Solving Systems of Linear Equations with Dirac Algebra

Worksheets 7, 9, and 11 of the course
“Advanced Mathematics (MQM110)”
of the Master program “Computer Science”
at iubh – Internationale Hochschule,
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Conceptual Background

One of the goals of this advanced mathematics course of computer science I have in mind is to discuss Conformal Geometric Algebra (CGA) with the students one day.

To be prepared for this discussion, the students should not only know how to deal with Pauli algebra, but they should also understand the basics of Dirac algebra. Dirac algebra describes the spacetime of extended special relativity. But this spacetime of extended special relativity with one time-like dimension and four space-like dimensions is identical to the five-dimensional space of Conformal Geometric Algebra.

Therefore problems which had been solved earlier with the help of Pauli algebra

> Martin Erik Horn (2018): Modern Linear Algebra: Geometric Algebra with GAALOP. Worksheets of the module „Mathematics for Business and Economics" of joint first-year bachelor lessons at Berlin School of Economics and Law/Hochschule für Wirtschaft und Recht Berlin, LV-Nr. 200691.01 & 400 691.01, Stand: 07. Jan. 2018. Wintersemester 2017/2018, BSEL/HWR Berlin,

and which can be downloaded at URL [20.12.2018]:

www.phydid.de/index.php/phydid-b/rt/suppFiles/881/0
www.phydid.de/index.php/phydid-b/article/downloadSuppFile/881/210
www.phydid.de/index.php/phydid-b/article/downloadSuppFile/881/Worksheets%20GAALOP

had been rewritten in the language of Dirac algebra. The new problems are identical to the problems of the older worksheets, but the solution strategy is different: The linear equations now are translated into vector equations of spacetime algebra (STA) making it again possible to find the solution values simply by comparing outer products.

Spacetime Generalized Matrix Inverses

Practice is everything. To get a good working knowledge of Dirac algebra, students should practice Dirac algebra. To get this practice, problems about the mathematics of Pauli algebra generalized matrix inverses are restated as problems of Dirac algebra generalized matrix inverses in worksheet 11.

The conceptual background of Pauli algebra generalized matrix inverses is simple: The scalar part of a Pauli algebra generalized matrix inverse is identical to the Moore-Penrose generalized matrix inverse, which can be discussed even with first-year students, e.g. see the business math slides

> Martin Erik Horn (2017): Modern Linear Algebra. A Geometric Algebra Crash Course. Part VII: Generalized Matrix Inverses. OHP slide of the course “Mathematics for Business and Economics““ of joint first-year bachelor lessons at Berlin School of Economics and Law/Hochschule für Wirtschaft und Recht Berlin, LV-Nr. 200691.01 & 400 691. 01, Stand: 19. Dez. 2017. Wintersemester 2017/2018, BSEL/HWR Berlin,

which can be downloaded at URL [20.12.2018]:

www.phydid.de/index.php/phydid-b/rt/suppFiles/851/0
www.phydid.de/index.php/phydid-b/article/downloadSuppFile/851/204
www.phydid.de/index.php/phydid-b/article/downloadSuppFile/851/OHP-Folien

In a similar way, the scalar part of a Dirac algebra generalized matrix inverse is identical to the spacetime matrix inverse, which follows modified Moore-Penrose conditions.

By the way: You will not find much about Dirac algebra generalized matrix inverses or spacetime generalized matrix inverses in the literature. I have invented them only a short time ago.

M. E. H.

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Advanced Mathematics (MQM110)

Worksheet 7 – Exercises

Problem 1:

Find the areas of the parallelograms if the two different sides are given by the following spacetime vectors and if the two base vectors γ_t and γ_x have lengths $|\gamma_t|$ and $|\gamma_x|$ of 1 cm. Please also draw a sketch of these spacetime parallelograms.

a) $\mathbf{a} = 5\gamma_t + 2\gamma_x$, $\mathbf{b} = 2\gamma_t + 6\gamma_x$

b) $\mathbf{a} = 8\gamma_t + 7\gamma_x$, $\mathbf{b} = 2\gamma_t + 20\gamma_x$

c) $\mathbf{a} = 5\gamma_t - 5\gamma_x$, $\mathbf{b} = 3\gamma_t + 7\gamma_x$

d) $\mathbf{a} = 4\gamma_t + 16\gamma_x$, $\mathbf{b} = 9\gamma_t + 2\gamma_x$

Problem 2:

Find the areas of the parallelograms if the two different sides are given by the following spacetime vectors and if the two base vectors γ_t and γ_x have lengths $|\gamma_t|$ and $|\gamma_x|$ of 1 cm. Please also draw a sketch of these spacetime parallelograms if possible and find the precise names of the given spacetime parallelograms.

a) $\mathbf{a} = 6\gamma_t + 4\gamma_x$, $\mathbf{b} = -4\gamma_t + 6\gamma_x$

b) $\mathbf{a} = -4.8\gamma_t - 3.4\gamma_x$, $\mathbf{b} = -5.1\gamma_t + 7.2\gamma_x$

c) $\mathbf{a} = 4\gamma_t + 3\gamma_x$, $\mathbf{b} = 12\gamma_t + 9\gamma_x$

d) $\mathbf{a} = 5\gamma_t + 20\gamma_x$, $\mathbf{b} = -\gamma_t - 4\gamma_x$

e) $\mathbf{a} = 6\gamma_t + 4\gamma_x$, $\mathbf{b} = 4\gamma_t + 6\gamma_x$

f) $\mathbf{a} = -4.8\gamma_t - 3.4\gamma_x$, $\mathbf{b} = 5.1\gamma_t + 7.2\gamma_x$

g) $\mathbf{a} = 4\gamma_t + 4\gamma_x$, $\mathbf{b} = 12\gamma_t + 9\gamma_x$

h) $\mathbf{a} = 8\gamma_t + 8\gamma_x$, $\mathbf{b} = -5\gamma_t + 5\gamma_x$

Problem 3:

Now please draw the sketches of the spacetime parallelograms of problems 1 & 2 into a coordinate system with a ct-axis, which points upwards, and an x-axis, which points to the right (called Minkowski diagram). Find the times, which are given by the γ_t-components.

Problem 4:

Find the squares of all spacetime vectors **a** and **b** of problems 1 & 2 and determine, whether they are space-like, time-like or light-like.

Find the magnitude squares of the spacetime parallelograms $\mathbf{a}\mathbf{b}$ of problems 1 & 2 and show that they are equal to the product of the squares $\mathbf{a}^2\mathbf{b}^2$ of the spacetime vectors.

Problem 5:

Solve the following systems of linear equations by using Dirac Algebra and check your results.

a) $3x + 8y = 28$, $6x + 2y = 28$

b) $4x + 9y = 29$, $5x + 6y = 31$

c) $6x + 4y = 6$, $2x + y = 3$

d) $5x - 2y = 6$, $-2x - 3y = 28$

Problem 6:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

3 units of R_1	and	6 units of R_2	to produce	1 unit of P_1
8 units of R_1	and	2 units of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 28 units of the first raw material R_1 and 28 units of the second raw material R_2 are consumed in the production process by using Dirac algebra. (Hint: The result of problem 3a) can be used.)

Problem 7:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

2 units of R_1	and	5 units of R_2	to produce	1 unit of P_1
7 units of R_1	and	1 unit of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 2050 units of the first raw material R_1 and 1000 units of the second raw material R_2 are consumed in the production process by using Dirac algebra.

Problem 8:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

4 units of R_1	and	1 unit of R_2	to produce	1 unit of P_1
3 units of R_1	and	5 units of R_2	to produce	1 unit of P_2

In the first quarter of a year exactly 33000 units of the first raw material R_1 and 38000 units of the second raw material R_2 are consumed in the production process. In the second quarter exactly 32000 units of the first raw material R_1 and 25000 units of the second raw material R_2 are consumed in the production process.

Find the quantities of final products P_1 and P_2 which will be produced in the first quarter, and find the quantities of final products P_1 and P_2 which will be produced in the second quarter by using Dirac algebra.

Problem 9:

A firm manufactures two different final products P_1 and P_2. To produce these final products two intermediate goods G_1 and G_2 are required. The production of the intermediate goods requires two different raw materials R_1 and R_2. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2
R_1	8	2
R_2	4	3

	P_1	P_2
R_1	42	28
R_2	23	26

Find the demand matrix of the second production step which shows the demand of intermediate goods to produce one unit of each final product by using Dirac algebra.

Problem 10:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products two intermediate goods G_1 and G_2 are required. The production of the intermediate goods requires two different raw materials R_1 and R_2. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2
R_1	9	3
R_2	2	2

	P_1	P_2	P_3
R_1	48	21	84
R_2	12	14	32

Find the demand matrix of the second production step which shows the demand of intermediate goods to produce one unit of each final product by using Dirac algebra.

Problem 11:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

7 units of R_1	and	4 units of R_2	to produce	1 unit of P_1
5 units of R_1	and	3 units of R_2	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the first raw material R_1 had been consumed in the production process.
And find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the second raw material R_2 had been consumed in the production process.

How can these results be understood?
Give an economic interpretation of the results.

Problem 12:

A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of two different raw materials R_1 and R_2 are required:

10 units of R_1	and	4 units of R_2	to produce	1 unit of P_1
12 units of R_1	and	5 units of R_2	to produce	1 unit of P_2

Please use Dirac algebra to find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the first raw material R_1 had been consumed in the production process.

And find the quantities of final products P_1 and P_2 which would have been produced in theory, if exactly one unit of the second raw material R_2 had been consumed in the production process by using Dirac algebra.

Find the inverse of the demand matrix and check your result.

Problem 13:

Find the inverses of the following matrices by using Dirac algebra and check your results.

a) $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$ b) $\mathbf{B} = \begin{bmatrix} 10 & 4 \\ 19 & 8 \end{bmatrix}$ c) $\mathbf{C} = \begin{bmatrix} 10 & 6 \\ 20 & 13 \end{bmatrix}$ d) $\mathbf{D} = \begin{bmatrix} 0 & -2.5 \\ 0.2 & 3.4 \end{bmatrix}$

Problem 14:

Why do we discuss Dirac algebra?
Why do we discuss Minkowski space?

Please have a look at the book of computer scientist and digital media specialist John Vince

> John Vince: Geometric Algebra for Computer Graphics. Springer-Verlag, London 2008,

search for the target word "Minkowski", and find out which topic of computer science is based mathematically on the ideas of Dirac, Minkowski – and of Einstein.

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Advanced Mathematics (MQM110)

Worksheet 9 – Exercises

Problem 1:

Find the volume of the parallelepipeds by direct calculation using Dirac algebra if the three different sides of the parallelepipeds are given by the following spacetime vectors and if the base vectors γ_t, γ_x, and γ_z have lengths $|\gamma_t|$, $|\gamma_x|$, and $|\gamma_y|$ of 1 cm.

a) $\mathbf{a} = 4\,\gamma_x + 2\,\gamma_y$, $\mathbf{b} = 2\,\gamma_x + 4\,\gamma_y$, $\mathbf{c} = 3\,\gamma_t$

b) $\mathbf{a} = 4\,\gamma_x + 2\,\gamma_y$, $\mathbf{b} = 2\,\gamma_x + 4\,\gamma_y$, $\mathbf{c} = 5\,\gamma_t + 5\,\gamma_y$

c) $\mathbf{a} = 4\,\gamma_x + 2\,\gamma_y$, $\mathbf{b} = 2\,\gamma_x + 4\,\gamma_y$, $\mathbf{c} = 7\,\gamma_t + 7\,\gamma_x + 7\,\gamma_y$

d) $\mathbf{a} = 5\,\gamma_t + 2\,\gamma_x + 5\,\gamma_y$, $\mathbf{b} = 6\,\gamma_t + 3\,\gamma_x + 3\,\gamma_y$, $\mathbf{c} = 4\,\gamma_t + 4\,\gamma_x + 4\,\gamma_y$

e) $\mathbf{a} = 10\,\gamma_t + 2\,\gamma_x + 6\,\gamma_y$, $\mathbf{b} = 12\,\gamma_t + 8\,\gamma_x + 3\,\gamma_y$, $\mathbf{c} = 4\,\gamma_t + 7\,\gamma_x + 9\,\gamma_y$

f) $\mathbf{a} = -5\,\gamma_t + 4\,\gamma_x + 8\,\gamma_y$, $\mathbf{b} = 6\,\gamma_t + 3\,\gamma_x - 7\,\gamma_y$, $\mathbf{c} = -\gamma_t - 2\,\gamma_x + 9\,\gamma_y$

Please also draw a sketch of these spacetime parallelepipeds of the first three exercise parts a), b), and c) and compare the results of the outer product volumes with the determinants of the coefficient matrices. Please also check your multiplication results by comparing the magnitude squares of the spacetime vectors and the spacetime parallelepipeds.

Problem 2:

Solve the following systems of linear equations by using Dirac algebra and check your results.

a)
$$\begin{aligned} 3x + 8y &= 28 \\ 6x + 2y &= 28 \\ 2x + 4y + 2z &= 28 \end{aligned}$$

b)
$$\begin{aligned} 8x + 5y + 10z &= 396 \\ 3x + 7y + 12z &= 375 \\ 2x + 6y + 14z &= 386 \end{aligned}$$

c)
$$\begin{aligned} 3x - 5y + 6z &= 41 \\ -2x + 5y + 8z &= 111 \\ 7x + y + 9z &= 185 \end{aligned}$$

d)
$$\begin{aligned} \tfrac{2}{5}x + \tfrac{7}{5}y + \tfrac{9}{5}z &= 210 \\ \tfrac{8}{5}x + \tfrac{1}{5}y + \tfrac{3}{5}z &= 138 \\ \tfrac{4}{5}x + \tfrac{12}{5}y + \tfrac{6}{5}z &= 282 \end{aligned}$$

Problem 3:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

7 units of R_1,	3 units of R_2,	and	4 units of R_3	to produce	1 unit of P_1
2 units of R_1,	9 units of R_2,	and	6 units of R_3	to produce	1 unit of P_2
5 units of R_1,		and	8 units of R_3	to produce	1 unit of P_3

Exactly 500 units of the first raw material R_1, 780 units of the second raw material R_2, and 880 units of the third raw material R_3 are consumed in the production process.

Find the output of final products P_1, P_2, and P_3 by using Dirac algebra.

Problem 4:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

12 units of R_1,	20 units of R_2,	and	16 units of R_3	to produce	1 unit of P_1
30 units of R_1,	15 units of R_2,	and	28 units of R_3	to produce	1 unit of P_2
10 units of R_1,	8 units of R_2,	and	25 units of R_3	to produce	1 unit of P_3

Exactly 12000 units of the first raw material R_1, 13900 units of the second raw material R_2, and 18300 units of the third raw material R_3 are consumed in the production process.

Find the output of final products P_1, P_2, and P_3 by using Dirac algebra.

Problem 5:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

9 units of R_1,	2 units of R_2,	and	7 units of R_3	to produce	1 unit of P_1
3 units of R_1,	2 units of R_2,	and	5 units of R_3	to produce	1 unit of P_2
4 units of R_1,	3 units of R_2,	and	2 units of R_3	to produce	1 unit of P_3

In the first quarter of a year exactly 98 units of the first raw material R_1, 35 units of the second raw material R_2, and 76 units of the third raw material R_3 are consumed in the production process.

In the second quarter exactly 61 units of the first raw material R_1, 30 units of the second raw material R_2, and 59 units of the third raw material R_3 are consumed in the production process.

Find the quantities of final products P_1, P_2, and P_3, which will be produced in the first quarter, and find the quantities of final products P_1, P_2, and P_3, which will be produced in the second quarter, by using Dirac algebra.

Problem 6:

A firm manufactures two different final products P_1 and P_2. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three different raw materials R_1, R_2, and R_3. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2	G_3
R_1	10	15	11
R_2	17	20	16
R_3	12	14	25

	P_1	P_2
R_1	964	814
R_2	1409	1184
R_3	1320	1093

Find the demand matrix of the second production step, which shows the demand of intermediate goods to produce one unit of each final product, by using Dirac algebra.

Problem 7:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three different raw materials R_1, R_2, and R_3. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2	G_3
R_1	8	6	6
R_2	7	5	7
R_3	5	4	0

	P_1	P_2	P_3
R_1	228	186	308
R_2	214	166	282
R_3	108	107	160

Find the demand matrix of the second production step, which shows the demand of intermediate goods to produce one unit of each final product, by using Dirac algebra.

Problem 8:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these final products three intermediate goods G_1, G_2, and G_3 are required. The production of the intermediate goods requires three different raw materials R_1, R_2, and R_3. The demand of raw materials to produce one unit of the intermediate goods and the total demand of raw materials to produce one unit of the final products is shown in the following tables:

	G_1	G_2	G_3
R_1	82	63	20
R_2	44	19	37
R_3	10	52	92

	P_1	P_2	P_3
R_1	4496	5462	4815
R_2	2530	3482	2801
R_3	3224	4062	4646

Find the demand matrix of the second production step, which shows the demand of intermediate goods to produce one unit of each final product, by using Dirac algebra.

Problem 9:

A firm manufactures three different final products P_1, P_2, and P_3. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

3 units of R_1,	2 units of R_2,	and	8 units of R_3	to produce	1 unit of P_1
5 units of R_1,	6 units of R_2,	and	7 units of R_3	to produce	1 unit of P_2
4 units of R_1,	3 units of R_2,	and	10 units of R_3	to produce	1 unit of P_3

Find the quantities of final products P_1, P_2, and P_3 which would have been produced in theory, if exactly one unit of the first raw material R_1 had been consumed in the production process, by using Dirac algebra.

Find the quantities of final products P_1, P_2, and P_3 which would have been produced in theory, if exactly one unit of the second raw material R_2 had been consumed in the production process, by using Dirac algebra.

And find the quantities of final products P_1, P_2, and P_3 which would have been produced in theory, if exactly one unit of the third raw material R_3 had been consumed in the production process, by using Dirac algebra.

Use the values just found to construct the inverse of the demand matrix and check your result.

Problem 10:

Find the inverses of the following matrices (if they exist) by using Dirac algebra and check your results.

a) $\mathbf{A} = \begin{bmatrix} 1 & 4 & 9 \\ 7 & 2 & 6 \\ 6 & 3 & 8 \end{bmatrix}$

b) $\mathbf{B} = \begin{bmatrix} 0 & 4 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

c) $\mathbf{C} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

d) $\mathbf{D} = \begin{bmatrix} 3 & 4 & 8 \\ 10 & 5 & 10 \\ 10 & 20 & 15 \end{bmatrix}$

Problem 11:

a) A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

5 units of R_1,	4 units of R_2,	and	3 units of R_3	to produce	1 unit of P_1
			2 units of R_3	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, by using Dirac algebra.

b) A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

5 units of R_1,	4 units of R_2,	and	3 units of R_3	to produce	1 unit of P_1
6 units of R_1,	7 units of R_2,	and	8 units of R_3	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, by using Dirac algebra.

