# Students Derive an Exact Solution of the Flatness Problem

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**Abstract**

In everyday life, time and space are essential. Moreover, space and time are fundamental concepts of physics. For it, Newton made a first proposal with flat space and time evolving at a constant rate as a basis. Einstein improved it with relativistic and curved spacetime. Accordingly to its relevance in everyday life and in physics, students are interested in the topic. Here, I present a learning process, by which learners can achieve the essential insights of special relativity and general relativity in an exact manner. Thereby, students experience basic principles directly at a free fall tower and by taking photographs in a school observatory. Using these basic principles, they achieve inspiring and exact results on their own, after an appropriate instruction. I present the learning process and a didactic analysis, so that you can directly use the concept for teaching. I tested the learning process in various learning groups, and I report about experiences.

## 1. Introduction

Students are interested in the evolution of space since the Big Bang. Accordingly, they observed the Big Bang with our school observatory by using three different methods, see e. g. Helmcke et al. (2018).

For instance, they provide a very basic method that can be understood even by students at the age 14 without any special knowledge about astronomy or astrophysics: In that method, our students used our school telescope with an aperture of 0.28 m, in order to take the picture in Fig. 1. So they obtained a photo of a quasar at a light-travel distance of 12.05 billion light years. For comparison, the aperture of the Hubble telescope is to 2.4 m. Thus, for each observed object, the amount of light received by the Hubble telescope exceeds the amount of light received by our telescope by the following factor:

$$\left(\frac{2.4}{0.28}\right)^2 \approx 73 \qquad \{1\}$$

In flat and constant space, as proposed by Newton (1687), the energy density of the light emitted by an object decreases proportional to the square of the distance. Correspondingly, the Hubble telescope should observe quasars at a light-travel distance that exceeds the light-travel distance of quasars observed by our telescope, 12.05 billion light years, by the factor $\sqrt{73} \approx 8.5$, at least. However, the Hubble telescope did never observe any object at a light-travel distance beyond 13.8 billion light years. We interpret this finding with the Big Bang that occurred 13.8 billion years ago: Light observed at Earth has propagated for less than 13.8 billion years, so that it propagated less than 13.8 billion light years.

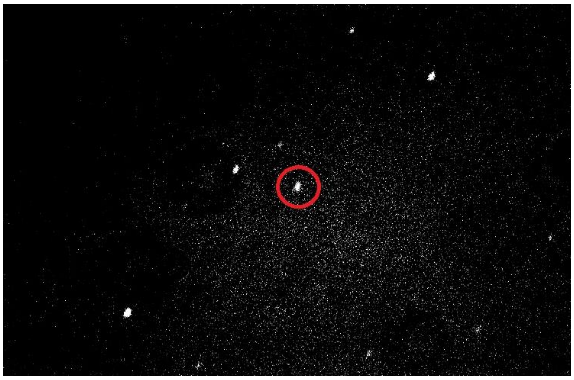

**Fig.1:** We observe the quasar APM08279+5255 at the light-travel distance of 12.05 billion light years. The quasar is near the light horizon. Objects can be observed at light-travel distance less than 13.8 billion light years. How can this finding be interpreted?

In a second method, our students observed the redshift and distance of galaxies and produced the Hubble diagram in Fig. 2. While in a third method, they used the supernova in Fig. 3. Of course, we humans want to understand our observations. Accordingly, the students ask the question: How can we understand, calculate and derive the time evolution of the expansion of space since the Big Bang?

### 1.1. Organization of the paper

We propose our learning process in part 2. Part 3 provides a didactic analysis. Experiences with teaching and a discussion are presented in parts 4 and 5.

## 2. Learning process

The learning process has been developed for the following groups: members of a research club with students in classes 5 to 13 and general studies courses at the university.

