projects and won prizes in the Jugend forscht competition. Altogether, it turned out that the large introductory part with 360 min has an intermediate learning barrier. This is optimal for a high participation in the learning process. An essential part is the cognitive conflict inherent to the space paradox. This provided a high motivation, as expected, see Kircher (2001). Students argued, that this provides an awareness of the problem and of the novelty of the solution. Thereby, the derivation of the paradox and of the solution have no technical complication. As a consequence, this part has an intermediate learning barrier. Based on the cognitive conflict, the additional volume was introduced. Hereby, the analogy to the mountains was helpful. Also in this part, the learning barrier has an optimal intermediate level. In the discussion, the students realized that the additional volume includes the volume portions that provide the solution of the space paradox. As a consequence, the additional volume should be essential in order to overcome the incompatibility between GR and QP. Indeed, the volume dynamics provides the Schrödinger equation in a very direct manner. This confirms the expectation that the VPs should overcome the incompatibility between GR and QP. For more details see e. g. Carmesin (2024a, 2025). In discussions, the students realize that the Schrödinger equation can be fundamentally understood with help of the VPs: The Schrödinger equation and the VD provide the most general propagation of volume portions. In addition, the wave function describes the time derivative of the relative additional volume of such a volume portion. In this manner, the VD and the Schrödinger equation clarify open questions of the present-day QP. Also in this part, the learning barrier has an optimal intermediate level. Moreover, many essential solutions of the DEQ of VPs and of the Schrödinger equation can be obtained with help of an Ansatz and verified by inserting. The postulates 1-3 about the algebraic structure of quantum physics have been introduced with help of several examples. The students liked that procedure, as they were able to derive many instructive solutions on their own. Hereby, the learning barrier has an optimal intermediate level, as linear algebra is relatively intuitive and can be supplemented by many geometric examples. Some students liked that part especially, as they were able to discover connections on their own.

The derivation of the exact gravity and of the energy density of the exact gravitational field was based on instructions. The students were able to solve exercises and discuss the field. Thereby, they realized the unifying power of the VD, as it provides quantum physics and gravity and curvature in an indivisible manner. This was confirmed when the energy density of the exact field turned out to provide the probabilistic outcomes and the probabilities in quantum physics. With it, the students derived the postulate about probabilistic outcomes on their own, for the case of an example. The general derivation was presented in the lecture and in the script. Also this topic has an optimal intermediate learning barrier, when examples are used in order to discover results. Altogether, the course showed how the exact, direct and general unification is motivated by the space paradox, and how the students can derive many results on their own, as the unification is very natural, intuitive and problem solving. In summary, the concept has been tested in several learning groups. The concept can be directly utilized, as it is highly elaborated. Additionally, the concept can be used in a seminar, as my students use the concept in order to do projects on their own.

12. Discussion

The space paradox clearly shows that the concept of a single space provides a contradiction. The solution shows that space consists of many volume portions. These should overcome the incompatibility of GR and QP, that Einstein et al. (1935) realized. Indeed, these VPs have a very general dynamics. This dynamics implies the Schrödinger equation and the quantum postulates. Additionally, the fact is derived that each signal or falling object has an invariant positive action S, which represents the invariant property of the object. The minimal value of S is universal, whereby the value of that minimal action, the Planck constant, must be measured. Moreover, the dynamics of the VPs provides exact gravity, curvature, the formation of space since the Big Bang, exact space navigation, exact adequate frames, and several solutions to fundamental problems of physics, see Carmesin (2017, 2018a-b, 2019, 2020, 2021a-d, 2022a-d, 2023a-d, 2024a-g, 2025). This shows the great unifying power of the concept. The space paradox is an ideal path to this unifying, problem solving, enlightening and fascinating field of physics. The students realize many of these advantages and express these in the discussions. The intensive use of previous knowledge provides a high learning efficiency, see Hattie (2009).

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Acknowledgement

I thank I. Carmesin for interesting discussions and proofreading the manuscript. I thank Matthias Carmesin for stimulating discussions.