As there are more linear equations than variable now, the systems of linear equations are over-constrained. But these three equations of a) and b) are consistent, and thus solutions will exist.

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Advanced Mathematics (MQM110)

Worksheet 11 – Exercises

Problem 1:

This year (2020) the British black hole scientist Roger Penrose was awarded the Noble Prize of physics together with Reinhard Genzel and Andrea Ghez. As Penrose is not only a physicists, but also a mathematician, this year's Noble Prize of Physics has partly become a Noble prize of mathematics. For example, Roger Penrose has reinvented the Moore-Penrose matrix inverse half a century ago.

Please have a look at the history of the Moore-Penrose matrix inverse at your math books or at the internet.

Problem 2:

Repeat and rethink the solution strategies of the following problem of previous worksheets, already solved twice there:

a) A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

5 units of R_1,	4 units of R_2,	and	3 units of R_3	to produce	1 unit of P_1
			2 units of R_3	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process.

b) A firm manufactures two different final products P_1 and P_2. To produce these products the following quantities of three different raw materials R_1, R_2, and R_3 are required:

5 units of R_1,	4 units of R_2,	and	3 units of R_3	to produce	1 unit of P_1
6 units of R_1,	7 units of R_2,	and	8 units of R_3	to produce	1 unit of P_2

Find the quantities of final products P_1 and P_2 which will be produced, if exactly 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process.

As there are more linear equations than variables now, the systems of linear equations are overconstrained. But these three equations of a) and b) are consistent, and thus solutions will exist.

Problem 3:

Construct the left-sided, non-square Pauli algebra generalized matrix inverses of the demand matrices of problem 2 by pre-multiplying the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ from the left and solve problem 2 with the help of these Pauli algebra generalized matrix inverses.

Problem 4:

Construct the left-sided, non-square Pauli algebra generalized matrix inverses of the demand matrices of problem 2 by post-multiplying the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ from the right and solve problem 2 with the help of these Pauli algebra generalized matrix inverses.

Problem 5:

Construct the left-sided, non-square Dirac algebra generalized matrix inverses of the demand matrices of problem 2 by pre-multiplying the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ from the left and solve problem 2 with the help of these Dirac algebra generalized matrix inverses.

Problem 6:

Construct the left-sided, non-square Dirac algebra generalized matrix inverses of the demand matrices of problem 2 by post-multiplying the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ from the right and solve problem 2 with the help of these Dirac algebra generalized matrix inverses.

Problem 7:

Construct the Moore-Penrose generalized matrix inverses of the demand matrices of problem 2 and check the Moore-Penrose conditions.

Then solve problem 2 with the help of these Moore-Penrose generalized matrix inverses.

Problem 8:

Construct spacetime generalized matrix inverses of the demand matrices of problem 2 with the help of the Dirac algebra generalized matrix inverses of problems 5 & 6 and check the Moore-Penrose conditions.

Try to find new conditions which describe the mathematics of these spacetime generalized matrix inverses.

Then solve problem 2 with the help of these spacetime generalized matrix inverses.

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Advanced Mathematics (MQM110)

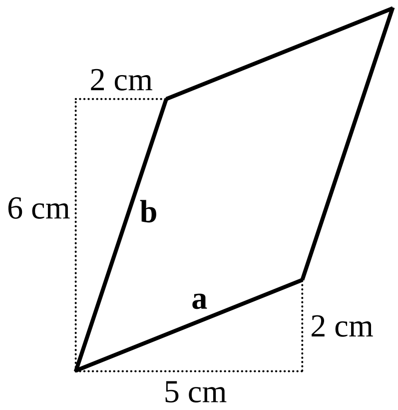
Worksheet 7 – Answers

Problem 1:

a) $\mathbf{a} = 5\,\gamma_t + 2\,\gamma_x$

$\mathbf{b} = 2\,\gamma_t + 6\,\gamma_x$

Sketch:



$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (5\,\gamma_t + 2\,\gamma_x)\,(2\,\gamma_t + 6\,\gamma_x)\\ &= 5\cdot 2\,\gamma_t^2 + 5\cdot 6\,\gamma_t\gamma_x + 2\cdot 2\,\gamma_x\gamma_t + 2\cdot 6\,\gamma_x^2\\ &= 10\,\gamma_t^2 + 30\,\gamma_t\gamma_x + 4\,\gamma_x\gamma_t + 12\,\gamma_x^2\\ &= 10\cdot 1 + 30\,\gamma_t\gamma_x + 4\,(-\,\gamma_t\gamma_x) + 12\cdot(-\,1)\\ &= 10 + 30\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_x - 12\\ &= -\,2 + 26\,\gamma_t\gamma_x\end{aligned}$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 26\,\gamma_t\gamma_x$

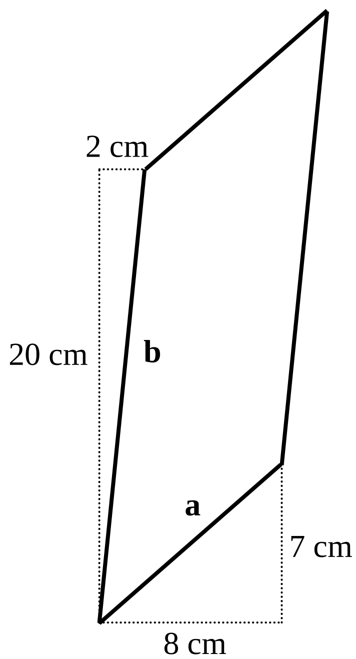
$\Rightarrow$ $|\mathbf{A}| = 26$

$\Rightarrow$ The area of the spacetime parallelogram is 26 cm^2.

b) $\mathbf{a} = 8\,\gamma_t + 7\,\gamma_x$

$\mathbf{b} = 2\,\gamma_t + 20\,\gamma_x$

Sketch:



$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (8\,\gamma_t + 7\,\gamma_x)\,(2\,\gamma_t + 20\,\gamma_x)\\ &= 8\cdot 2\,\gamma_t^2 + 8\cdot 20\,\gamma_t\gamma_x + 7\cdot 2\,\gamma_x\gamma_t + 7\cdot 20\,\gamma_x^2\\ &= 16\,\gamma_t^2 + 160\,\gamma_t\gamma_x + 14\,\gamma_x\gamma_t + 140\,\gamma_x^2\\ &= 16\cdot 1 + 160\,\gamma_t\gamma_x + 14\,(-\,\gamma_t\gamma_x) + 140\cdot(-\,1)\\ &= 16 + 160\,\gamma_t\gamma_x - 14\,\gamma_t\gamma_x - 140\\ &= -\,124 + 146\,\gamma_t\gamma_x\end{aligned}$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 146\,\gamma_t\gamma_x$

$\Rightarrow$ $|\mathbf{A}| = 146$

$\Rightarrow$ The area of the spacetime parallelogram is 146 cm^2.

c) $\mathbf{a} = 5\,\gamma_t - 5\,\gamma_x$

$\mathbf{b} = 3\,\gamma_t + 7\,\gamma_x$

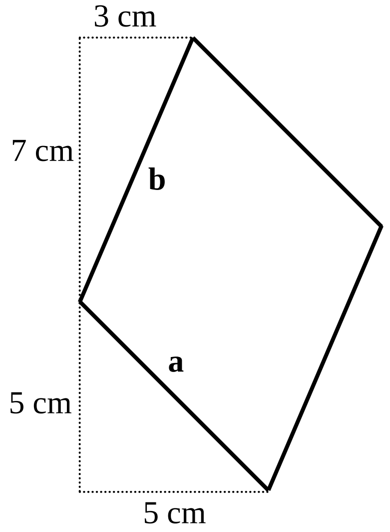
$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (5\,\gamma_t - 5\,\gamma_x)\,(3\,\gamma_t + 7\,\gamma_x)\\ &= 5\cdot 3\,\gamma_t^2 + 5\cdot 7\,\gamma_t\gamma_x - 5\cdot 3\,\gamma_x\gamma_t - 5\cdot 7\,\gamma_x^2\\ &= 15\,\gamma_t^2 + 35\,\gamma_t\gamma_x - 15\,\gamma_x\gamma_t - 35\,\gamma_x^2\\ &= 15\cdot 1 + 35\,\gamma_t\gamma_x - 15\,(-\gamma_t\gamma_x) - 35\cdot(-1)\\ &= 15 + 35\,\gamma_t\gamma_x + 15\,\gamma_t\gamma_x + 35\\ &= 50 + 50\,\gamma_t\gamma_x\end{aligned}$$

$\Rightarrow\ \mathbf{a}\wedge\mathbf{b} = 50\,\gamma_t\gamma_x$

$\Rightarrow\ |\mathbf{A}| = 50$

$\Rightarrow$ The area of the spacetime parallelogram is 50 cm^2.

Sketch:



d) $\mathbf{a} = 4\,\gamma_t + 16\,\gamma_x$

$\mathbf{b} = 9\,\gamma_t + 2\,\gamma_x$

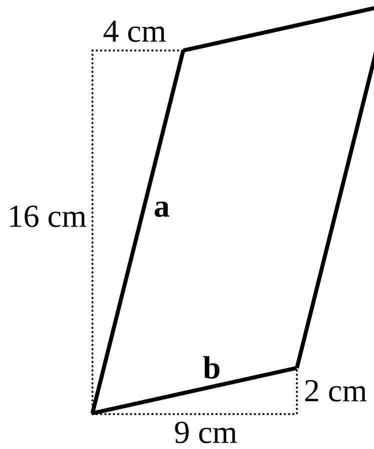
$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (4\,\gamma_t + 16\,\gamma_x)\,(9\,\gamma_t + 2\,\gamma_x)\\ &= 4\cdot 9\,\gamma_t^2 + 4\cdot 2\,\gamma_t\gamma_x + 16\cdot 9\,\gamma_x\gamma_t + 16\cdot 2\,\gamma_x^2\\ &= 36\,\gamma_t^2 + 8\,\gamma_t\gamma_x + 144\,\gamma_x\gamma_t + 32\,\gamma_x^2\\ &= 36\cdot 1 + 8\,\gamma_t\gamma_x + 144\,(-\gamma_t\gamma_x) + 32\cdot(-1)\\ &= 36 + 8\,\gamma_t\gamma_x - 144\,\gamma_t\gamma_x - 32\\ &= 4 - 136\,\gamma_t\gamma_x\end{aligned}$$

$\Rightarrow\ \mathbf{a}\wedge\mathbf{b} = -136\,\gamma_t\gamma_x$

$\Rightarrow\ |\mathbf{A}| = 136$

$\Rightarrow$ The area of the spacetime parallelogram is 136 cm^2.

Sketch:



Problem 2:

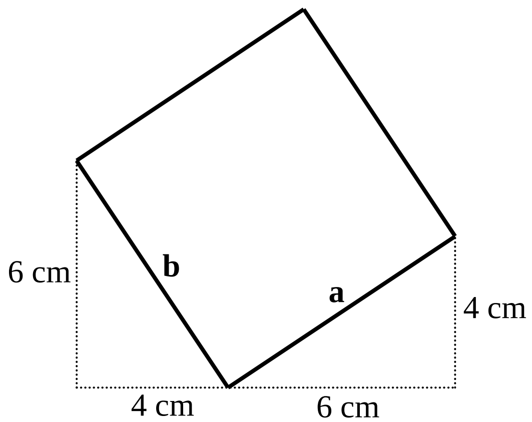
a) $\mathbf{a} = 6\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = -4\,\gamma_t + 6\,\gamma_x$

$$\begin{aligned}\mathbf{a}\,\mathbf{b} &= (6\,\gamma_t + 4\,\gamma_x)\,(-4\,\gamma_t + 6\,\gamma_x)\\ &= 6\cdot(-4)\,\gamma_t^2 + 6\cdot 6\,\gamma_t\gamma_x + 4\cdot(-4)\,\gamma_x\gamma_t + 4\cdot 6\,\gamma_x^2\\ &= -24\,\gamma_t^2 + 36\,\gamma_t\gamma_x - 16\,\gamma_x\gamma_t + 24\,\gamma_x^2\\ &= -24\cdot 1 + 36\,\gamma_t\gamma_x - 16\,(-\gamma_t\gamma_x) + 24\cdot(-1)\\ &= -24 + 36\,\gamma_t\gamma_x + 16\,\gamma_t\gamma_x - 24\\ &= -48 + 52\,\gamma_t\gamma_x\end{aligned}$$

$\Rightarrow\ \mathbf{a}\wedge\mathbf{b} = 52\,\gamma_t\gamma_x$

Sketch:



$\Rightarrow$ $|\mathbf{A}| = 52$

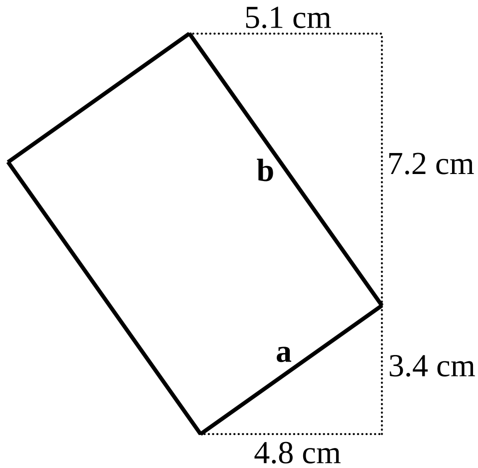
$\Rightarrow$ The area of the spacetime parallelogram is 52 cm^2.

As the inner product does not vanish, the sides of the spacetime parallelogram are not perpendicular to each other. Thus this spacetime parallelogram is not a spacetime square, but a spacetime rhombus.

b) $\mathbf{a} = -4.8\,\gamma_t - 3.4\,\gamma_x$
$\mathbf{b} = -5.1\,\gamma_t + 7.2\,\gamma_x$

$\mathbf{a}\,\mathbf{b} = (-4.8\,\gamma_t - 3.4\,\gamma_x)\,(-5.1\,\gamma_t + 7.2\,\gamma_x)$
$= -4.8 \cdot (-5.1)\,\gamma_t^2 - 4.8 \cdot 7.2\,\gamma_t\gamma_x - 3.4 \cdot (-5.1)\,\gamma_x\gamma_t - 3.4 \cdot 7.2\,\gamma_x^2$
$= 24.48\,\gamma_t^2 - 34.56\,\gamma_t\gamma_x + 17.34\,\gamma_x\gamma_t - 24.48\,\gamma_x^2$
$= 24.48 \cdot 1 - 34.56\,\gamma_t\gamma_x + 17.34\,(-\gamma_t\gamma_x) - 24.48 \cdot (-1)$
$= 24.48 - 34.56\,\gamma_t\gamma_x - 17.34\,\gamma_t\gamma_x + 24.48$
$= 48.96 - 51.90\,\gamma_t\gamma_x$

Sketch:



$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = -51.90\,\gamma_t\gamma_x$

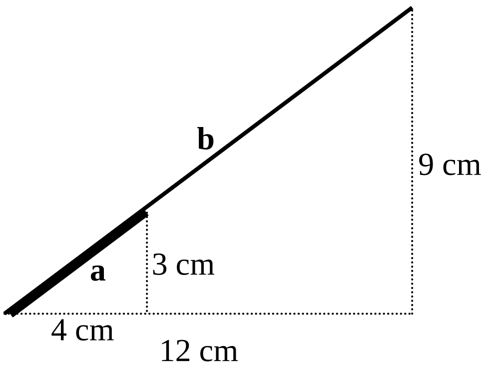
$\Rightarrow$ $|\mathbf{A}| = 51.90$

$\Rightarrow$ The area of the spacetime parallelogram is 51.90 cm^2.

As the inner product does not vanish, the sides of the spacetime parallelogram are not perpendicular to each other. Thus this spacetime parallelogram is not a spacetime rectangle.

c) $\mathbf{a} = 4\,\gamma_t + 3\,\gamma_x$
$\mathbf{b} = 12\,\gamma_t + 9\,\gamma_x$

Sketch:



$\mathbf{a}\,\mathbf{b} = (4\,\gamma_t + 3\,\gamma_x)\,(12\,\gamma_t + 9\,\gamma_x)$
$= 4 \cdot 12\,\gamma_t^2 + 4 \cdot 9\,\gamma_t\gamma_x + 3 \cdot 12\,\gamma_x\gamma_t + 3 \cdot 9\,\gamma_x^2$
$= 48\,\gamma_t^2 + 36\,\gamma_t\gamma_x + 36\,\gamma_x\gamma_t + 27\,\gamma_x^2$
$= 48 \cdot 1 + 36\,\gamma_t\gamma_x + 36\,(-\gamma_t\gamma_x) + 27 \cdot (-1)$
$= 48 + 36\,\gamma_t\gamma_x - 36\,\gamma_t\gamma_x - 27$
$= 21 + 0\,\gamma_t\gamma_x$
$= 21$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 0\,\gamma_t\gamma_x = 0$

$\Rightarrow$ $|\mathbf{A}| = 0$

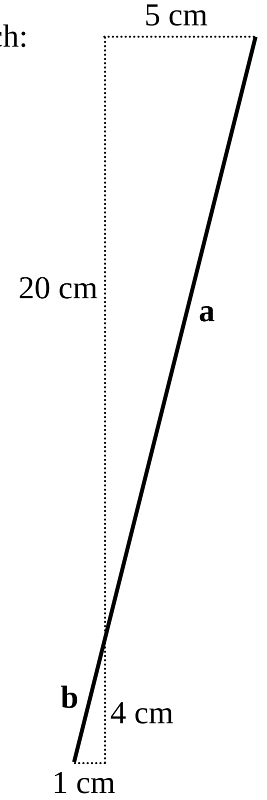
$\Rightarrow$ The area of the spacetime parallelogram equals 0 cm^2. Thus there is no area.