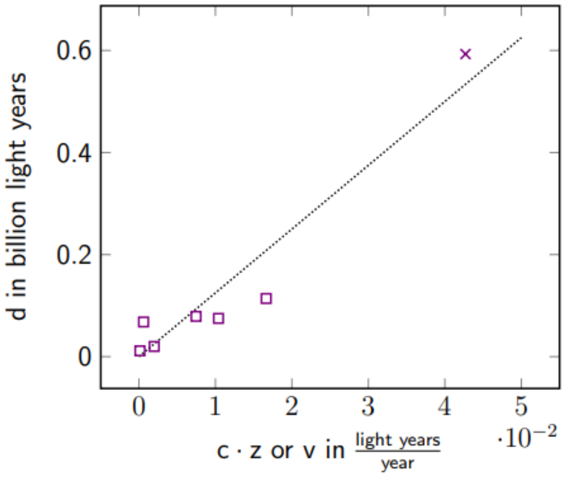


**Fig.2:** We observe galaxies at redshifts z (squares and triangle). Moreover, the learners observe the light-travel distances d (squares). In the case of the galaxy UGC 8058, the light-travel distance is obtained from the literature. The straight line has the slope 12.5, corresponding to a measured age of the universe of 12.5 billion years. The linear law shown in this diagram is called Hubble law, and the inverse of the slope in this diagram is called Hubble constant $H_0$.

## 2.1. **Preconditions of learning**

As a precondition of the learning process, the students treated already basic facts in gravity and special relativity, see e. g. Newton (1687), Einstein (1905), Carmesin (2023a, pp 102-129). Moreover, they have basic competences about the Schwarzschild metric, including the position factor and findings in general relativity, see e. g. Einstein (1915), Schwarzschild (1916), Carmesin (2023b) or Burisch (2022, pp 484-489):

In the vicinity of a field generating mass M, a probe mass m has the following energy:

$$E(r,v) = m_0 c^2 \varepsilon(r)\gamma(v) = m_0 c^2 = E_0 \quad \{2\}$$

Thereby, $\gamma(v)$ is the Lorentz factor,

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \{3\}$$

And $\varepsilon(r)$ is the position factor:

$$\varepsilon(r) = \sqrt{1 - \frac{R_S}{r}} \text{ with } R_S = \frac{2GM}{c^2} \quad \{4\}$$

Hereby, the energy is conserved, as the vicinity of M is stationary, so that the Noether (1918) theorem can be applied.

Moreover, the students are familiar with basic facts about the expansion of space since the Big Bang, see e. g. Carmesin (2020a, pp 296-301). In particular, they know that the Hubble constant $H_0$ is a time dependent Hubble parameter $H(t)$, more generally, see Fig. 2. Hereby, the square of the Hubble parameter is described by the Friedmann (1922) Lemaitre (1927) equation, FLE:

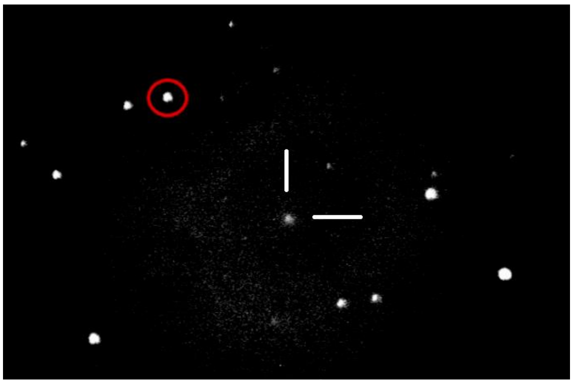

**Fig.3:** We observed the supernova SN 2017eaw (red circle) in the fireworks galaxy NGC 6946 (at white bars).

$$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - k\frac{c^2}{R^2} \quad \{5\}$$

Hereby, R is a radius of a prototypical ball in the universe, see Fig. 4. That ball increases as a function of time, whereby it describes the expansion of space. As usual, $\rho$ describes the density. The possible curvature of space is described by the curvature parameter $k$. Thereby, the FLE has been derived on the basis of the Einstein field equation, the dynamical equation of the usual theory of relativity proposed by Einstein (1915) and Hilbert (1915), see e. g. Hobson (2006).

If $k = 0$, then the space is not curved or flat.

If $k > 0$, then the space is a hyperball.

If $k < 0$, then the space is curved like a saddle.

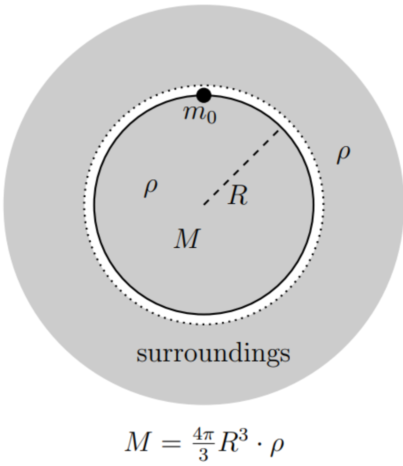


**Fig.4:** Prototypical ball in the universe. Many such balls form space as a whole. It is like a hyperball, if the curvature parameter is positive. Space is flat and unlimited, if the curvature parameter is zero. Space is curved like a saddle and unlimited, if the curvature parameter is negative.