It is not possible to form a parallelogram, because all sides are parallel.

d) $\mathbf{a} = 5\,\gamma_t + 20\,\gamma_x$

$\mathbf{b} = -\gamma_t - 4\,\gamma_x$

Sketch:



$$\begin{aligned}
\mathbf{a\,b} &= (5\,\gamma_t + 20\,\gamma_x)\,(-\gamma_t - 4\,\gamma_x) \\
&= 5 \cdot (-1)\,\gamma_t^2 + 5 \cdot (-4)\,\gamma_t\gamma_x + 20 \cdot (-1)\,\gamma_x\gamma_t + 20 \cdot (-4)\,\gamma_x^2 \\
&= -5\,\gamma_t^2 + -20\,\gamma_t\gamma_x - 20\,\gamma_x\gamma_t - 80\,\gamma_x^2 \\
&= -5 \cdot 1 - 20\,\gamma_t\gamma_x - 20\,(-\gamma_t\gamma_x) - 80 \cdot (-1) \\
&= -5 - 20\,\gamma_t\gamma_x + 20\,\gamma_t\gamma_x + 80 \\
&= 75 + 0\,\gamma_t\gamma_x \\
&= 75
\end{aligned}$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 0\,\gamma_t\gamma_x = 0$

$\Rightarrow$ $|\mathbf{A}| = 0$

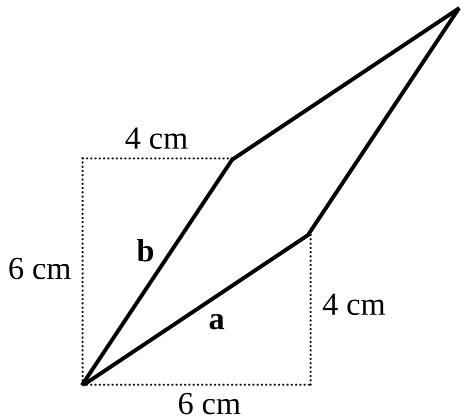
$\Rightarrow$ The area of the spacetime parallelogram equals 0 cm^2. Thus there is no area.

It is not possible to form a parallelogram, because all sides are parallel.

e) $\mathbf{a} = 6\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = 4\,\gamma_t + 6\,\gamma_x$

Sketch:



$$\begin{aligned}
\mathbf{a\,b} &= (6\,\gamma_t + 4\,\gamma_x)\,(4\,\gamma_t + 6\,\gamma_x) \\
&= 6 \cdot 4\,\gamma_t^2 + 6 \cdot 6\,\gamma_t\gamma_x + 4 \cdot 4\,\gamma_x\gamma_t + 4 \cdot 6\,\gamma_x^2 \\
&= 24\,\gamma_t^2 + 36\,\gamma_t\gamma_x + 16\,\gamma_x\gamma_t + 24\,\gamma_x^2 \\
&= 24 \cdot 1 + 36\,\gamma_t\gamma_x + 16\,(-\gamma_t\gamma_x) + 24 \cdot (-1) \\
&= 24 + 36\,\gamma_t\gamma_x - 16\,\gamma_t\gamma_x - 24 \\
&= 0 + 20\,\gamma_t\gamma_x \\
&= 20\,\gamma_t\gamma_x
\end{aligned}$$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 20\,\gamma_t\gamma_x$

$\Rightarrow$ $|\mathbf{A}| = 20$

$\Rightarrow$ The area of the spacetime parallelogram is 20 cm^2.

Now the inner product vanishes. Thus the sides of the spacetime parallelogram are perpendicular to each other, and this spacetime parallelogram is a spacetime square.

f) $\mathbf{a} = -4.8\,\gamma_t - 3.4\,\gamma_x$

$\mathbf{b} = \ \ 5.1\,\gamma_t + 7.2\,\gamma_x$

$$\begin{aligned}
\mathbf{a\,b} &= (-4.8\,\gamma_t - 3.4\,\gamma_x)\,(5.1\,\gamma_t + 7.2\,\gamma_x) \\
&= -4.8 \cdot 5.1\,\gamma_t^2 - 4.8 \cdot 7.2\,\gamma_t\gamma_x - 3.4 \cdot 5.1\,\gamma_x\gamma_t - 3.4 \cdot 7.2\,\gamma_x^2
\end{aligned}$$

$= -24.48\ \gamma_t^2 - 34.56\ \gamma_t\gamma_x - 17.34\ \gamma_x\gamma_t - 24.48\ \gamma_x^2$

$= -24.48 \cdot 1 - 34.56\ \gamma_t\gamma_x - 17.34\ (-\gamma_t\gamma_x) - 24.48 \cdot (-1)$

$= -24.48 - 34.56\ \gamma_t\gamma_x + 17.34\ \gamma_t\gamma_x + 24.48$

$= 0 - 17.22\ \gamma_t\gamma_x$

$= -17.22\ \gamma_t\gamma_x$

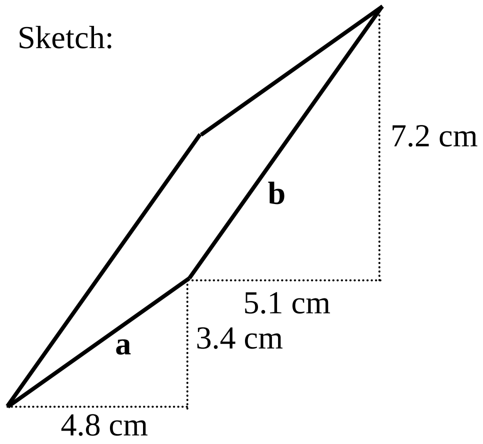
$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = -17.22\ \gamma_t\gamma_x$

$\Rightarrow$ $|\mathbf{A}| = 17.22$

$\Rightarrow$ The area of the spacetime parallelogram is 17.22 cm^2.

Now the inner product vanishes. Thus the sides of the spacetime parallelogram are perpendicular to each other, and this spacetime parallelogram is a spacetime rectangle.

Sketch:



g) $\mathbf{a} = \ \ 4\,\gamma_t + 4\,\gamma_x$
$\mathbf{b} = 12\,\gamma_t + 9\,\gamma_x$

$\mathbf{a}\ \mathbf{b} = (4\,\gamma_t + 4\,\gamma_x)\ (12\,\gamma_t + 9\,\gamma_x)$

$= 4 \cdot 12\ \gamma_t^2 + 4 \cdot 9\ \gamma_t\gamma_x + 4 \cdot 12\ \gamma_x\gamma_t + 4 \cdot 9\ \gamma_x^2$

$= 48\ \gamma_t^2 + 36\ \gamma_t\gamma_x + 48\ \gamma_x\gamma_t + 36\ \gamma_x^2$

$= 48 \cdot 1 + 36\ \gamma_t\gamma_x + 48\ (-\gamma_t\gamma_x) + 36 \cdot (-1)$

$= 48 + 36\ \gamma_t\gamma_x - 48\ \gamma_t\gamma_x - 36$

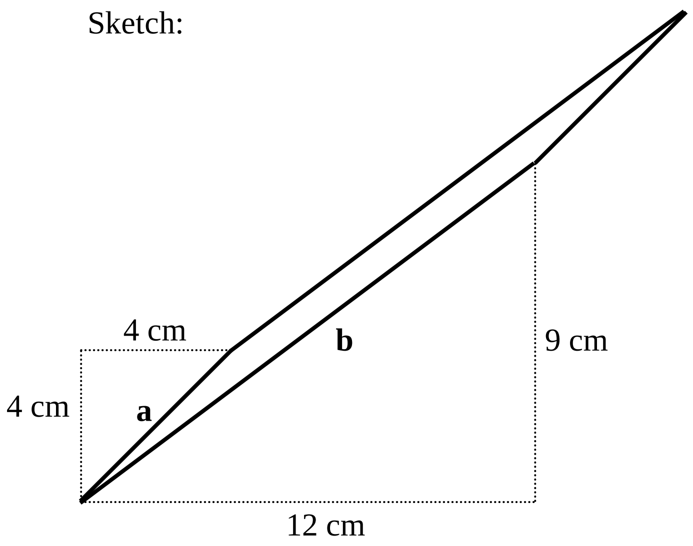
$= 12 - 12\ \gamma_t\gamma_x$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = -12\ \gamma_t\gamma_x$

$\Rightarrow$ $|\mathbf{A}| = 12$

$\Rightarrow$ The area of the spacetime parallelogram is 12 cm^2.

Sketch:



h) $\mathbf{a} = \ \ 8\,\gamma_t + 8\,\gamma_x$
$\mathbf{b} = -5\,\gamma_t + 5\,\gamma_x$

$\mathbf{a}\ \mathbf{b} = (8\,\gamma_t + 8\,\gamma_x)\ (-5\,\gamma_t + 5\,\gamma_x)$

$= 8 \cdot (-5)\ \gamma_t^2 + 8 \cdot 5\ \gamma_t\gamma_x + 8 \cdot (-5)\ \gamma_x\gamma_t + 8 \cdot 5\ \gamma_x^2$

$= -40\ \gamma_t^2 + 40\ \gamma_t\gamma_x - 40\ \gamma_x\gamma_t + 40\ \gamma_x^2$

$= -40 \cdot 1 + 40\ \gamma_t\gamma_x - 40\ (-\gamma_t\gamma_x) + 40 \cdot (-1)$

$= -40 + 40\ \gamma_t\gamma_x + 40\ \gamma_t\gamma_x - 40$

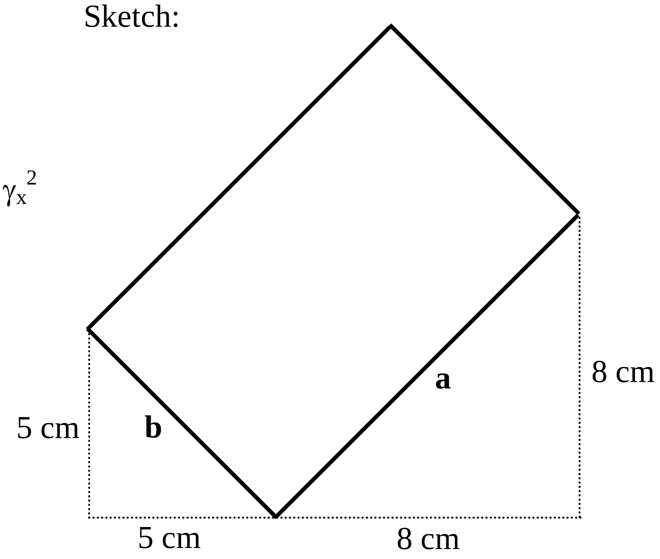
$= -80 + 80\ \gamma_t\gamma_x$

$\Rightarrow$ $\mathbf{a} \wedge \mathbf{b} = 80\ \gamma_t\gamma_x$

$\Rightarrow$ $|\mathbf{A}| = 80$

$\Rightarrow$ The area of the spacetime parallelogram is 80 cm^2.

Sketch:



Problem 3:

Measuring relativistic time intervals:

$$\Delta(ct) = c\,\Delta t \quad \Rightarrow \quad \Delta t = \frac{\Delta(ct)}{c} = \frac{\Delta(ct)}{3 \cdot 10^8\,\mathrm{m/s}} = \frac{\Delta(ct)}{3 \cdot 10^{10}\,\mathrm{cm/s}}$$

Light needs 1 nanosecond = 1 ns to cover the length of a ruler: $\Delta(ct) = 30$ cm

$$\Delta t = \frac{\Delta(ct)}{c} = \frac{30\,\mathrm{cm}}{3 \cdot 10^{10}\,\mathrm{cm/s}} = 10^{-9}\ \mathrm{s} = 1\ \mathrm{ns}$$

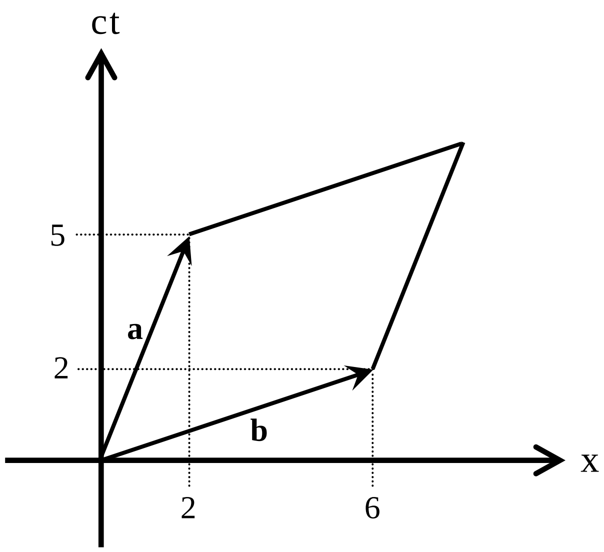
1 a) $\mathbf{a} = 5\,\gamma_t + 2\,\gamma_x$

$\mathbf{b} = 2\,\gamma_t + 6\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = 5$ cm

$\Delta(ct_b) = 2$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{5\,\mathrm{cm}}{3 \cdot 10^{10}\,\mathrm{cm/s}} = 1.67 \cdot 10^{-10}\ \mathrm{s} = 0.167\ \mathrm{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{2\,\mathrm{cm}}{3 \cdot 10^{10}\,\mathrm{cm/s}} = 6.67 \cdot 10^{-11}\ \mathrm{s} = 0.0667\ \mathrm{ns}$$

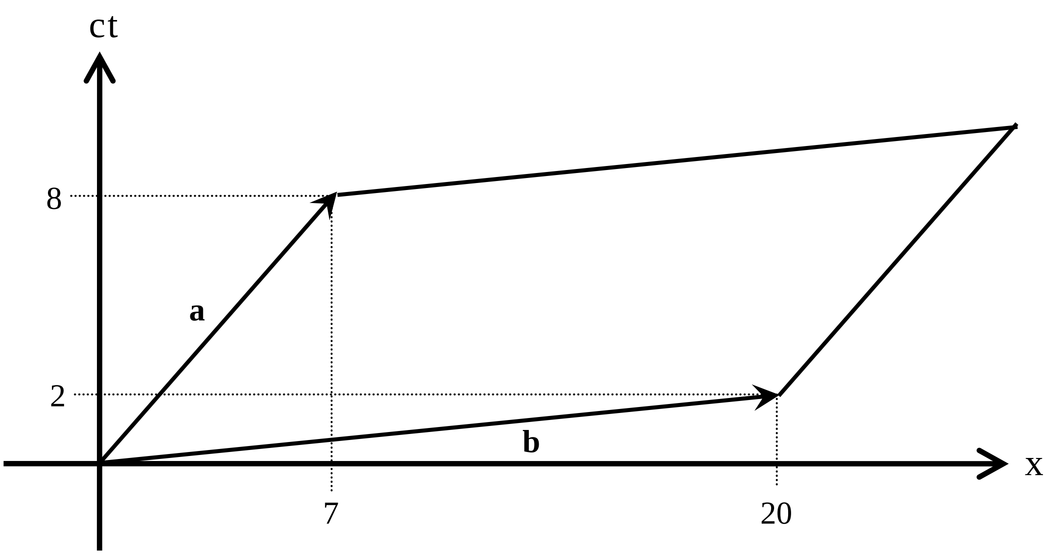
1 b) $\mathbf{a} = 8\,\gamma_t + \ \ 7\,\gamma_x$

$\mathbf{b} = 2\,\gamma_t + 20\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = 8$ cm

$\Delta(ct_b) = 2$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{8\,\mathrm{cm}}{3 \cdot 10^{10}\,\mathrm{cm/s}} = 2.67 \cdot 10^{-10}\ \mathrm{s} = 0.267\ \mathrm{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{2\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = 6.67 \cdot 10^{-11}\ \text{s} = 0.0667\ \text{ns}$$

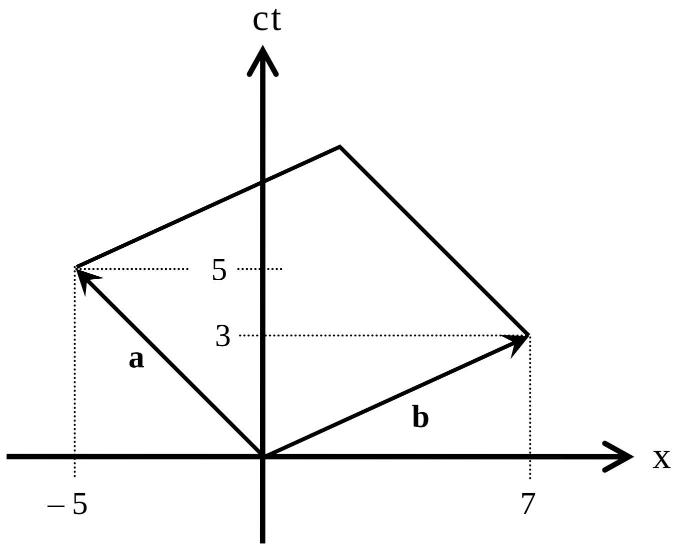
1 c) $\mathbf{a} = 5\,\gamma_t - 5\,\gamma_x$

$\mathbf{b} = 3\,\gamma_t + 7\,\gamma_x$

↑

$\Delta(ct_a) = 5$ cm

$\Delta(ct_b) = 3$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{5\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = 1.67 \cdot 10^{-10}\ \text{s} = 0.167\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{3\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = 1 \cdot 10^{-10}\ \text{s} = 0.1\ \text{ns}$$

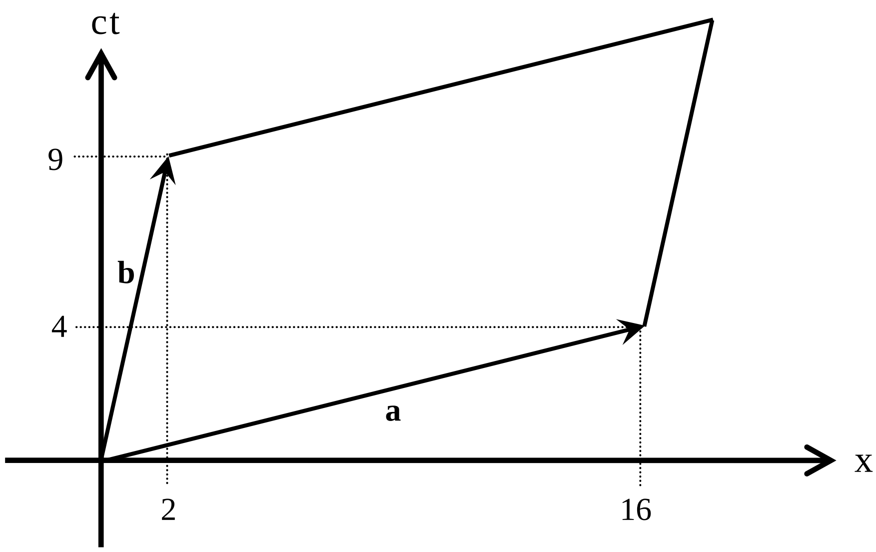
1 d) $\mathbf{a} = 4\,\gamma_t + 16\,\gamma_x$