## 2.2. **Flatness problem**

When the students discuss the FLE, they realize that the FLE includes the above three possibilities of curvature. However, the universe cannot exhibit all three possibilities simultaneously. Using observations, see e. g. Planck collaboration (2020), the learners realize that the observed curvature is zero within the error of measurement. This fact of observation should be explained. That missing explanation is called flatness problem, see e. g. Hobson (2006, p. 418) or Guth (1981).

### 2.3. **Derivation of the squared Hubble parameter**

In order to solve the flatness problem, we plan to derive the squared Hubble parameter $H^2$ of the prototypical ball in Fig. 4 on our own. Thereby, we should derive a value of the curvature parameter. If that value is zero, then we solved the flatness problem, and we can analyse the reason for flatness. If we derive a non-zero value of the curvature parameter, then the flatness problem remains.

#### 2.3.1. **Plan of transformation**

We plan to derive the dynamics of the radius of the ball in Fig. 4 on the basis of the Schwarzschild metric in Eqs. {2-4}, as this metric has been confirmed by many observations of present-day objects, see e. g. Will (2014). Thus, we plan to transform Eqs. {2-4}.

For it, we analyse a probe mass $m_0$ that marks the radius $R$ of the ball in Fig. 4.

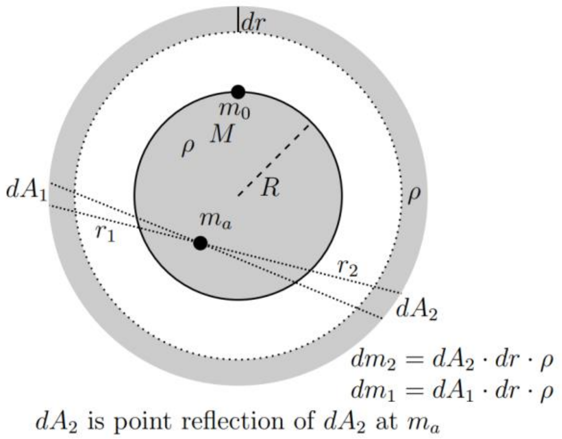


**Fig.5:** Prototypical ball in the universe. A shell of the surroundings causes forces at a test mass $m_a$ in the ball. Thereby, the forces caused by the two masses $dm_1$ and $dm_2$ cancel each other. Thus, all forces caused by the shell cancel each other at $m_a$. Hence, the forces caused by all shells cancel each other at $m_a$. Thence, the forces caused by the surroundings cancel each other at $m_a$. As the location of the test mass is arbitrary, the homogeneous and isotropic surroundings cause no field in the ball.

#### 2.3.2. **Fields of surroundings cancel in the ball**

As the surroundings of the ball are homogeneous, they do not cause any gravitational field within the ball. This fact has already been derived by Newton (1687).

The students derive it with help of the shell and the test mass $m_a$ in Fig. 5. That test mass is at an arbitrary location in the ball. An area $dA_1$ has a mass $dm_1$ and a distance $r_1$ to the test mass. The learners derive the absolute value of the force of interaction between the test mass and $dm_1$:

$$F_1 = \frac{G \cdot m_a \cdot dm_1}{r_1^2} \qquad \{6\}$$

The area $dA_2$ in Fig. 5 is the point reflection of the area $dA_1$ at the test mass. The students derive the absolute value of the force of interaction between the test mass and $dm_2$ in Fig. 5:

$$F_2 = \frac{G \cdot m_a \cdot dm_2}{r_2^2} \quad \text{with} \qquad \{7\}$$

$$dA_2 = dA_1 \cdot \frac{r_2^2}{r_1^2} \quad \text{and} \qquad \{8\}$$

$$dm_2 = dm_1 \cdot \frac{r_2^2}{r_1^2} \qquad \{9\}$$

The students combine the above Eqs. {6-9} and realize that the two forces cancel each other:

$$F_1 = F_2 \quad \text{and} \quad \vec{F}_1 = -\vec{F}_2 \qquad \{10\}$$

As the area $dA_1$ has been chosen arbitrarily, each pair of such an area and its point reflection cause forces that cancel each other. Thus, there is no remaining force or field at each such test mass. Hence, the shell in Fig. 5 does not cause any field in the ball. As the homogeneous surroundings of the ball (see Fig. 4) can be partitioned into shells similar to that in Fig. 5, the surroundings of the ball do not cause any field in the ball.