$\mathbf{b} = 9\,\gamma_t + \ \ 2\,\gamma_x$

↑

$\Delta(ct_a) = 4$ cm

$\Delta(ct_b) = 9$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{4\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = 1.33 \cdot 10^{-10}\ \text{s} = 0.133\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{9\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = 3 \cdot 10^{-10}\ \text{s} = 0.3\ \text{ns}$$

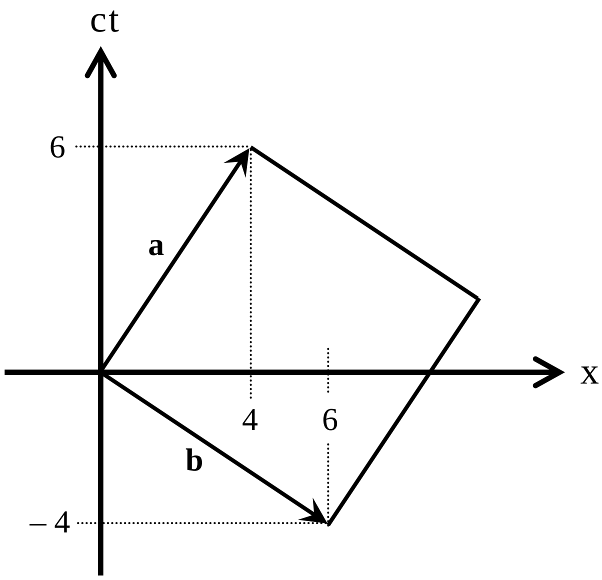
2 a) $\mathbf{a} = \;\; 6\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = -\,4\,\gamma_t + 6\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = \;\; 6$ cm

$\Delta(ct_b) = -\,4$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{6\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 2 \cdot 10^{-10}\ \text{s} = 0.2\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{-\,4\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = -\,1.33 \cdot 10^{-10}\ \text{s} = -\,0.133\ \text{ns}$$

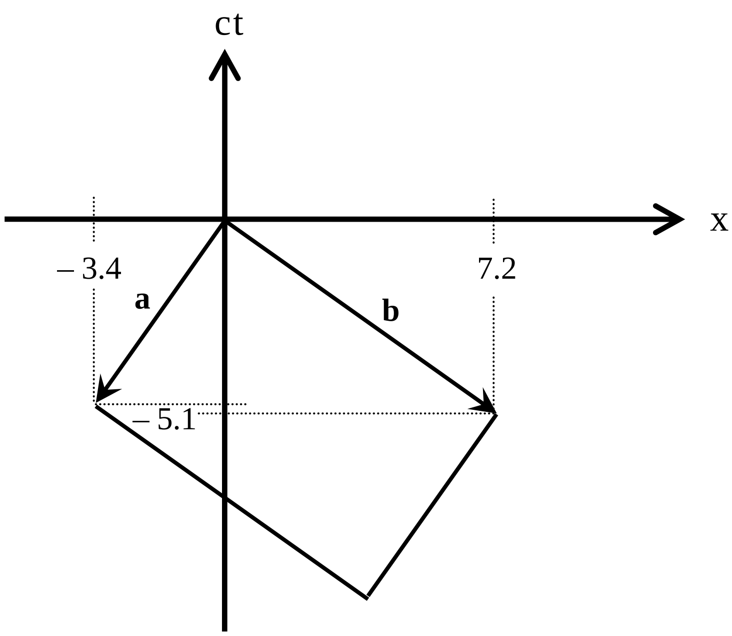
2 b) $\mathbf{a} = -\,4.8\,\gamma_t - 3.4\,\gamma_x$

$\mathbf{b} = -\,5.1\,\gamma_t + 7.2\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = -\,4.8$ cm

$\Delta(ct_b) = -\,5.1$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{-\,4.8\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = -\,1.6 \cdot 10^{-10}\ \text{s} = -\,0.16\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{-\,5.1\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = -\,1.7 \cdot 10^{-10}\ \text{s} = -\,0.17\ \text{ns}$$

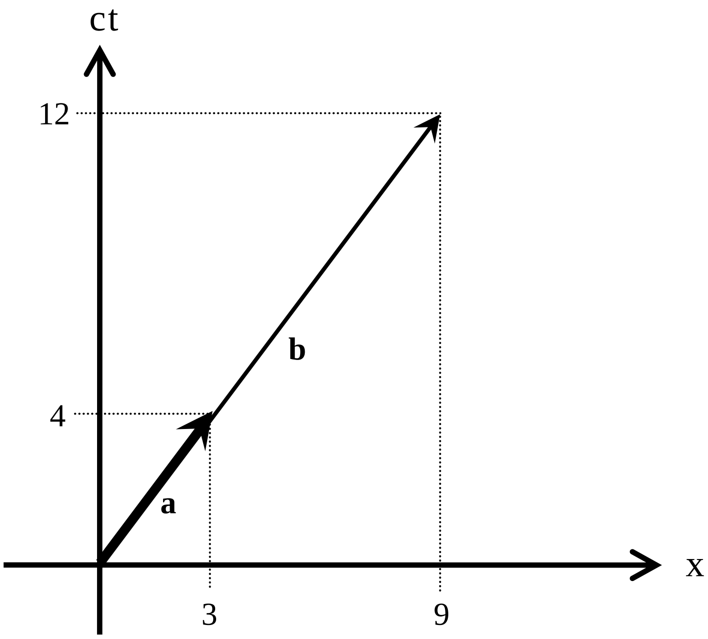
2 c) $\mathbf{a} = \ 4\,\gamma_t + 3\,\gamma_x$

$\mathbf{b} = 12\,\gamma_t + 9\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = \ 4$ cm

$\Delta(ct_b) = 12$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{4\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 1.33 \cdot 10^{-10}\ \text{s} = 0.133\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{12\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 4 \cdot 10^{-10}\ \text{s} = 0.4\ \text{ns}$$

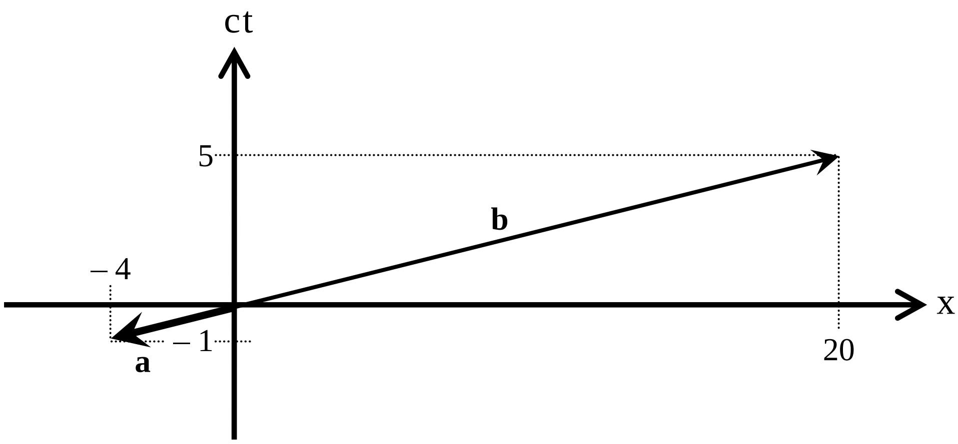
2 d) $\mathbf{a} = \ 5\,\gamma_t + 20\,\gamma_x$

$\mathbf{b} = -\gamma_t - \ 4\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = \ 5$ cm

$\Delta(ct_b) = -1$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{5\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 1.67 \cdot 10^{-10}\ \text{s} = 0.167\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{-1\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = -\,3.33 \cdot 10^{-11}\ \text{s} = -\,0.033\ \text{ns}$$

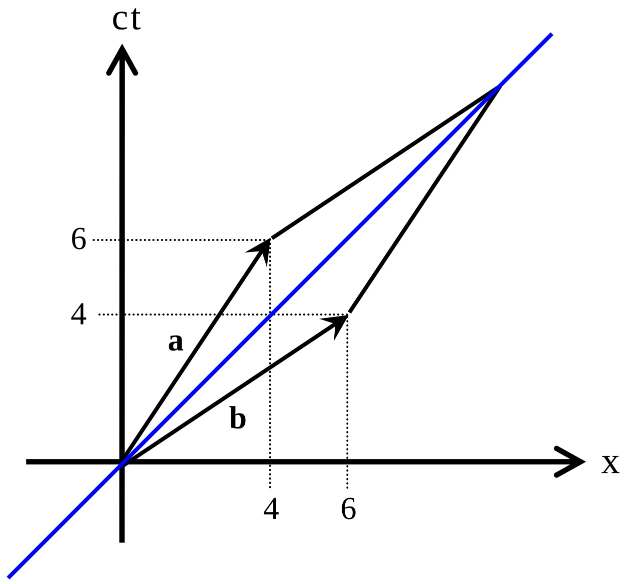
2 e) $\mathbf{a} = 6\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = 4\,\gamma_t + 6\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = 6$ cm

$\Delta(ct_b) = 4$ cm



As the **diagonal line (world line of light)** can be drawn right through the end points of the parallelogram, this parallelogram is a spacetime square. The diagonal line looks like a bisector of the angle between spacetime vectors **a** and **b**.

$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{6\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 2\cdot 10^{-10}\,\text{s} = 0.2\,\text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{4\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 1.33\cdot 10^{-10}\,\text{s} = 0.133\,\text{ns}$$

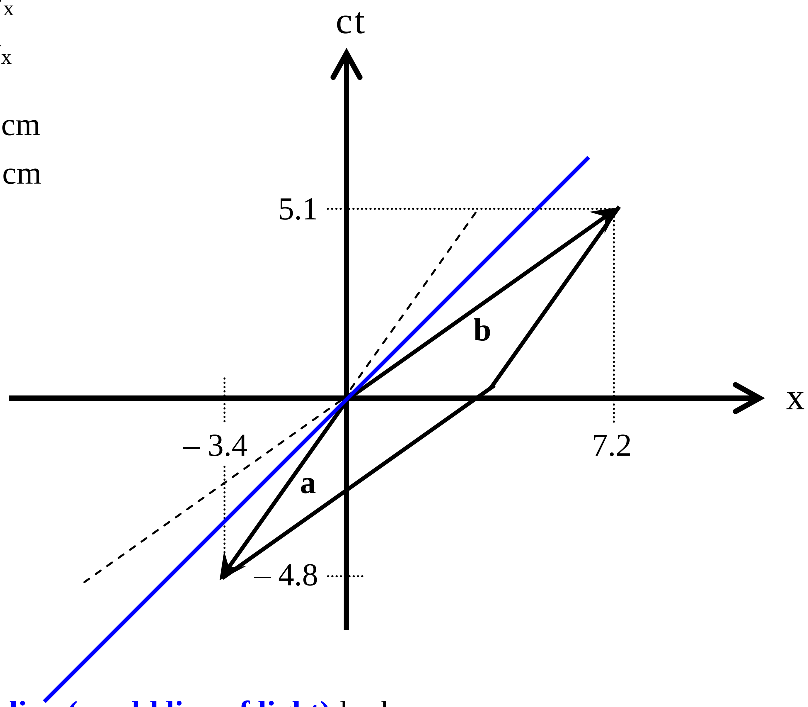
2 f) $\mathbf{a} = -4.8\,\gamma_t - 3.4\,\gamma_x$

$\mathbf{b} = \;\; 5.1\,\gamma_t + 7.2\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = -4.8$ cm

$\Delta(ct_b) = \;\; 5.1$ cm



As the **diagonal line (world line of light)** looks like the bisector of the angle between spacetime vectors **a** and **b**, this parallelogram is a spacetime rectangle.

$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{-4.8\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = -1.6 \cdot 10^{-10}\ \text{s} = -0.16\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{5.1\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 1.7 \cdot 10^{-10}\ \text{s} = 0.17\ \text{ns}$$

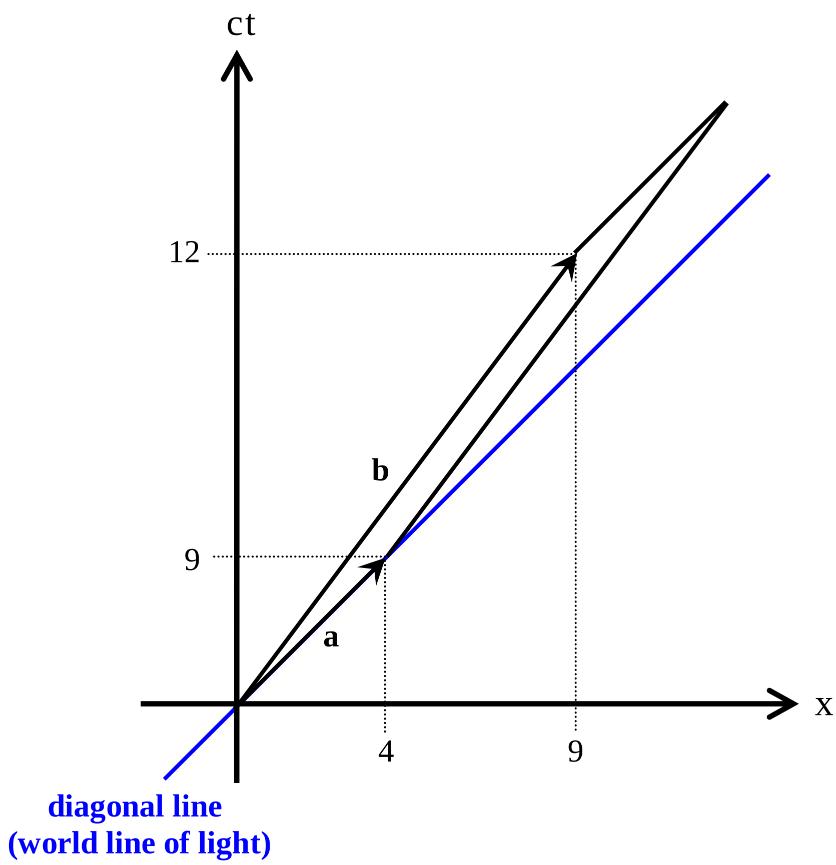
2 g) $\mathbf{a} = 4\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = 12\,\gamma_t + 9\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = 4$ cm

$\Delta(ct_b) = 12$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{4\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 1.33 \cdot 10^{-10}\ \text{s} = 0.133\ \text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{12\,\text{cm}}{3\cdot 10^{10}\,\text{cm/s}} = 4 \cdot 10^{-10}\ \text{s} = 0.4\ \text{ns}$$

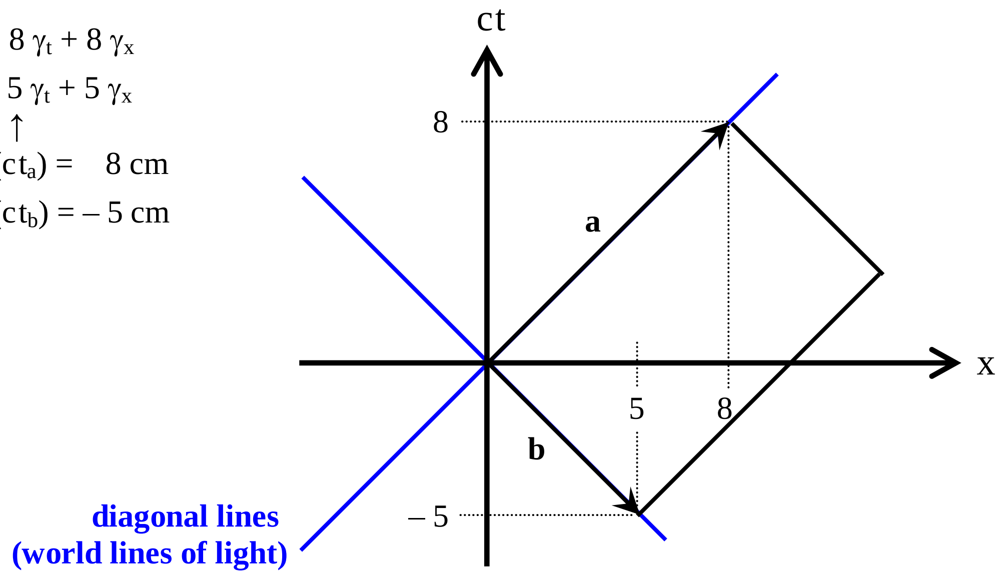
2 h) $\mathbf{a} = 8\,\gamma_t + 8\,\gamma_x$

$\mathbf{b} = -5\,\gamma_t + 5\,\gamma_x$

$\uparrow$

$\Delta(ct_a) = 8$ cm

$\Delta(ct_b) = -5$ cm



$$\Delta t_a = \frac{\Delta(ct_a)}{c} = \frac{8\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = 2.67 \cdot 10^{-10}\,\text{s} = 0.267\,\text{ns}$$

$$\Delta t_b = \frac{\Delta(ct_b)}{c} = \frac{-5\,\text{cm}}{3 \cdot 10^{10}\,\text{cm/s}} = -1.67 \cdot 10^{-10}\,\text{s} = -0.167\,\text{ns}$$

Problem 4:

1 a) $\mathbf{a} = 5\,\gamma_t + 2\,\gamma_x$

$\mathbf{b} = 2\,\gamma_t + 6\,\gamma_x$ $\qquad \mathbf{a\,b} = -2 + 26\,\gamma_t\gamma_x$

$$\mathbf{a}^2 = (5\,\gamma_t + 2\,\gamma_x)^2 = (5\,\gamma_t + 2\,\gamma_x)\,(5\,\gamma_t + 2\,\gamma_x)$$
$$= 25\,\gamma_t^2 + 10\,\gamma_t\gamma_x + 10\,\gamma_x\gamma_t + 4\,\gamma_x^2 = 25 - 4 = 21 > 0 \;\Rightarrow\; \text{time-like vector}$$

$$\mathbf{b}^2 = (2\,\gamma_t + 6\,\gamma_x)^2 = (2\,\gamma_t + 6\,\gamma_x)\,(2\,\gamma_t + 6\,\gamma_x)$$
$$= 4\,\gamma_t^2 + 12\,\gamma_t\gamma_x + 12\,\gamma_x\gamma_t + 36\,\gamma_x^2 = 4 - 36 = -32 < 0 \;\Rightarrow\; \text{space-like vector}$$

$$(\mathbf{ab})^2 = (-2 + 26\,\gamma_t\gamma_x)^2 = (-2 + 26\,\gamma_t\gamma_x)\,(-2 + 26\,\gamma_t\gamma_x)$$
$$= 4 - 52\,\gamma_t\gamma_x - 52\,\gamma_t\gamma_x + 676\,\gamma_t\gamma_x\gamma_t\gamma_x$$
$$= 4 - 104\,\gamma_t\gamma_x + 676$$
$$= 680 - 104\,\gamma_t\gamma_x$$

$\Rightarrow$ This is the square of the spacetime parallelogram **a b**, but this is not the magnitude square of spacetime parallelogram **a b**. To find the magnitude square of the spacetime parallelogram, $\mathbf{a\,b} = -2 + 26\,\gamma_t\gamma_x$ must be multiplied by its reverse $(\mathbf{a\,b})^\sim = \mathbf{b\,a} = -2 - 26\,\gamma_t\gamma_x$.

$$(\mathbf{ab})\,(\mathbf{a\,b})^\sim = (-2 + 26\,\gamma_t\gamma_x)\,(-2 - 26\,\gamma_t\gamma_x)$$
$$= 4 - 52\,\gamma_t\gamma_x + 52\,\gamma_t\gamma_x - 676\,\gamma_t\gamma_x\gamma_t\gamma_x$$
$$= 4 - 676$$
$$= -672$$

$\Rightarrow$ This is the magnitude square of the given spacetime parallelogram.

$$\Rightarrow\; \mathbf{a}^2\,\mathbf{b}^2 = 21 \cdot (-32) = -672 = (\mathbf{ab})\,(\mathbf{a\,b})^\sim$$

1 b) $\mathbf{a} = 8\,\gamma_t + 7\,\gamma_x$

$\mathbf{b} = 2\,\gamma_t + 20\,\gamma_x$ $\qquad \mathbf{a\,b} = -124 + 146\,\gamma_t\gamma_x$

$$\mathbf{a}^2 = (8\,\gamma_t + 7\,\gamma_x)^2 = (8\,\gamma_t + 7\,\gamma_x)\,(8\,\gamma_t + 7\,\gamma_x)$$
$$= 64\,\gamma_t^2 + 56\,\gamma_t\gamma_x + 56\,\gamma_x\gamma_t + 49\,\gamma_x^2 = 64 - 49 = 15 > 0 \;\Rightarrow\; \text{time-like vector}$$

$$\mathbf{b}^2 = (2\,\gamma_t + 20\,\gamma_x)^2 = (2\,\gamma_t + 20\,\gamma_x)\,(2\,\gamma_t + 20\,\gamma_x)$$
$$= 4\,\gamma_t^2 + 40\,\gamma_t\gamma_x + 40\,\gamma_x\gamma_t + 400\,\gamma_x^2 = 4 - 400 = -396 < 0 \;\Rightarrow\; \text{space-like vector}$$

$$(\mathbf{ab})^2 = (-124 + 146\,\gamma_t\gamma_x)^2 = (-124 + 146\,\gamma_t\gamma_x)\,(-124 + 146\,\gamma_t\gamma_x)$$
$$= 15376 - 18104\,\gamma_t\gamma_x - 18104\,\gamma_t\gamma_x + 21316\,\gamma_t\gamma_x\gamma_t\gamma_x$$

$= 15376 - 36208\, \gamma_t\gamma_x + 21316$

$= 36692 - 36208\, \gamma_t\gamma_x \quad \Rightarrow$ This is the square of the spacetime parallelogram **a b**, but this is not the magnitude square of spacetime parallelogram **a b**. To find the magnitude square of the spacetime parallelogram, **a b** $= -124 + 146\, \gamma_t\gamma_x$ must be multiplied by its reverse $(\mathbf{a\,b})^{\sim} = \mathbf{b\,a} = -124 - 146\, \gamma_t\gamma_x$.

$(\mathbf{ab})\,(\mathbf{a\,b})^{\sim} = (-124 + 146\, \gamma_t\gamma_x)\,(-124 - 146\, \gamma_t\gamma_x)$

$= 15376 + 18104\, \gamma_t\gamma_x - 18104\, \gamma_t\gamma_x - 21316\, \gamma_t\gamma_x\gamma_t\gamma_x$

$= 15376 - 21316$

$= -5940 \quad \Rightarrow$ This is the magnitude square of the given spacetime parallelogram.

$\Rightarrow \quad \mathbf{a}^2\, \mathbf{b}^2 = 15 \cdot (-396) = -5940 = (\mathbf{ab})\,(\mathbf{a\,b})^{\sim}$

1 c) $\mathbf{a} = 5\,\gamma_t - 5\,\gamma_x$

$\mathbf{b} = 3\,\gamma_t + 7\,\gamma_x \qquad \mathbf{a\,b} = 50 + 50\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (5\,\gamma_t - 5\,\gamma_x)^2 = (5\,\gamma_t - 5\,\gamma_x)\,(5\,\gamma_t - 5\,\gamma_x)$

$= 25\,\gamma_t^2 - 25\,\gamma_t\gamma_x - 25\,\gamma_x\gamma_t + 25\,\gamma_x^2 = 25 - 25 = 0 \quad \Rightarrow$ light-like vector

$\mathbf{b}^2 = (3\,\gamma_t + 7\,\gamma_x)^2 = (3\,\gamma_t + 7\,\gamma_x)\,(3\,\gamma_t + 7\,\gamma_x)$

$= 9\,\gamma_t^2 + 21\,\gamma_t\gamma_x + 21\,\gamma_x\gamma_t + 49\,\gamma_x^2 = 9 - 49 = -40 < 0 \quad \Rightarrow$ space-like vector

$(\mathbf{ab})^2 = (50 + 50\,\gamma_t\gamma_x)^2 = (50 + 50\,\gamma_t\gamma_x)\,(50 + 50\,\gamma_t\gamma_x)$

$= 2500 + 2500\,\gamma_t\gamma_x + 2500\,\gamma_t\gamma_x + 2500\,\gamma_t\gamma_x\gamma_t\gamma_x$

$= 2500 + 5000\,\gamma_t\gamma_x + 2500$

$= 5000 + 5000\,\gamma_t\gamma_x \quad \Rightarrow$ This is the square of the spacetime parallelogram **a b**, but this is not the magnitude square of spacetime parallelogram **a b**. To find the magnitude square of the spacetime parallelogram, **a b** $= 50 + 50\,\gamma_t\gamma_x$ must be multiplied by its reverse $(\mathbf{a\,b})^{\sim} = \mathbf{b\,a} = 50 - 50\,\gamma_t\gamma_x$.

$(\mathbf{ab})\,(\mathbf{a\,b})^{\sim} = (50 + 50\,\gamma_t\gamma_x)\,(50 - 50\,\gamma_t\gamma_x)$

$= 2500 - 2500\,\gamma_t\gamma_x + 2500\,\gamma_t\gamma_x - 2500\,\gamma_t\gamma_x\gamma_t\gamma_x$

$= 2500 - 2500$

$= 0 \quad \Rightarrow$ This is the magnitude square of the given spacetime parallelogram.

$\Rightarrow \quad \mathbf{a}^2\, \mathbf{b}^2 = 0 \cdot (-40) = 0 = (\mathbf{ab})\,(\mathbf{a\,b})^{\sim}$

1 d) $\mathbf{a} = 4\,\gamma_t + 16\,\gamma_x$

$\mathbf{b} = 9\,\gamma_t + \;\; 2\,\gamma_x \qquad\qquad \mathbf{a}\,\mathbf{b} = 4 - 136\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (4\,\gamma_t + 16\,\gamma_x)^2 = (4\,\gamma_t + 16\,\gamma_x)\,(4\,\gamma_t + 16\,\gamma_x)$

$= 16\,\gamma_t^{\,2} + 64\,\gamma_t\gamma_x + 64\,\gamma_x\gamma_t + 256\,\gamma_x^{\,2} = 16 - 256 = -\,240 < 0 \;\;\Rightarrow$ space-like vector

$\mathbf{b}^2 = (9\,\gamma_t + 2\,\gamma_x)^2 = (9\,\gamma_t + 2\,\gamma_x)\,(9\,\gamma_t + 2\,\gamma_x)$

$= 81\,\gamma_t^{\,2} + 18\,\gamma_t\gamma_x + 18\,\gamma_x\gamma_t + 4\,\gamma_x^{\,2} = 81 - 4 = 77 > 0 \;\;\Rightarrow$ time-like vector

$(\mathbf{a}\mathbf{b})^2 = (4 - 136\,\gamma_t\gamma_x)^2 = (4 - 136\,\gamma_t\gamma_x)\,(4 - 136\,\gamma_t\gamma_x)$

$= 16 - 544\,\gamma_t\gamma_x - 544\,\gamma_t\gamma_x + 18496\,\gamma_t\gamma_x\gamma_t\gamma_x$

$= 16 - 1088\,\gamma_t\gamma_x + 18496$

$= 18512 - 1088\,\gamma_t\gamma_x \;\;\Rightarrow$ This is the square of the spacetime parallelogram **a b**, but this is not the magnitude square of spacetime parallelogram **a b**. To find the magnitude square of the spacetime parallelogram, $\mathbf{a}\,\mathbf{b} = 4 - 136\,\gamma_t\gamma_x$ must be multiplied by its reverse $(\mathbf{a}\,\mathbf{b})^{\sim} = \mathbf{b}\,\mathbf{a} = 4 + 136\,\gamma_t\gamma_x$.

$(\mathbf{a}\mathbf{b})\,(\mathbf{a}\,\mathbf{b})^{\sim} = (4 - 136\,\gamma_t\gamma_x)\,(4 + 136\,\gamma_t\gamma_x)$

$= 16 + 544\,\gamma_t\gamma_x - 544\,\gamma_t\gamma_x - 18496\,\gamma_t\gamma_x\gamma_t\gamma_x$

$= 16 - 18496$

$= -\,18480 \qquad \Rightarrow$ This is the magnitude square of the given spacetime parallelogram.

$\Rightarrow\;\; \mathbf{a}^2\,\mathbf{b}^2 = -\,240 \cdot 77 = -\,18480 = (\mathbf{a}\mathbf{b})\,(\mathbf{a}\,\mathbf{b})^{\sim}$

2 a) $\mathbf{a} = \;\; 6\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = -\,4\,\gamma_t + 6\,\gamma x \qquad\qquad \mathbf{a}\,\mathbf{b} = -\,48 + 52\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (6\,\gamma_t + 4\,\gamma_x)^2 = 36 - 16 = 20 > 0 \qquad \Rightarrow$ time-like vector

$\mathbf{b}^2 = (-\,4\,\gamma_t + 6\,\gamma_x)^2 = 16 - 36 = -\,20 < 0 \qquad \Rightarrow$ space-like vector

$(\mathbf{a}\mathbf{b})^2 = (-\,48 + 52\,\gamma_t\gamma_x)^2 = 5008 - 4992\,\gamma_t\gamma_x \qquad \Rightarrow$ square of the spacetime parallelogram

$(\mathbf{a}\mathbf{b})\,(\mathbf{a}\,\mathbf{b})^{\sim} = (-\,48 + 52\,\gamma_t\gamma_x)\,(-\,48 - 52\,\gamma_t\gamma_x) = -\,400 \qquad \Rightarrow$ magnitude square

$\Rightarrow\;\; \mathbf{a}^2\,\mathbf{b}^2 = 20 \cdot (-\,20) = -\,400 = (\mathbf{a}\mathbf{b})\,(\mathbf{a}\,\mathbf{b})^{\sim}$

2 b) $\mathbf{a} = -\,4.8\,\gamma_t - 3.4\,\gamma_x$

$\mathbf{b} = -\,5.1\,\gamma_t + 7.2\,\gamma_x \qquad \mathbf{a}\,\mathbf{b} = 48.96 - 51.90\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (-\,4.8\,\gamma_t - 3.4\,\gamma_x)^2 = 23.04 - 11.56 = 11.48 > 0 \qquad \Rightarrow$ time-like vector

$\mathbf{b}^2 = (-\,5.1\,\gamma_t + 7.2\,\gamma_x)^2 = 26.01 - 51.84 = -\,25.83 < 0 \qquad \Rightarrow$ space-like vector

$(\mathbf{a}\mathbf{b})^2 = (48.96 - 51.90\,\gamma_t\gamma_x)^2 = 5090.6916 - 5082.0480\,\gamma_t\gamma_x \qquad \Rightarrow$ square of the spacetime parallelogram

$(\mathbf{a}\mathbf{b})\,(\mathbf{a}\,\mathbf{b})^{\sim} = (48.96 - 51.90\,\gamma_t\gamma_x)\,(48.96 + 51.90\,\gamma_t\gamma_x) = -\,296.5284 \;\;\Rightarrow$ magnitude square

$\Rightarrow\;\; \mathbf{a}^2\,\mathbf{b}^2 = 11.48 \cdot (-\,25.83) = -\,296.5284 = (\mathbf{a}\mathbf{b})\,(\mathbf{a}\,\mathbf{b})^{\sim}$

2 c) $\mathbf{a} = 4\,\gamma_t + 3\,\gamma_x$

$\mathbf{b} = 12\,\gamma_t + 9\,\gamma_x$ $\quad\quad$ $\mathbf{a}\,\mathbf{b} = 21 + 0\,\gamma_t\gamma_x = 21$

$\mathbf{a}^2 = (4\,\gamma_t + 3\,\gamma_x)^2 = 16 - 9 = 7 > 0$ $\quad\Rightarrow$ time-like vector

$\mathbf{b}^2 = (12\,\gamma_t + 9\,\gamma_x)^2 = 144 - 81 = 63 > 0$ $\quad\Rightarrow$ time-like vector

$(\mathbf{ab})^2 = 21^2 = 441$ $\quad\Rightarrow$ square of geometric product

$(\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim = (21 + 0\,\gamma_t\gamma_x)\,(21 - 0\,\gamma_t\gamma_x) = 441$ $\quad\Rightarrow$ magnitude square

$\Rightarrow\ \mathbf{a}^2\,\mathbf{b}^2 = 7 \cdot 63 = 441 = (\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim$

2 d) $\mathbf{a} = 5\,\gamma_t + 20\,\gamma_x$

$\mathbf{b} = -\gamma_t - 4\,\gamma_x$ $\quad\quad$ $\mathbf{a}\,\mathbf{b} = 75 + 0\,\gamma_t\gamma_x = 75$

$\mathbf{a}^2 = (5\,\gamma_t + 20\,\gamma_x)^2 = 25 - 400 = -375 < 0$ $\quad\Rightarrow$ space-like vector

$\mathbf{b}^2 = (-\gamma_t - 4\,\gamma_x)^2 = 1 - 16 = -15 < 0$ $\quad\Rightarrow$ space-like vector

$(\mathbf{ab})^2 = 75^2 = 5625$ $\quad\Rightarrow$ square of geometric product

$(\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim = (75 + 0\,\gamma_t\gamma_x)\,(75 - 0\,\gamma_t\gamma_x) = 5625$ $\quad\Rightarrow$ magnitude square

$\Rightarrow\ \mathbf{a}^2\,\mathbf{b}^2 = (-375) \cdot (-15) = 5625 = (\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim$

2 e) $\mathbf{a} = 6\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = 4\,\gamma_t + 6\,\gamma_x$ $\quad\quad$ $\mathbf{a}\,\mathbf{b} = 0 + 20\,\gamma_t\gamma_x = 20\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (6\,\gamma_t + 4\,\gamma_x)^2 = 36 - 16 = 20 > 0$ $\quad\Rightarrow$ time-like vector

$\mathbf{b}^2 = (4\,\gamma_t + 6\,\gamma_x)^2 = 16 - 36 = -20 < 0$ $\quad\Rightarrow$ space-like vector

$(\mathbf{ab})^2 = (20\,\gamma_t\gamma_x)^2 = 400$ $\quad\Rightarrow$ square of spacetime square

$(\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim = (20\,\gamma_t\gamma_x)\,(-20\,\gamma_t\gamma_x) = -400$ $\quad\Rightarrow$ magnitude square

$\Rightarrow\ \mathbf{a}^2\,\mathbf{b}^2 = 20 \cdot (-20) = -400 = (\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim$

2 f) $\mathbf{a} = -4.8\,\gamma_t - 3.4\,\gamma_x$

$\mathbf{b} = \ \ 5.1\,\gamma_t + 7.2\,\gamma_x$ $\quad\quad$ $\mathbf{a}\,\mathbf{b} = 0 - 17.22\,\gamma_t\gamma_x = -17.22\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (-4.8\,\gamma_t - 3.4\,\gamma_x)^2 = 23.04 - 11.56 = 11.48 > 0$ $\quad\Rightarrow$ time-like vector

$\mathbf{b}^2 = (5.1\,\gamma_t + 7.2\,\gamma_x)^2 = 26.01 - 51.84 = -25.83 < 0$ $\quad\Rightarrow$ space-like vector

$(\mathbf{ab})^2 = (-17.22\,\gamma_t\gamma_x)^2 = 296.5284$ $\quad\Rightarrow$ square of spacetime rectangle

$(\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim = (-17.22\,\gamma_t\gamma_x)\,(17.22\,\gamma_t\gamma_x) = -296.5284$ $\quad\Rightarrow$ magnitude square

$\Rightarrow\ \mathbf{a}^2\,\mathbf{b}^2 = 11.48 \cdot (-25.83) = -296.5284 = (\mathbf{ab})\,(\mathbf{a}\,\mathbf{b})^\sim$

2 g) $\mathbf{a} = \ \ 4\,\gamma_t + 4\,\gamma_x$

$\mathbf{b} = 12\,\gamma_t + 9\,\gamma_x$ $\quad\quad$ $\mathbf{a}\,\mathbf{b} = 12 - 12\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (4\,\gamma_t + 4\,\gamma_x)^2 = 16 - 16 = 0$ $\quad\Rightarrow$ light-like vector

$\mathbf{b}^2 = (12\,\gamma_t + 9\,\gamma_x)^2 = 144 - 81 = 63 > 0$ $\Rightarrow$ time-like vector

$(\mathbf{ab})^2 = (12 - 12\,\gamma_t\gamma_x)^2 = 288 - 288\,\gamma_t\gamma_x$ $\Rightarrow$ square of spacetime parallelogram

$(\mathbf{ab})\,(\mathbf{a\,b})^{\sim} = (12 - 12\,\gamma_t\gamma_x)\,(12 + 12\,\gamma_t\gamma_x) = 0$ $\Rightarrow$ magnitude square

$\Rightarrow\ \mathbf{a}^2\,\mathbf{b}^2 = 0 \cdot 63 = 0 = (\mathbf{ab})\,(\mathbf{a\,b})^{\sim}$