#### 2.3.3. **Field at the probe mass**

The students remind that the field at the surface of Earth is derived from the mass of Earth $M_E$ at the distance of the radius $R_E$, see e. g. Carmesin (2023a, pp 102-129):

$$G^*(R_E) = \frac{G \cdot M_E}{R_E^2} \qquad \{11\}$$

With it, they realize that the field at the surface of the ball, at the probe mass $m_0$ is derived similarly:

$$G^*(R) = \frac{G \cdot M}{R^2} \qquad \{12\}$$

With it, the conditions for the application of the derivation of the position factor in Carmesin (2023b) are fulfilled. Thus, Eqs. {2-4} can be applied .

#### 2.3.4. **Transformations with the position factor**

In order to derive the square of the Hubble constant in Eq. {5}, we plan to form the square of Eq. {2}, as a first step of the transformation planned at the beginning:

$$E^2(r,v) = m_0^2 c^4 \varepsilon^2(r) \gamma^2(v) \qquad \{13\}$$

As we are interested in the energy of the dynamics, not in the relativistic energy $E_0$ of the probe mass, we subtract the square $E_0^2$, as a second step of the transformation:

$$E^2(r,v) - E_0^2 = E_0^2(\varepsilon^2(r)\gamma^2(v) - 1) \qquad \{14\}$$

The above difference is divided by the squared available energy of the probe mass $E_0^2\gamma^2$, so that we obtain a dimensionless term scaled by the locally measurable available energy:

$$\frac{E^2(r,v) - E_0^2}{E_0^2 \gamma^2} = (\varepsilon^2(r) - \gamma^{-2}(v)) \qquad \{15\}$$

The students simplify the above scaled difference by using Eqs. {3-4}:

$$\frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2} = \frac{v^2}{c^2} - \frac{R_S}{R} \qquad \{16\}$$

According to our plan, the learners use the above scaled difference in order to derive the squared Hubble parameter. For it, they realize that the velocity $v$ is equal to the time derivative $\dot{R}$ of $R$, as the probe mass is at the radius of the ball. In order to derive the square of the Hubble parameter, the students multiply by $\frac{c^2}{R^2}$. Correspondingly, they insert for the Schwarzschild radius $R_S$ (Eq. {4}):

$$\frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2}\,\frac{c^2}{R^2} = H^2 - \frac{2GM}{R^3} \qquad \{17\}$$

The learners use the density in Fig. 4:

$$\frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2}\,\frac{c^2}{R^2} = H^2 - \frac{8\pi G}{3}\rho \qquad \{18\}$$

According to our plan, the students solve for $H^2$:

$$H^2 = \frac{8\pi G}{3}\rho + \frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2}\,\frac{c^2}{R^2} \qquad \{19\}$$

Basically, according to the principle of energy conservation, the energy $\mathrm{E}(r,v)$ is constant and equal to $E_0$, see Eq. {4}. Thus the fraction $\frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2}$ in the above equation is basically zero. Hence, the learners derive the following term for the squared Hubble parameter:

$$H^2 = \frac{8\pi G}{3}\rho \qquad \{20\}$$

The learners compare with the FLE in Eq. {5}. With it, they conclude that the curvature parameter is basically zero:

$$basically: k = 0 \qquad \{21\}$$

This result is a first basic solution of the flatness problem.

### 2.3.5. Basic principle underlying our solution

A discussion of our solution shows that we used the principle of energy conservation. This principle is applicable according to the stationarity of the field generating mass $M$ and its vicinity, according to the Noether (1918) theorem.

### 2.3.6. Generalization

In a discussion, we realize that the field generating mass is nearly constant in the matter era, see Carmesin (2020a, pp 296-301) and Carmesin (2020b). As our universe changed from the radiation era to the matter era already at a redshift of $z_{eq} = 3411$, see Planck collaboration (2020), and as it is just changing towards the vacuum era, our universe is still dominated by the matter era. Thus, our universe exhibits the curvature parameter zero within the errors of measurement.

More generally, the scaled difference $\frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2}$ in Eq. {19} is identified with the negative curvature parameter in Eq. {5}.Thus, we derive the usual FLE, and we derive a term for the curvature parameter in addition:

$$H^2 = \frac{8\pi G}{3}\rho - k\,\frac{c^2}{R^2} \quad \text{with } k = -\frac{E^2(r,v)-E_0^2}{E_0^2\gamma^2} \qquad \{22\}$$

In order to analyse the curvature parameter directly, we insert Eqs. {2-4} and cancel out $E_0^2\gamma^2$:

$$k = -\varepsilon^2(r) + \gamma^{-2} = -\varepsilon^2(r) + \varepsilon^2(r) = 0 \qquad \{23\}$$

This result represents a second solution of the flatness problem. Hereby, we consider a nonzero curvature parameter. By using energy conservation inherent to Eqs. {2-4}, we solve the flatness problem again.

In our next solution, we do not use energy conservation. Of course, we do not state energy conservation would be violated, but we do not apply energy conservation. In principle, one might think that energy could be lost at a redshift of radiation, if one does not consider a corresponding gravitational potential, for instance. Accordingly, we introduce the density of radiation and all essential densities in cosmology:

### 2.3.7. Notations in cosmology

In cosmology, dynamically essential densities are denoted as follows, see e. g. Hobson (2006) or Carmesin (2019) or Carmesin (2020a, pp 296-301): The density is a sum of the density of radiation $\rho_r$, the density of matter $\rho_m$ and the density of the cosmological constant the density of radiation $\rho_\Lambda$:

$$\rho = \rho_r + \rho_m + \rho_\Lambda \qquad \{24\}$$

In the time evolution, present-day values are marked by the subscript zero. For instance, the present-day value of the Hubble parameter is the Hubble constant $H_0 = H(t_0)$, see Fig. 2. The inverse of the Hubble constant is called Hubble time, at a good approximation, see Carmesin (2019) or Hobson (2006), it is the age of the universe:

$$t_{H_0} = \frac{1}{H_0} \qquad \{25\}$$

During the Hubble time, light travelled the light-travel distance $c \cdot t_{H_0}$, it is called Hubble radius:

$$R_{H_0} = c \cdot t_{H_0} = \frac{c}{H_0} \qquad \{26\}$$

The density of flat space is called critical density, we derive with Eq. {22}:

$$H^2 = \frac{8\pi G}{3}\rho_{cr} \quad \text{and} \quad H_0^2 = \frac{8\pi G}{3}\rho_{cr.0} \text{ or} \qquad \{27\}$$

$$\rho_{cr.0} = \frac{3H_0^2}{8\pi G} \quad \{28\}$$

The present-day curvature parameter is expressed with a density:

$$\rho_k = -\rho_{cr}\frac{c^2}{H^2R^2}k \quad \text{or} \quad \Omega_k = -\frac{c^2}{H^2R^2}k \quad \{29\}$$

The ratio of a density and the critical density is called density parameter:

$$\frac{\rho_j}{\rho_{cr}} = \Omega_\mathrm{j} \text{ and } \frac{\rho_{j,0}}{\rho_{cr.0}} = \Omega_{\mathrm{j},0} \text{ with } j \in \{r, m, \Lambda, \mathrm{k}\} \quad \{30\}$$

Einstein (1917) introduced the cosmological constant $\Lambda$. Accordingly, the corresponding density is a constant:

$$\rho_\Lambda(t) = \rho_{\Lambda,0} \quad \{31\}$$

As the volume is proportional to the third power of the radius $R^3(t)$, and as matter does not change as a consequence of expansion, the density of matter is proportional to $R^{-3}(t)$:

$$\rho_\mathrm{m}(t) = \rho_{\mathrm{m},0}\cdot\left(\frac{R}{R_{H_0}}\right)^{-3} = \frac{\rho_{\mathrm{m},0}}{a^3} \quad \{32\}$$

Hereby, we introduce the scaled radius and the redshift $z = \frac{\Delta\lambda}{\lambda}$:

$$a(t) = \frac{R(t)}{R_{H_0}} = \frac{1}{z+1} \quad \{33\}$$

As the volume is proportional to the third power of the radius $R^3(t)$, and as the energy or dynamical density of radiation changes as a consequence of expansion by the redshift proportional to $R^{-1}(t)$, the density of radiation is proportional to $R^{-4}(t)$:

$$\rho_\mathrm{r}(t) = \rho_{\mathrm{r},0}\cdot\left(\frac{R}{R_{H_0}}\right)^{-4} = \frac{\rho_{\mathrm{m},0}}{a^4} \quad \{34\}$$

Using the above definitions, the students express the density in terms of the density parameters as follows:

$$\rho = \rho_{cr.0}(\Omega_\Lambda + \Omega_{k,0}a^{-2} + \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4})\{35\}$$