2 h) $\mathbf{a} = 8\,\gamma_t + 8\,\gamma_x$

$\mathbf{b} = -5\,\gamma_t + 5\,\gamma_x$ $\qquad \mathbf{a\,b} = -80 + 80\,\gamma_t\gamma_x$

$\mathbf{a}^2 = (8\,\gamma_t + 8\,\gamma_x)^2 = 64 - 64 = 0$ $\Rightarrow$ light-like vector

$\mathbf{b}^2 = (-5\,\gamma_t + 5\,\gamma_x)^2 = 25 - 25 = 0$ $\Rightarrow$ light-like vector

$(\mathbf{ab})^2 = (80 + 80\,\gamma_t\gamma_x)^2 = 12800 - 12800\,\gamma_t\gamma_x$ $\Rightarrow$ square of spacetime parallelogram

$(\mathbf{ab})\,(\mathbf{a\,b})^{\sim} = (80 + 80\,\gamma_t\gamma_x)\,(80 - 80\,\gamma_t\gamma_x) = 0$ $\Rightarrow$ magnitude square

$\Rightarrow\ \mathbf{a}^2\,\mathbf{b}^2 = 0 \cdot 0 = 0 = (\mathbf{ab})\,(\mathbf{a\,b})^{\sim}$

Problem 5:

a) $3\,x + 8\,y = 28$ $\qquad \Rightarrow \quad \mathbf{a} = 3\,\gamma_t + 6\,\gamma_x$

$6\,x + 2\,y = 28$ $\qquad \mathbf{b} = 8\,\gamma_t + 2\,\gamma_x$

$\mathbf{r} = 28\,\gamma_t + 28\,\gamma_x$

$$\Rightarrow \quad \mathbf{a\,b} = (3\,\gamma_t + 6\,\gamma_x)\,(8\,\gamma_t + 2\,\gamma_x)$$
$$= 24\,\gamma_t^2 + 6\,\gamma_t\gamma_x + 48\,\gamma_x\gamma_t + 12\,\gamma_x^2$$
$$= 12 - 42\,\gamma_t\gamma_x$$

$$\mathbf{a} \wedge \mathbf{b} = -42\,\gamma_t\gamma_x$$

$$\Rightarrow \quad \mathbf{r\,b} = (28\,\gamma_t + 28\,\gamma_x)\,(8\,\gamma_t + 2\,\gamma_x)$$
$$= 224\,\gamma_t^2 + 56\,\gamma_t\gamma_x + 224\,\gamma_x\gamma_t + 56\,\gamma_x^2$$
$$= 168 - 168\,\gamma_t\gamma_x$$

$$\mathbf{r} \wedge \mathbf{b} = -168\,\gamma_t\gamma_x$$

$$(\mathbf{a} \wedge \mathbf{b})\,x = \mathbf{r} \wedge \mathbf{b}$$
$$-42\,\gamma_t\gamma_x\,x = -168\,\gamma_t\gamma_x$$
$$\Rightarrow \quad x = 4$$

$$\Rightarrow \quad \mathbf{a\,r} = (3\,\gamma_t + 6\,\gamma_x)\,(28\,\gamma_t + 28\,\gamma_x)$$
$$= 84\,\gamma_t^2 + 84\,\gamma_t\gamma_x + 168\,\gamma_x\gamma_t + 168\,\gamma_x^2$$
$$= -84 - 84\,\gamma_t\gamma_x$$

$$\mathbf{a} \wedge \mathbf{r} = -84\,\gamma_t\gamma_x$$

$$(\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r}$$
$$-42\,\gamma_t\gamma_x\,y = -84\,\gamma_t\gamma_x$$
$$\Rightarrow \quad y = 2$$

Check: $3 \cdot 4 + 8 \cdot 2 = 12 + 16 = 28$

$6 \cdot 4 + 2 \cdot 2 = 24 + 4 = 28$

b) $4\,x + 9\,y = 29$ $\qquad \Rightarrow \quad \mathbf{a} = 4\,\gamma_t + 5\,\gamma_x$

$5\,x + 6\,y = 31$ $\qquad \mathbf{b} = 9\,\gamma_t + 6\,\gamma_x$

$\mathbf{r} = 29\,\gamma_t + 31\,\gamma_x$

$\Rightarrow \quad \mathbf{a\,b} = (4\,\gamma_t + 5\,\gamma_x)\,(9\,\gamma_t + 6\,\gamma_x)$
$= 36\,\gamma_t^2 + 24\,\gamma_t\gamma_x + 45\,\gamma_x\gamma_t + 30\,\gamma_x^2$
$= 6 - 21\,\gamma_t\gamma_x$

$\mathbf{a} \wedge \mathbf{b} = -21\,\gamma_t\gamma_x$

$\Rightarrow \quad \mathbf{r\,b} = (29\,\gamma_t + 31\,\gamma_x)\,(9\,\gamma_t + 6\,\gamma_x)$
$= 261\,\gamma_t^2 + 174\,\gamma_t\gamma_x + 279\,\gamma_x\gamma_t + 186\,\gamma_x^2$
$= 75 - 105\,\gamma_t\gamma_x$

$\mathbf{r} \wedge \mathbf{b} = -105\,\gamma_t\gamma_x$

$(\mathbf{a} \wedge \mathbf{b})\,x = \mathbf{r} \wedge \mathbf{b}$
$-21\,\gamma_t\gamma_x\,x = -105\,\gamma_t\gamma_x$
$\Rightarrow \quad x = 5$

$\Rightarrow \quad \mathbf{a\,r} = (4\,\gamma_t + 5\,\gamma_x)\,(29\,\gamma_t + 31\,\gamma_x)$
$= 116\,\gamma_t^2 + 124\,\gamma_t\gamma_x + 145\,\gamma_x\gamma_t + 155\,\gamma_x^2$
$= -39 - 21\,\gamma_t\gamma_x$

$\mathbf{a} \wedge \mathbf{r} = -21\,\gamma_t\gamma_x$

$(\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r}$
$-21\,\gamma_t\gamma_x\,y = -21\,\gamma_t\gamma_x$
$\Rightarrow \quad y = 1$

Check: $4 \cdot 5 + 9 \cdot 1 = 20 + 9 = 29$
$5 \cdot 5 + 6 \cdot 1 = 25 + 6 = 31$

c) $6\,x + 4\,y = 6$
$2\,x + \,y = 3$

$\Rightarrow \quad \mathbf{a} = 6\,\gamma_t + 2\,\gamma_x$
$\mathbf{b} = 4\,\gamma_t + \gamma_x$
$\mathbf{r} = 6\,\gamma_t + 3\,\gamma_x$

$\Rightarrow \quad \mathbf{a\,b} = (6\,\gamma_t + 2\,\gamma_x)\,(4\,\gamma_t + \gamma_x)$
$= 24\,\gamma_t^2 + 6\,\gamma_t\gamma_x + 8\,\gamma_x\gamma_t + 2\,\gamma_x^2$
$= 22 - 2\,\gamma_t\gamma_x$

$\mathbf{a} \wedge \mathbf{b} = -2\,\gamma_t\gamma_x$

$\Rightarrow \quad \mathbf{r\,b} = (6\,\gamma_t + 3\,\gamma_x)\,(4\,\gamma_t + \gamma_x)$
$= 24\,\gamma_t^2 + 6\,\gamma_t\gamma_x + 12\,\gamma_x\gamma_t + 3\,\gamma_x^2$
$= 21 - 6\,\gamma_t\gamma_x$

$\mathbf{r} \wedge \mathbf{b} = -6\,\gamma_t\gamma_x$

$(\mathbf{a} \wedge \mathbf{b})\,x = \mathbf{r} \wedge \mathbf{b}$
$-2\,\gamma_t\gamma_x\,x = -6\,\gamma_t\gamma_x$
$\Rightarrow \quad x = 3$

$\Rightarrow \quad \mathbf{a\,r} = (6\,\gamma_t + 2\,\gamma_x)\,(6\,\gamma_t + 3\,\gamma_x)$
$= 36\,\gamma_t^2 + 18\,\gamma_t\gamma_x + 12\,\gamma_x\gamma_t + 6\,\gamma_x^2$
$= 30 + 6\,\gamma_t\gamma_x$

$\mathbf{a} \wedge \mathbf{r} = 6\,\gamma_t\gamma_x$

$(\mathbf{a} \wedge \mathbf{b})\,y = \mathbf{a} \wedge \mathbf{r}$
$-2\,\gamma_t\gamma_x\,y = 6\,\gamma_t\gamma_x$
$\Rightarrow \quad y = -3$

Check: $6 \cdot 3 + 4 \cdot (-3) = 18 - 12 = 6$
$2 \cdot 3 + (-3) = 6 - 3 = 3$

d) $\begin{aligned} 5x - 2y &= 6 \\ -2x - 3y &= 28 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 5\,\gamma_t - 2\,\gamma_x \\ \mathbf{b} &= -2\,\gamma_t - 3\,\gamma_x \\ \mathbf{r} &= 6\,\gamma_t + 28\,\gamma_x \end{aligned}$

$$\begin{aligned} \Rightarrow \quad \mathbf{a\,b} &= (5\,\gamma_t - 2\,\gamma_x)\,(-2\,\gamma_t - 3\,\gamma_x) \\ &= -10\,\gamma_t^2 - 15\,\gamma_t\gamma_x + 4\,\gamma_x\gamma_t + 6\,\gamma_x^2 \\ &= -16 - 19\,\gamma_t\gamma_x \end{aligned}$$

$$\mathbf{a} \wedge \mathbf{b} = -19\,\gamma_t\gamma_x$$

$$\begin{aligned} \Rightarrow \quad \mathbf{r\,b} &= (6\,\gamma_t + 28\,\gamma_x)\,(-2\,\gamma_t - 3\,\gamma_x) \\ &= -12\,\gamma_t^2 - 18\,\gamma_t\gamma_x - 56\,\gamma_t\gamma_x - 84\,\gamma_x^2 \\ &= 72 + 38\,\gamma_t\gamma_x \end{aligned}$$

$$\mathbf{r} \wedge \mathbf{b} = 38\,\gamma_t\gamma_x$$

$$\left.\right\} \quad \begin{aligned} (\mathbf{a} \wedge \mathbf{b})\,x &= \mathbf{r} \wedge \mathbf{b} \\ -19\,\gamma_t\gamma_x\,x &= 38\,\gamma_t\gamma_x \\ \Rightarrow \quad x &= -2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \mathbf{a\,r} &= (5\,\gamma_t - 2\,\gamma_x)\,(6\,\gamma_t + 28\,\gamma_x) \\ &= 30\,\gamma_t^2 + 140\,\gamma_t\gamma_x - 12\,\gamma_x\gamma_t - 56\,\gamma_x^2 \\ &= 86 + 152\,\gamma_t\gamma_x \end{aligned}$$

$$\mathbf{a} \wedge \mathbf{r} = 152\,\gamma_t\gamma_x$$

$$\left.\right\} \quad \begin{aligned} (\mathbf{a} \wedge \mathbf{b})\,y &= \mathbf{a} \wedge \mathbf{r} \\ -19\,\gamma_t\gamma_x\,y &= 152\,\gamma_t\gamma_x \\ \Rightarrow \quad y &= -8 \end{aligned}$$

Check: $\begin{aligned} 5 \cdot (-2) - 2 \cdot (-8) &= -10 + 16 = 6 \\ -2 \cdot (-2) - 3 \cdot (-8) &= 4 + 24 = 28 \end{aligned}$

Problem 6:

The system of two linear equations of this text problem is identical to the system of linear equations of problem 3 a). Therefore the results of that problem can be used.

$\begin{aligned} 3x + 8y &= 28 \\ 6x + 2y &= 28 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 3\,\gamma_t + 6\,\gamma_x \\ \mathbf{b} &= 8\,\gamma_t + 2\,\gamma_x \\ \mathbf{r} &= 28\,\gamma_t + 28\,\gamma_x \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} \wedge \mathbf{b} &= -42\,\gamma_t\gamma_x \\ \mathbf{r} \wedge \mathbf{b} &= -168\,\gamma_t\gamma_x \\ \mathbf{a} \wedge \mathbf{r} &= -84\,\gamma_t\gamma_x \end{aligned}$

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \frac{-168}{-42} = 4 \qquad y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \frac{-84}{-42} = 2$$

Check:

		4
		2
3	8	28
6	2	28

If 28 units of the first raw material R_1 and 28 units of the second raw material R_2 are consumed in the production process, 4 units of the first final product P_1 and 2 units of the second final product P_2 will be produced.

Problem 7:

$$
\begin{array}{lll}
2x + 7y = 2050 & \Rightarrow\ \mathbf{a} = 2\,\gamma_t + 5\,\gamma_x & \Rightarrow\ \mathbf{a} \wedge \mathbf{b} = -33\,\gamma_t\gamma_x \\
5x + \ \ y = 1000 & \qquad \mathbf{b} = 7\,\gamma_t + \gamma_x & \qquad \mathbf{r} \wedge \mathbf{b} = -4950\,\gamma_t\gamma_x \\
 & \qquad \mathbf{r} = 2050\,\gamma_t + 1000\,\gamma_x & \qquad \mathbf{a} \wedge \mathbf{r} = -8250\,\gamma_t\gamma_x
\end{array}
$$

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = \frac{-4950}{-33} = 150 \qquad y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = \frac{-8250}{-33} = 250$$

Check:

$$
\begin{array}{cc|c}
 & & 150 \\
 & & 250 \\
\hline
2 & 7 & 2050 \\
5 & 1 & 1000
\end{array}
$$

If 2050 units of the first raw material R_1 and 1000 units of the second raw material R_2 are consumed in the production process, 150 units of the first final product P_1 and 250 units of the second final product P_2 will be produced.

Problem 8:

$$
\begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}
\underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}}_{\mathbf{P}}
=
\underbrace{
\begin{array}{cc}
\text{1st quarter} & \text{2nd quarter} \\
\downarrow & \downarrow
\end{array}
\begin{bmatrix} 33\,000 & 32\,000 \\ 38\,000 & 25\,000 \end{bmatrix}}_{\mathbf{R}}
$$

R .... matrix of quarterly consumption of raw materials (consumption matrix)

P .... matrix of quarterly production (production matrix)

$$
\begin{array}{lll}
4x_1 + 3y_1 = 33000 & \Rightarrow\ \mathbf{a} = 4\,\gamma_t + \gamma_x & \Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 17\,\gamma_t\gamma_x \\
\ \ x_1 + 5y_1 = 38000 & \qquad \mathbf{b} = 3\,\gamma_t + 5\,\gamma_x & \qquad \mathbf{r_1} \wedge \mathbf{b} = 51000\,\gamma_t\gamma_x \\
 & \qquad \mathbf{r_1} = 33000\,\gamma_t + 38000\,\gamma_x & \qquad \mathbf{a} \wedge \mathbf{r_1} = 119000\,\gamma_t\gamma_x
\end{array}
$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{51000}{17} = 3000 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{119000}{17} = 7000$$

$$
\begin{array}{lll}
4x_2 + 3y_2 = 32000 & \Rightarrow\ \mathbf{a} = 4\,\gamma_t + \gamma_x & \Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 17\,\gamma_t\gamma_x \\
\ \ x_2 + 5y_2 = 25000 & \qquad \mathbf{b} = 3\,\gamma_t + 5\,\gamma_x & \qquad \mathbf{r_2} \wedge \mathbf{b} = 85000\,\gamma_t\gamma_x \\
 & \qquad \mathbf{r_2} = 32000\,\gamma_t + 25000\,\gamma_x & \qquad \mathbf{a} \wedge \mathbf{r_2} = 68000\,\gamma_t\gamma_x
\end{array}
$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{85000}{17} = 5000 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{68000}{17} = 4000$$

$\Rightarrow$ matrix of quarterly production: $\mathbf{P} = \begin{bmatrix} 3\,000 & 5\,000 \\ 7\,000 & 4\,000 \end{bmatrix}$

Check:				
			3000	5000
			7000	4000
4	3		33000	32000
1	5		38000	25000

3000 units of the first final product P_1 and 7000 units of the second final product P_2 will be produced in the first quarter.

5000 units of the first final product P_1 and 4000 units of the second final product P_2 will be produced in the second quarter.