And the dynamics in Eq. {22} are as follows:

$$H^2 = H_0^2(\Omega_\Lambda + \Omega_{k,0}a^{-2} + \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4})\{36\}$$

Hereby, the sum of the density parameters is one:

$$\Omega_\Lambda + \Omega_{k,0} + \Omega_{m,0} + \Omega_{r,0} = 1 \quad \{37\}$$

The Planck collaboration (2020) measured the following parameters, see the TT-mode in table 2, the abstract and Carmesin (2019) for an evaluation of $\Omega_{r,0}$:

$$\Omega_\Lambda = 0{,}679 \pm 0.013 \quad \{38\}$$

$$\Omega_{m,0} = 0.321 \pm 0.013 \quad \{39\}$$

$$\Omega_{k,0} = 0.001 \pm 0.002 \quad \{40\}$$

$$\Omega_{r,0} = 9.265\cdot 10^{-5} \pm 3.1\% \quad \{41\}$$

$$\mathrm{H}_0 = 66.88 \pm 0.92\ \frac{km}{s\cdot Mpc} \quad \{42\}$$

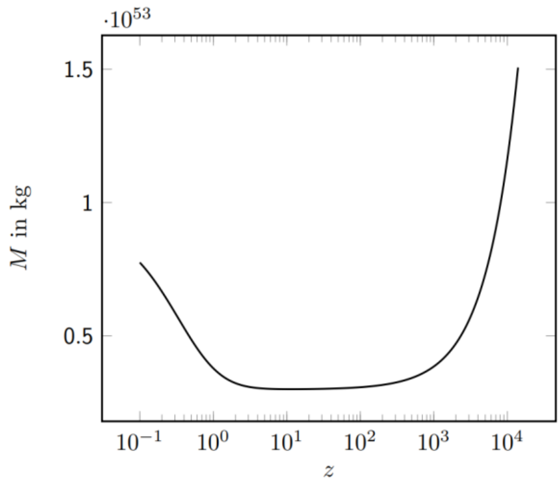


**Fig.6:** Mass or dynamical mass $M$ in the ball with present-day radius $R_{H_0}$ as a function of the redshift $z$. Thereby, the density $\rho(z)$ in Eq. {44} is a function of the redshift $z$, see Eqs. {33,35}.

### 2.3.8. **Time evolution of M**

As the only possibility for a nonzero curvature parameter is a time evolution of the field generating mass $M$ in Figs. 4 and 5 and in Eq. {4}, we analyse that time evolution.

For instance, we consider a ball with the present-day radius equal to the Hubble radius. At a scaled radius $a(t)$, that ball had the radius $a(t)$. Thus, that ball had the following mass or dynamical mass, with the density in Eq. {34}, the critical density in Eq. {28} and the parameters in Eqs. {38-42}:

$$M(a) = \rho(a)\cdot\frac{4\pi}{3}a^3\cdot R_{H_0}^3 \quad \text{or} \quad \{43\}$$

$$M(z) = \rho(z)\cdot\frac{4\pi}{3}\frac{1}{(1+z)^3}\cdot R_{H_0}^3 \quad \{44\}$$

That mass or dynamical mass in Eq. 44 as a function of the redshift is shown in Fig. 6. That function exhibits a local minimum. Thus, that (dynamical) mass $M$ is constant at the local minimum $z_{const}$, so that the density parameter of curvature in Eq. {28} is zero at that redshift:

$$\Omega_k = -\frac{c^2}{H^2R^2}k \text{ and } \Omega_k(z_{const}) = 0 \quad \{45\}$$

That result implies that the curvature is zero also at smaller redshifts $z \le z_{const}$, or at later times $t \ge t_{const}$, see Carmesin (2023c) or section 2.3.9.

### 2.3.9. **Time evolution of curvature**

We apply the method of the analysis of the curvature parameter $k$ and of the density parameter $\Omega_k$ as a function of the radius $r$ of the ball.

Firstly, at very small $r$ or in the very early universe, the radiation was blue-shifted compared to the present-day primordial radiation, for details see Carmesin (2021a). In principle, that could have caused an increased field generating mass or dynamic mass $M$ in the ball in Fig. 4. In principle, that could have caused a large positive curvature, as proposed by Hobson (2006, p. 417) or by Guth (1981). This could be the case even if that effect was made smaller by the era of cosmic inflation, see e. g. Guth (1981), Carmesin (2019). We will show that such possible values of curvature in the early universe are not essential for the curvature in the present-day universe:

Secondly, the curvature is zero at $z_{const}$ or at the time $t_{const}$, see Eq. {45}.

Thirdly, the density parameter $\Omega_k$ in Eq. {45} is a function of time. With it and the FLE, the time derivative of that density parameter can be derived, see e. g. Hobson (2006, Eq. 15.48) or Carmesin (2023c):

$$\frac{d\Omega_k}{dt} = \Omega_k[H(\Omega_m + 2\Omega_r - 2\Omega_\Lambda)] \qquad \{46\}$$

As the term in the rectangular bracket does not become infinite at redshifts $z \leq z_{const}$, or at times $t \geq t_{const}$, that bracket is limited by its maximum and by its minimum:

$$[H(\Omega_m + 2\Omega_r - 2\Omega_\Lambda)] \leq B_{max} \qquad \{47\}$$

$$[H(\Omega_m + 2\Omega_r - 2\Omega_\Lambda)] \geq B_{min} \qquad \{48\}$$

With it, we derive an upper limit $\Omega_{k,upper}$ and a lower limit $\Omega_{k,lower}$ of the density parameter as follows:

$$\Omega_{k,lower} \leq \Omega_k \leq \Omega_{k,upper} \qquad \{49\}$$

$$\frac{d\Omega_{k,upper}}{dt} = \Omega_{k,upper} \cdot B_{max} \qquad \{50\}$$

$$\frac{d\Omega_{k,lower}}{dt} = \Omega_{k,lower} \cdot B_{min} \qquad \{51\}$$

The solutions are exponential functions with the initial value $\Omega_k(z_{const})$:

$$\Omega_{k,upper} = \Omega_k(z_{const}) \cdot e^{B_{max}\cdot(t-t_{const})} = 0 \quad \{52\}$$

$$\Omega_{k,lower} = \Omega_k(z_{const}) \cdot e^{B_{min}\cdot(t-t_{const})} = 0 \quad \{53\}$$

As both functions are zero, and as the density parameter is limited by these, see Eq. {49}, the density parameter is zero at times $t \geq t_{const}$, or at redshifts $z \leq z_{const}$:

$$\Omega_k = 0 \text{ for } t \geq t_{const} \text{ or } z \leq z_{const} \qquad \{54\}$$

Altogether, we derived that the curvature is zero at the present-day universe. Thus, we solved the flatness problem.

## 3. Didactic analysis

We provide three solutions of the flatness problem, in order to cover a wide range of discussed physical situations, proposed for instance by Hobson (2006, p. 417) or Guth (1981).

### 3.1. Didactic steps in the first solution

In a first didactic step, the students realize in a discussion that the solution of the curvature problem provided by general relativity and the FLE includes the observed value $k = 0$ of the curvature parameter as a possibility. However, these theories do not predict the observed value. The students realize that there is an element missing in these theories. Moreover, they realize that such a missing explanation of the observed flatness ($k = 0$) of global space is a problem of the theory. They understand that it makes sense to identify the problem and to call it flatness problem. In this manner, the students accept the cognitive conflict and are motivated to solve it in the course. This didactic step does not provide any technical learning barrier. So the step is directly executed in a discussion.

In a second didactic step, during a discussion, the students plan that we derive the squared Hubble parameter on our own, in order to analyse or solve the flatness problem. The learners realize that the Schwarzschild metric is an ideal starting point, as it is confirmed by many experiments, as the local dynamics should explain the global dynamics, similarly as the molecules in a gas explain the universal gas equation via the kinetic gas theory. Moreover, the students feel confident, as they did already derive the Schwarzschild metric with help of the free fall tower, see Carmesin (2023b). This didactic step does not provide any technical learning barrier. So the step is directly executed in a discussion.

In a third didactic step, we show that there is no field in the prototypical ball. Hereby, the mathematical barrier is high. That barrier consists of two parts: the idea of the separation of caused fields as shown in Fig. 5, and the analysis on the basis of that idea. The first part of the barrier is very high, so Fig. 5 and the idea are presented to the learners.

In the didactic step four, the field at the probe mass is derived. Hereby, the learners can activate their knowledge about the field at Earth. Thus the learning barrier is intermediate. Accordingly, the step is directly executed in a discussion.