Problem 9:

$$\underbrace{\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 42 & 28 \\ 23 & 26 \end{bmatrix}}_{\mathbf{D}} \qquad \mathbf{A}\,\mathbf{B} = \mathbf{D}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$8\,x_1 + 2\,y_1 = 42 \quad \Rightarrow \quad \mathbf{a} = 8\,\gamma_t + 4\,\gamma_x \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 16\,\gamma_t\gamma_x$

$4\,x_1 + 3\,y_1 = 23 \qquad \mathbf{b} = 2\,\gamma_t + 3\,\gamma_x \qquad \mathbf{r_1} \wedge \mathbf{b} = 80\,\gamma_t\gamma_x$

$\mathbf{r_1} = 42\,\gamma_t + 23\,\gamma_x \qquad \mathbf{a} \wedge \mathbf{r_1} = 16\,\gamma_t\gamma_x$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{80}{16} = 5 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{16}{16} = 1$$

$8\,x_2 + 2\,y_2 = 28 \quad \Rightarrow \quad \mathbf{a} = 8\,\gamma_t + 4\,\gamma_x \quad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 16\,\gamma_t\gamma_x$

$4\,x_2 + 3\,y_2 = 26 \qquad \mathbf{b} = 2\,\gamma_t + 3\,\gamma_x \qquad \mathbf{r_2} \wedge \mathbf{b} = 32\,\gamma_t\gamma_x$

$\mathbf{r_2} = 28\,\gamma_t + 26\,\gamma_x \qquad \mathbf{a} \wedge \mathbf{r_2} = 96\,\gamma_t\gamma_x$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{32}{16} = 2 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{96}{16} = 6$$

Check:				
			5	2
			1	6
8	2		42	28
4	3		23	26

$\Rightarrow$ demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix}$

Problem 10:

$$\underbrace{\begin{bmatrix} 9 & 3 \\ 2 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 48 & 21 & 84 \\ 12 & 14 & 32 \end{bmatrix}}_{\mathbf{D}}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$$\begin{array}{llll} 9\,x_1 + 3\,y_1 = 48 & \Rightarrow \quad \mathbf{a} = 9\,\gamma_t + 2\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 12\,\gamma_t\gamma_x \\ 2\,x_1 + 2\,y_1 = 12 & \qquad \mathbf{b} = 3\,\gamma_t + 2\,\gamma_x & & \mathbf{r_1} \wedge \mathbf{b} = 60\,\gamma_t\gamma_x \\ & \qquad \mathbf{r_1} = 48\,\gamma_t + 12\,\gamma_x & & \mathbf{a} \wedge \mathbf{r_1} = 12\,\gamma_t\gamma_x \end{array}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{60}{12} = 5 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{12}{12} = 1$$

$$\begin{array}{llll} 9\,x_2 + 3\,y_2 = 21 & \Rightarrow \quad \mathbf{a} = 9\,\gamma_t + 2\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 12\,\gamma_t\gamma_x \\ 2\,x_2 + 2\,y_2 = 14 & \qquad \mathbf{b} = 3\,\gamma_t + 2\,\gamma_x & & \mathbf{r_2} \wedge \mathbf{b} = 0\,\gamma_t\gamma_x \\ & \qquad \mathbf{r_2} = 21\,\gamma_t + 14\,\gamma_x & & \mathbf{a} \wedge \mathbf{r_2} = 84\,\gamma_t\gamma_x \end{array}$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{0}{12} = 0 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{84}{12} = 7$$

$$\begin{array}{llll} 9\,x_3 + 3\,y_3 = 84 & \Rightarrow \quad \mathbf{a} = 9\,\gamma_t + 2\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 12\,\gamma_t\gamma_x \\ 2\,x_3 + 2\,y_3 = 32 & \qquad \mathbf{b} = 3\,\gamma_t + 2\,\gamma_x & & \mathbf{r_3} \wedge \mathbf{b} = 72\,\gamma_t\gamma_x \\ & \qquad \mathbf{r_3} = 84\,\gamma_t + 32\,\gamma_x & & \mathbf{a} \wedge \mathbf{r_3} = 120\,\gamma_t\gamma_x \end{array}$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b}) = \frac{72}{12} = 6 \qquad y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3}) = \frac{120}{12} = 10$$

Check:				
		5	0	6
		1	7	10
9	3	48	21	84
2	2	12	14	32

$\Rightarrow$ demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 5 & 0 & 6 \\ 1 & 7 & 10 \end{bmatrix}$

Problem 11:

First part of problem 11: Consumption of exactly one unit of the first raw material R_1

$$\begin{array}{llll} 7\,x + 5\,y = 1 & \Rightarrow \quad \mathbf{a} = 7\,\gamma_t + 4\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 1\,\gamma_t\gamma_x = \gamma_t\gamma_x \\ 4\,x + 3\,y = 0 & \qquad \mathbf{b} = 5\,\gamma_t + 3\,\gamma_x & & \mathbf{r_1} \wedge \mathbf{b} = 3\,\gamma_t\gamma_x \\ & \qquad \mathbf{r_1} = 1\,\gamma_t = \gamma_t & & \mathbf{a} \wedge \mathbf{r_1} = -\,4\,\gamma_t\gamma_x \end{array}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{3}{1} = 3 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-4}{1} = -4$$

Economic interpretation:

If exactly one unit of the first raw material R_1 had been consumed in the production process, 3 units of the first final product P_1 and (-4) units of the second final product P_2 would have been produced. However, the production of a negative number of final products is problematic.

Producing (-4) units means that in addition to an already produced quantity (-4) units are added. Mathematically, the negative number “minus four” is added or alternatively, the positive number “four” is subtracted. Thus after the production process the quantity is reduced by four units.

Therefore these four units will not be produced, but consumed and (in theory completely) split again into the initial raw materials R_1 and R_2.

The correct economic interpretation will then be:

If exactly one unit of the first raw material R_1 had been consumed in the production process, 3 units of the first final product P_1 would have been produced and additionally 4 units of the second final product P_2 would have been consumed.

Second part of problem 11: Consumption of exactly one unit of the second raw material R_2

$$\begin{array}{lclcl}
7\,x + 5\,y = 0 & \Rightarrow & \mathbf{a} = 7\,\gamma_t + 4\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 1\,\gamma_t\gamma_x = \gamma_t\gamma_x \\
4\,x + 3\,y = 1 & & \mathbf{b} = 5\,\gamma_t + 3\,\gamma_x & & \mathbf{r_2} \wedge \mathbf{b} = -5\,\gamma_t\gamma_x \\
 & & \mathbf{r_2} = 1\,\gamma_x = \gamma_x & & \mathbf{a} \wedge \mathbf{r_2} = 7\,\gamma_t\gamma_x
\end{array}$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-5}{1} = -5 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{7}{1} = 7$$

Economic interpretation:

If exactly one unit of the second raw material R_2 had been consumed in the production process, in addition 5 units of the first final product P_1 would have been consumed and 7 units of the second final product P_2 would have been produced.

As a complete splitting of products into the initial raw materials is hardly possible (and then usually connected with higher costs), negative production quantities or a negative output will only very rarely be part of realistic economical situations.

But **mathematically** the results just found are of enormous importance, which can be seen at the following check of the results.

Check:

$$\begin{array}{rr|rl}
 & & \begin{matrix}3 & -5\\ -4 & 7\end{matrix} & \left.\vphantom{\begin{matrix}3\\4\end{matrix}}\right\}\text{inverse } \mathbf{A}^{-1}\text{ of matrix }\mathbf{A} \\ \hline
\text{initial matrix }\mathbf{A}\left\{\vphantom{\begin{matrix}3\\4\end{matrix}}\right. & \begin{matrix}7 & 5\\ 4 & 3\end{matrix} & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & \left.\vphantom{\begin{matrix}3\\4\end{matrix}}\right\}\text{identity matrix }\mathbf{I}
\end{array}$$

Mathematical interpretation:

The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$.

Problem 12:

First part of problem 12: Consumption of exactly one unit of the first raw material R_1

$$\begin{array}{lclcl} 10\,x + 12\,y = 1 & \Rightarrow & \mathbf{a} = 10\,\gamma_t + 4\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 2\,\gamma_t\gamma_x \\ \;\;4\,x + \;\;5\,y = 0 & & \mathbf{b} = 12\,\gamma_t + 5\,\gamma_x & & \mathbf{r_1} \wedge \mathbf{b} = 5\,\gamma_t\gamma_x \\ & & \mathbf{r_1} = 1\,\gamma_t = \gamma_t & & \mathbf{a} \wedge \mathbf{r_1} = -\,4\,\gamma_t\gamma_x \end{array}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{5}{2} = 2.5 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-4}{2} = -\,2$$

Economic interpretation:

If exactly one unit of the first raw material R_1 had been consumed in the production process, 2.5 units of the first final product P_1 would have been produced and additionally 2 units of the second final product P_2 would have been consumed.

Second part of problem 12: Consumption of exactly one unit of the second raw material R_2

$$\begin{array}{lclcl} 10\,x + 12\,y = 0 & \Rightarrow & \mathbf{a} = 10\,\gamma_t + 4\,\gamma_x & \Rightarrow & \mathbf{a} \wedge \mathbf{b} = 2\,\gamma_t\gamma_x \\ \;\;4\,x + \;\;5\,y = 1 & & \mathbf{b} = 12\,\gamma_t + 5\,\gamma_x & & \mathbf{r_2} \wedge \mathbf{b} = -\,12\,\gamma_t\gamma_x \\ & & \mathbf{r_2} = 1\,\gamma_x = \gamma_x & & \mathbf{a} \wedge \mathbf{r_2} = 10\,\gamma_t\gamma_x \end{array}$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-12}{2} = -\,6 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{10}{2} = 5$$

Economic interpretation:

If exactly one unit of the second raw material R_2 had been consumed in the production process, in addition 6 units of the first final product P_1 would have been consumed and 5 units of the second final product P_2 would have been produced.

Check:

$$\begin{array}{rcc|ccl} & & & 2.5 & -\,6 & \multirow{2}{*}{$\left.\right\}$ inverse $\mathbf{A}^{-1}$ of matrix $\mathbf{A}$} \\ & & & -\,2 & 5 & \\ \hline \text{initial matrix } \mathbf{A} \left\{\right. & 10 & 12 & 1 & 0 & \left.\right\} \text{ identity matrix } \mathbf{I} \\ & 4 & 5 & 0 & 1 & \end{array}$$

Result:

The inverse of the initial demand matrix $\mathbf{A} = \begin{bmatrix} 10 & 12 \\ 4 & 5 \end{bmatrix}$ is $\mathbf{A}^{-1} = \begin{bmatrix} 2.5 & -6 \\ -2 & 5 \end{bmatrix}$.

Problem 13:

a)

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 5\,\gamma_t + 9\,\gamma_x \\ \mathbf{b} &= 4\,\gamma_t + 7\,\gamma_x \\ \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\mathbf{a} \wedge \mathbf{b} = -\,\gamma_t\gamma_x && \\ &\mathbf{r_1} \wedge \mathbf{b} = 7\,\gamma_t\gamma_x && \mathbf{r_2} \wedge \mathbf{b} = -\,4\,\gamma_t\gamma_x \\ &\mathbf{a} \wedge \mathbf{r_1} = -\,9\,\gamma_t\gamma_x && \mathbf{a} \wedge \mathbf{r_2} = 5\,\gamma_t\gamma_x \end{aligned}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{7}{-1} = -\,7 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-9}{-1} = 9$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-4}{-1} = 4 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{5}{-1} = -\,5$$

Check:

		– 7	4
		9	– 5
5	4	1	0
9	7	0	1

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} -7 & 4 \\ 9 & -5 \end{bmatrix}$$

b)

$$\mathbf{B} = \begin{bmatrix} 10 & 4 \\ 19 & 8 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 10\,\gamma_t + 19\,\gamma_x \\ \mathbf{b} &= 4\,\gamma_t + 8\,\gamma_x \\ \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\mathbf{a} \wedge \mathbf{b} = 4\,\gamma_t\gamma_x && \\ &\mathbf{r_1} \wedge \mathbf{b} = 8\,\gamma_t\gamma_x && \mathbf{r_2} \wedge \mathbf{b} = -\,4\,\gamma_t\gamma_x \\ &\mathbf{a} \wedge \mathbf{r_1} = -\,19\,\gamma_t\gamma_x && \mathbf{a} \wedge \mathbf{r_2} = 10\,\gamma_t\gamma_x \end{aligned}$$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{8}{4} = 2 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-19}{4} = -\,4.75$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-4}{4} = -\,1 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{10}{4} = 2.5$$

Check:

		2	– 1
		– 4.75	2.5
10	4	1	0
19	8	0	1

$$\Rightarrow \quad \mathbf{B}^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -19 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4.75 & 2.5 \end{bmatrix}$$

c) $\mathbf{C} = \begin{bmatrix} 10 & 6 \\ 20 & 13 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 10\,\gamma_t + 20\,\gamma_x \\ \mathbf{b} &= 6\,\gamma_t + 13\,\gamma_x \\ \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} \wedge \mathbf{b} &= 10\,\gamma_t\gamma_x \\ \mathbf{r_1} \wedge \mathbf{b} &= 13\,\gamma_t\gamma_x \\ \mathbf{a} \wedge \mathbf{r_1} &= -20\,\gamma_t\gamma_x \end{aligned} \qquad \begin{aligned} \mathbf{r_2} \wedge \mathbf{b} &= -6\,\gamma_t\gamma_x \\ \mathbf{a} \wedge \mathbf{r_2} &= 10\,\gamma_t\gamma_x \end{aligned}$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{13}{10} = 1.3 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-20}{10} = -2$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{-6}{10} = -0.6 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{10}{10} = 1$$

Check:

		1.3	– 0.6
		– 2	1
10	6	1	0
20	13	0	1

$$\Rightarrow \quad \mathbf{C}^{-1} = \frac{1}{10}\begin{bmatrix} 13 & -6 \\ -20 & 10 \end{bmatrix} = \begin{bmatrix} 1.3 & -0.6 \\ -2 & 1 \end{bmatrix}$$

d) $\mathbf{D} = \begin{bmatrix} 0 & -2.5 \\ 0.2 & 3.4 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 0.2\,\gamma_x \\ \mathbf{b} &= -2.5\,\gamma_t + 3.4\,\gamma_x \\ \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} \wedge \mathbf{b} &= 0.5\,\gamma_t\gamma_x \\ \mathbf{r_1} \wedge \mathbf{b} &= 3.4\,\gamma_t\gamma_x \\ \mathbf{a} \wedge \mathbf{r_1} &= -0.2\,\gamma_t\gamma_x \end{aligned} \qquad \begin{aligned} \mathbf{r_2} \wedge \mathbf{b} &= 2.5\,\gamma_t\gamma_x \\ \mathbf{a} \wedge \mathbf{r_2} &= 0 \end{aligned}$

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) = \frac{3.4}{0.5} = 2 \cdot 3.4 = 6.8 \qquad y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1}) = \frac{-0.2}{0.5} = 2 \cdot (-0.2) = -0.4$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b}) = \frac{2.5}{0.5} = 2 \cdot 2.5 = 5 \qquad y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2}) = \frac{0}{0.5} = 2 \cdot 0 = 0$$

Check:

		6.8	5
		– 0.4	0
0	– 2.5	1	0
0.2	3.4	0	1

$$\Rightarrow \quad \mathbf{D}^{-1} = 2 \cdot \begin{bmatrix} 3.4 & 2.5 \\ -0.2 & 0 \end{bmatrix} = \begin{bmatrix} 6.8 & 5 \\ -0.4 & 0 \end{bmatrix}$$

Problem 14:

In the introduction of his book Vince explains on page 3: “Chapter 11 addresses the conformal model developed by David Hestenes et al. Its use of 5D Minkowski space is a recent development and has natural applications to quantum physics and electrodynamics, but is also being applied to computer graphics." And he goes on explaining at the beginning of chap. 11 on page 199: "(…) conformal space requires five dimensions, (…) one of the dimensions has, what is called a negative signature, which transforms the space into a Minkowski space."

Thus we discuss Minkowski space, because Minkowski space is the space of conformal geometry. And later we will discuss conformal geometry, because it is an important mathematical tool in computer graphics.

To be able to understand Minkowski space, we will use a very important idea, stated in clear words by David Hestenes already half a century ago: "The reader is asked to think of the γ_μ as a frame of four orthonormal vectors in space-time." (See: David Hestenes: Real Spinor Fields. Journal of Mathematical Physics, Vol. 8, No. 4 (1967), pp. 798 – 808).

Thus we discuss Dirac algebra because this algebra describes the mathematical foundations of Minkowski space – and the spacetime of Einstein's special theory of relativity.

When travelling the universe one day, you will need it, as it is also the mathematics of starship Voyager. So please have an additional look on the dialogue between Neelix and Tuvok, the Vulcan, in:

> Miroslav Josipović: Geometric Multiplication of Vectors. An Introduction to Geometric Algebra in Physics. Birkhäuser / Springer Nature Switzerland, Cham 2019, corrected publication 2020.

Josipović there gives a nice overview of the conformal model and its connection to Minkowski space in sec. 2.12, pp. 117 – 119.

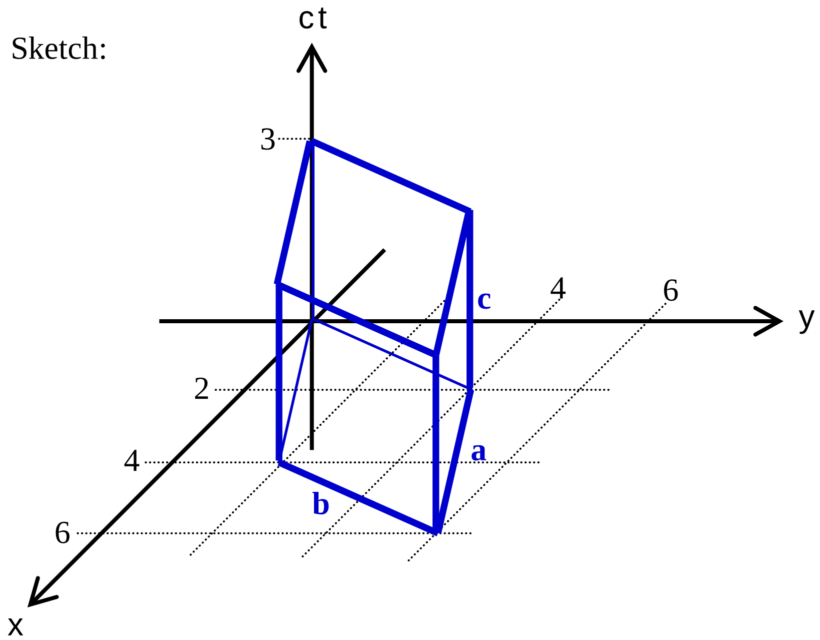
Advanced Mathematics (MQM110)

Worksheet 9 – Answers

Problem 1:

a) $\mathbf{a} = 4\,\gamma_x + 2\,\gamma_y$
$\mathbf{b} = 2\,\gamma_x + 4\,\gamma_y$
$\mathbf{c} = 3\,\gamma_t$

Sketch:



Detailed calculation:

$$\mathbf{a}\,\mathbf{b} = (4\,\gamma_x + 2\,\gamma_y)\,(2\,\gamma_x + 4\,\gamma_y)$$

$$= 8\,\gamma_x^2 + 16\,\gamma_x\gamma_y + 4\,\gamma_y\gamma_x + 8\,\gamma_y^2$$

$$= -\,8 + 16\,\gamma_x\gamma_y - 4\,\gamma_x\gamma_y - 8$$

$$= -\,16 + 12\,\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 12\,\gamma_x\gamma_y$$

$$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (-\,16 + 12\,\gamma_x\gamma_y)\,(3\,\gamma_t) = -\,48\,\gamma_t + 36\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 36\,\gamma_t\gamma_x\gamma_y$$

$$\Rightarrow \quad |\mathbf{V}| = 36$$

$\Rightarrow$ The spacetime volume of the parallelepiped is 36 cm^3.