In a fifth didactic step, we transform the energy function $E(r, v)$ so that we derive the squared Hubble parameter. As we know the desired product of the transformation, we can plan each of the steps of the transformation. At each step, the learning mathematical learning barrier is relatively low, as only simple equivalent transformations are required. Each equivalent transformation is planned in a short discussion, then everybody can execute it, finally, the result is reflected in a short discussion. In this manner, we achieve the first solution of the flatness problem, see Eq. {21}.

### 3.2. Didactic steps in the second solution

In the didactic step six, we identify the term of the curvature parameter in Eq. {22} and evaluate it in Eq. {23}. This provides the second solution of the flatness problem. Thereby, there occurs no essential learning barrier. So the step is directly executed in a discussion. Hereby, the learners should realize that energy conservation is used again.

### 3.3. Didactic steps in the third solution

In the seventh didactic step, the mass or dynamic mass is calculated as a function of time. That process is straight forward, but time consuming. Accordingly, the result in Fig. 6 can be presented directly. Then the students can identify the minimum, with slope zero, the corresponding energy conservation and the implied flatness at the time or redshift of the minimum.

Each of these arguments has a low barrier, however, the chain of arguments is complex. Thus, this step is at best achieved in an interactive discussion.

In the didactic step eight, the differential equation {46} is presented and discussed. Thereby, the plan to introduce an upper and lower bound is introduced in an interactive manner. These bounds can be derived by the learners. Hereby, the mathematical barrier is high. So that solution and the conclusion can be derived interactively as well. Thus, the third solution of the flatness problem is derived. Hereby, energy conservation has not been used.

## 4. Experiences with teaching

I used this solution of the flatness problem in several courses in the framework of a research club. Moreover I used it in several courses in general studies at the university.

In all groups, we used and needed the preconditions of the learning process. Thereby, we use the transformation as the start of the formation of the theory of the Big Bang. Accordingly, the learners have an especially high motivation, as they want to understand how the dynamics of the Big Bang works. Thereby, the first solution of the flatness problem arises as a by-product of the theory. The cognitive conflict about the curvature provides an additional motivation, of course. As the transformation has no essential mathematical barrier, all students can explain the transformation and interpret the results, including the first solution of the flatness problem. The second solution provides a term for the curvature constant as another by-product, and the derivation of its value is easy.

The third solution provides the introduction of the densities as a by-product. These densities are essential for the following treatment of dark energy and the $H_0$ tension, see e. g. Carmesin (2018, 2019, 2021a,b, 2022a, 2023c). As the students know that application of the densities, there is an additional high motivation hereby. So far, I treated the introduction and solution of the differential equations only in the form of a discussion, whereby I provided the derivation in written form, in order to save time. The students appeared very interested also in that topic.

## 5. Discussion

Spacetime and its curvature are very interesting to many students, as space and time are very fundamental concepts. Moreover, the dynamics of the Big Bang is very motivating to learners.

In the proposed learning process, the flatness problem serves as a cognitive conflict. This is especially motivating. Accordingly, the students participate in the solution at various levels: They can derive the dynamics of the Big Bang on their own, if they are supported by common phases of planning the steps and interpreting the results. Thus, the learning process provides an experience of competence and self-esteem.

Moreover, the learning process shows how the local dynamics of the Schwarzschild metric implies the global dynamics of the expansion of space since the Big Bang. Furthermore, the learning process provides clear insights into the role of energy conservation and global curvature of spacetime, see e. g. Carmesin (2019, 2021c). The students can experience, how energy conservation can solve the flatness problem. Additionally, they can realize that it suffices to identify a minimum of dynamic mass or energy in Fig. 6, in order to derive and explain the flatness problem.

Moreover, all results are derived from first principles in an exact manner. So, the new results are fully connected with previous knowledge, and a high learning efficiency is achieved, see Hattie (2009). Furthermore, the results explain observations, partially obtained by the learners, see Figs. 1-3, such explanations of observations also provide a high learning efficiency, see Hattie (2009).

I tested the learning process in several learning groups several times. I showed that most learning barriers are low. Moreover, the results are exact, very elucidating, rich in content and useful for many topics of the Big Bang, spacetime and dark energy. Thereby, the theories of general relativity and of the FLE are already very good, however, the solution of the flatness problem is beyond these theories. Similarly, the theories of general relativity and of the FLE describe the evolution of spacetime already in a very good manner, however, the prepared derivation and explanation of dark energy, see Carmesin (2018, 2019, 2021a,b,d 2022a, 2023c) is beyond these theories.

The learning process is very robust and transparent. I provide a description of the learning process that can be used directly for teaching.

## 6. Literature

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