Check by applying the rule of Sarrus:

$$\begin{aligned} \mathbf{a} &= 0\,\gamma_t + 4\,\gamma_x + 2\,\gamma_y \\ \mathbf{b} &= 0\,\gamma_t + 2\,\gamma_x + 4\,\gamma_y \\ \mathbf{c} &= 3\,\gamma_t + 0\,\gamma_x + 0\,\gamma_y \end{aligned} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 3 \\ 4 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix} \quad \Rightarrow \quad \det \mathbf{A} = 36$$

Magnitude check: $\mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2 = (-\,4^2 - 2^2)\cdot(-\,2^2 - 4^2)\cdot 3^2 = 3600$

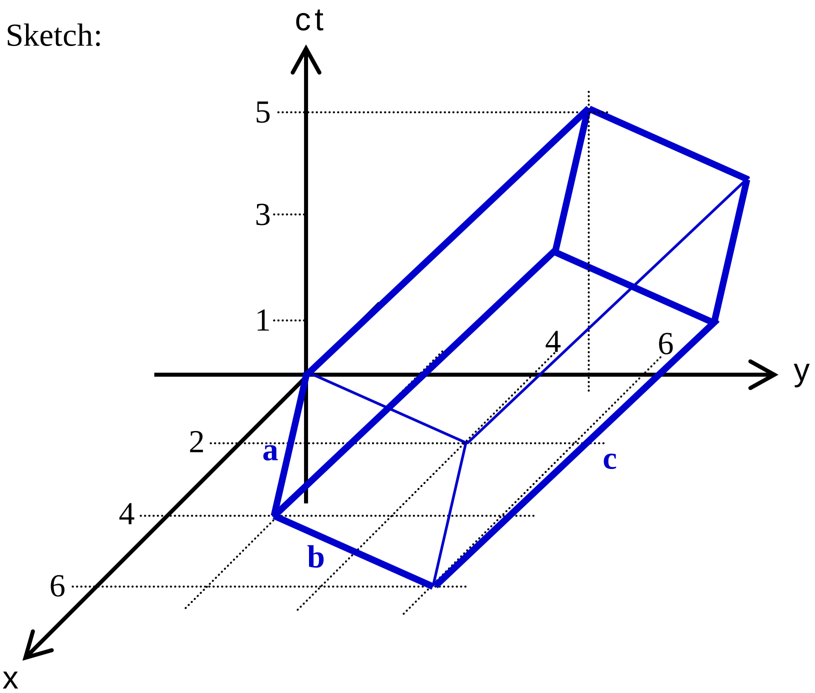
$(\mathbf{a}\,\mathbf{b}\,\mathbf{c})\,(\mathbf{a}\,\mathbf{b}\,\mathbf{c})^{\sim} = (-\,48)^2 + 36^2 = 3600 \qquad \surd$

b) $\mathbf{a} = 4\,\gamma_x + 2\,\gamma_y$

$\mathbf{b} = 2\,\gamma_x + 4\,\gamma_y$

$\mathbf{c} = 5\,\gamma_t + 5\,\gamma_y$

Sketch:



Detailed calculation:

$$\begin{aligned}
\mathbf{a}\,\mathbf{b} &= (4\,\gamma_x + 2\,\gamma_y)\,(2\,\gamma_x + 4\,\gamma_y) \\
&= 8\,\gamma_x^{\,2} + 16\,\gamma_x\gamma_y + 4\,\gamma_y\gamma_x + 8\,\gamma_y^{\,2} \\
&= -\,8 + 16\,\gamma_x\,\gamma_y - 4\,\gamma_x\gamma_y - 8 \\
&= -\,16 + 12\,\gamma_x\gamma_y
\end{aligned}
\qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 12\,\gamma_x\gamma_y$$

$$\begin{aligned}
\mathbf{a}\,\mathbf{b}\,\mathbf{c} &= (-\,16 + 12\,\gamma_x\gamma_y)\,(5\,\gamma_t + 5\,\gamma_y) \\
&= -\,80\,\gamma_t - 80\,\gamma_y + 60\,\gamma_x\gamma_y\gamma_t + 60\,\gamma_x\gamma_y\gamma_y \\
&= -\,80\,\gamma_t - 80\,\gamma_y + 60\,\gamma_t\gamma_x\gamma_y - 60\,\gamma_x \\
&= -\,80\,\gamma_t\; - 60\,\gamma_x - 80\,\gamma_y + 60\,\gamma_t\gamma_x\gamma_y
\end{aligned}
\qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 60\,\gamma_t\gamma_x\gamma_y$$

$$\Rightarrow \quad |\mathbf{V}| = 60$$

$\Rightarrow$ The spacetime volume of the parallelepiped is 60 cm^3.

Check by applying the rule of Sarrus:

$$\begin{aligned}
\mathbf{a} &= 0\,\gamma_t + 4\,\gamma_x + 2\,\gamma_y \\
\mathbf{b} &= 0\,\gamma_t + 2\,\gamma_x + 4\,\gamma_y \\
\mathbf{c} &= 5\,\gamma_t + 0\,\gamma_x + 5\,\gamma_y
\end{aligned}
\qquad \Rightarrow \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 5 \\ 4 & 2 & 0 \\ 2 & 4 & 5 \end{bmatrix} \qquad \Rightarrow \qquad \det \mathbf{B} = 60$$

Magnitude check: $\mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2 = (-\,4^2 - 2^2)\cdot(-\,2^2 - 4^2)\cdot(5^2 - 5^2) = 0$

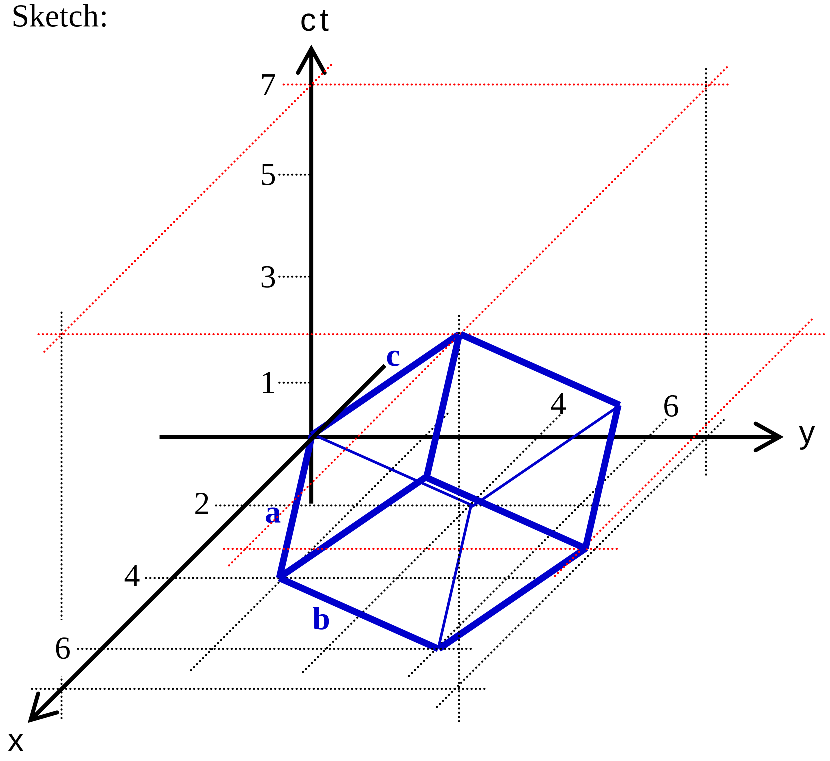
$$(\mathbf{a}\,\mathbf{b}\,\mathbf{c})\,(\mathbf{a}\,\mathbf{b}\,\mathbf{c})^{\sim} = (-\,80)^2 - (-\,60)^2) - (-\,80)^2 + 60^2 = 0 \qquad \surd$$

c) $\mathbf{a} = 4\,\gamma_x + 2\,\gamma_y$

$\mathbf{b} = 2\,\gamma_x + 4\,\gamma_y$

$\mathbf{c} = 7\,\gamma_t + 7\,\gamma_x + 7\,\gamma_y$

Sketch:



Detailed calculation:

$$\mathbf{a}\,\mathbf{b} = (4\,\gamma_x + 2\,\gamma_y)\,(2\,\gamma_x + 4\,\gamma_y)$$
$$= 8\,\gamma_x^{\;2} + 16\,\gamma_x\gamma_y + 4\,\gamma_y\gamma_x + 8\,\gamma_y^{\;2}$$
$$= -\,8 + 16\,\gamma_x\,\gamma_y - 4\,\gamma_x\gamma_y - 8$$
$$= -\,16 + 12\,\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a}\wedge\mathbf{b} = 12\,\gamma_x\gamma_y$$

$$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (-\,16 + 12\,\gamma_x\gamma_y)\,(7\,\gamma_t + 7\,\gamma_x + 7\,\gamma_y)$$
$$= -\,112\,\gamma_t - 112\,\gamma_x - 112\,\gamma_y + 84\,\gamma_x\gamma_y\gamma_t + 84\,\gamma_x\gamma_y\gamma_x + 84\,\gamma_x\gamma_y\gamma_y$$
$$= -\,112\,\gamma_t - 112\,\gamma_x - 112\,\gamma_y + 84\,\gamma_t\gamma_x\gamma_y + 84\,\gamma_y - 84\,\gamma_x$$
$$= -\,112\,\gamma_t - 196\,\gamma_x - 28\,\gamma_y + 84\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = 84\,\gamma_t\gamma_x\gamma_y$$
$$\Rightarrow \quad |\mathbf{V}| = 84$$

$\Rightarrow$ The spacetime volume of the parallelepiped is 84 cm^3.

Check by applying the rule of Sarrus:

$$\begin{aligned} \mathbf{a} &= 0\,\gamma_t + 4\,\gamma_x + 2\,\gamma_y \\ \mathbf{b} &= 0\,\gamma_t + 2\,\gamma_x + 4\,\gamma_y \\ \mathbf{c} &= 7\,\gamma_t + 7\,\gamma_x + 7\,\gamma_y \end{aligned} \quad \Rightarrow \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 7 \\ 4 & 2 & 7 \\ 2 & 4 & 7 \end{bmatrix} \quad \Rightarrow \quad \det \mathbf{C} = 84$$

Magnitude check: $\mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2 = (-\,4^2 - 2^2)\cdot(-\,2^2 - 4^2)\cdot(7^2 - 7^2 - 7^2) = -\,19600$

$(\mathbf{a}\,\mathbf{b}\,\mathbf{c})\,(\mathbf{a}\,\mathbf{b}\,\mathbf{c})^{\sim} = (-\,112)^2 - (-\,196)^2) - (-\,28)^2 + 84^2 = -\,19600 \quad \surd$

d) $\mathbf{a} = 5\,\gamma_t + 2\,\gamma_x + 5\,\gamma_y$
$\mathbf{b} = 6\,\gamma_t + 3\,\gamma_x + 3\,\gamma_y$
$\mathbf{c} = 4\,\gamma_t + 4\,\gamma_x + 4\,\gamma_y$

Detailed calculation:

$$\mathbf{a}\,\mathbf{b} = (5\,\gamma_t + 2\,\gamma_x + 5\,\gamma_y)\,(6\,\gamma_t + 3\,\gamma_x + 3\,\gamma_y)$$

$$= 30\,\gamma_t^2 + 15\,\gamma_t\gamma_x + 15\,\gamma_t\gamma_y + 12\,\gamma_x\gamma_t + 6\,\gamma_x^2 + 6\,\gamma_x\gamma_y + 30\,\gamma_y\gamma_t + 15\,\gamma_y\gamma_x + 15\,\gamma_y^2$$

$$= 30 + 15\,\gamma_t\gamma_x + 15\,\gamma_t\gamma_y - 12\,\gamma_t\gamma_x - 6 + 6\,\gamma_x\gamma_y - 30\,\gamma_t\gamma_y - 15\,\gamma_x\gamma_y - 15$$

$$= 9 + 3\,\gamma_t\gamma_x - 15\,\gamma_t\gamma_y - 9\,\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 3\,\gamma_t\gamma_x - 15\,\gamma_t\gamma_y - 9\,\gamma_x\gamma_y$$

$$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = (9 + 3\,\gamma_t\gamma_x - 15\,\gamma_t\gamma_y - 9\,\gamma_x\gamma_y)\,(4\,\gamma_t + 4\,\gamma_x + 4\,\gamma_y)$$

$$= 36\,\gamma_t + 36\,\gamma_x + 36\,\gamma_y + 12\,\gamma_t\gamma_x\gamma_t + 12\,\gamma_t\gamma_x\gamma_x + 12\,\gamma_t\gamma_x\gamma_y - 60\,\gamma_t\gamma_y\gamma_t - 60\,\gamma_t\gamma_y\gamma_x - 60\,\gamma_t\gamma_y\gamma_y$$
$$- 36\,\gamma_x\gamma_y\gamma_t - 36\,\gamma_x\gamma_y\gamma_x - 36\,\gamma_x\gamma_y\gamma_y$$

$$= 36\,\gamma_t + 36\,\gamma_x + 36\,\gamma_y - 12\,\gamma_x - 12\,\gamma_t + 12\,\gamma_t\gamma_x\gamma_y + 60\,\gamma_y + 60\,\gamma_t\gamma_x\gamma_y + 60\,\gamma_t$$
$$- 36\,\gamma_t\gamma_x\gamma_y - 36\,\gamma_y + 36\,\gamma_x$$

$$= 84\,\gamma_t + 60\,\gamma_x + 60\,\gamma_y + 36\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 36\,\gamma_t\gamma_x\gamma_y$$

$$\Rightarrow \quad |\mathbf{V}| = 36$$

$\Rightarrow$ The spacetime volume of the parallelepiped is 63 cm^3.

Check by applying the rule of Sarrus:

$$\begin{aligned} \mathbf{a} &= 5\,\gamma_t + 2\,\gamma_x + 5\,\gamma_y \\ \mathbf{b} &= 6\,\gamma_t + 3\,\gamma_x + 3\,\gamma_y \\ \mathbf{c} &= 4\,\gamma_t + 4\,\gamma_x + 4\,\gamma_y \end{aligned} \qquad \Rightarrow \qquad \mathbf{D} = \begin{bmatrix} 5 & 6 & 4 \\ 2 & 3 & 4 \\ 5 & 3 & 4 \end{bmatrix} \qquad \Rightarrow \qquad \det \mathbf{D} = 36$$

Magnitude check: $\mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2 = (5^2 - 2^2 - 5^2) \cdot (6^2 - 3^2 - 3^2) \cdot (4^2 - 4^2 - 4^2) = 1152$

$$(\mathbf{a}\,\mathbf{b}\,\mathbf{c})\,(\mathbf{a}\,\mathbf{b}\,\mathbf{c})^\sim = 84^2 - 60^2 - 60^2 + 36^2 = 1152 \qquad \surd$$

e) $\mathbf{a} = 10\,\gamma_t + 2\,\gamma_x + 6\,\gamma_y$
$\mathbf{b} = 12\,\gamma_t + 8\,\gamma_x + 3\,\gamma_y$
$\mathbf{c} = \ \ 4\,\gamma_t + 7\,\gamma_x + 9\,\gamma_y$

Detailed calculation:

$$\mathbf{a}\,\mathbf{b} = (10\,\gamma_t + 2\,\gamma_x + 6\,\gamma_y)\,(12\,\gamma_t + 8\,\gamma_x + 3\,\gamma_y)$$

$$= 120\,\gamma_t^2 + 80\,\gamma_t\gamma_x + 30\,\gamma_t\gamma_y + 24\,\gamma_x\gamma_t + 16\,\gamma_x^2 + 6\,\gamma_x\gamma_y + 72\,\gamma_y\gamma_t + 48\,\gamma_y\gamma_x + 18\,\gamma_y^2$$

$$= 120 + 80\,\gamma_t\gamma_x + 30\,\gamma_t\gamma_y - 24\,\gamma_t\gamma_x - 16 + 6\,\gamma_x\gamma_y - 72\,\gamma_t\gamma_y - 48\,\gamma_x\gamma_y - 18$$

$$= 86 + 56\,\gamma_t\gamma_x - 42\,\gamma_t\gamma_y - 42\,\gamma_x\gamma_y \qquad \Rightarrow \quad \mathbf{a} \wedge \mathbf{b} = 56\,\gamma_t\gamma_x - 42\,\gamma_t\gamma_y - 42\,\gamma_x\gamma_y$$

$$\mathbf{a\,b\,c} = (86 + 56\,\gamma_t\gamma_x - 42\,\gamma_t\gamma_y - 42\,\gamma_x\gamma_y)\,(4\,\gamma_t + 7\,\gamma_x + 9\,\gamma_y)$$

$$= 344\,\gamma_t + 602\,\gamma_x + 774\,\gamma_y + 224\,\gamma_t\gamma_x\gamma_t + 392\,\gamma_t\gamma_x\gamma_x + 504\,\gamma_t\gamma_x\gamma_y - 168\,\gamma_t\gamma_y\gamma_t - 294\,\gamma_t\gamma_y\gamma_x$$
$$- 378\,\gamma_t\gamma_y\gamma_y - 168\,\gamma_x\gamma_y\gamma_t - 294\,\gamma_x\gamma_y\gamma_x - 378\,\gamma_x\gamma_y\gamma_y$$

$$= 344\,\gamma_t + 602\,\gamma_x + 774\,\gamma_y - 224\,\gamma_x - 392\,\gamma_t + 504\,\gamma_t\gamma_x\gamma_y + 168\,\gamma_y + 294\,\gamma_t\gamma_x\gamma_y$$
$$+ 378\,\gamma_t - 168\,\gamma_t\gamma_x\gamma_y - 294\,\gamma_y + 378\,\gamma_x$$

$$= 330\,\gamma_t + 756\,\gamma_x + 648\,\gamma_y + 630\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow\ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 630\,\gamma_t\gamma_x\gamma_y$$

$$\Rightarrow\ \left|\mathbf{V}\right| = 630$$

$\Rightarrow$ The spacetime volume of the parallelepiped is 630 cm^3.

Check by applying the rule of Sarrus:

$$\begin{aligned} \mathbf{a} &= 10\,\gamma_t + 2\,\gamma_x + 6\,\gamma_y \\ \mathbf{b} &= 12\,\gamma_t + 8\,\gamma_x + 3\,\gamma_y \\ \mathbf{c} &= 4\,\gamma_t + 7\,\gamma_x + 9\,\gamma_y \end{aligned} \quad \Rightarrow \quad \mathbf{D} = \begin{bmatrix} 10 & 12 & 4 \\ 2 & 8 & 7 \\ 6 & 3 & 9 \end{bmatrix} \quad \Rightarrow \quad \det \mathbf{D} = 630$$

Magnitude check: $\mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2 = (10^2 - 2^2 - 6^2) \cdot (12^2 - 8^2 - 3^2) \cdot (4^2 - 7^2 - 9^2) = -\,485640$

$$(\mathbf{a\,b\,c})\,(\mathbf{a\,b\,c})^{\sim} = 330^2 - 756^2 - 648^2 + 630^2 = -\,485640 \qquad \surd$$

f) $\mathbf{a} = -\,5\,\gamma_t + 4\,\gamma_x + 8\,\gamma_y$
$\mathbf{b} = 6\,\gamma_t + 3\,\gamma_x - 7\,\gamma_y$
$\mathbf{c} = -\,\gamma_t - 2\,\gamma_x + 9\,\gamma_y$

Detailed calculation:

$$\mathbf{a\,b} = (-\,5\,\gamma_t + 4\,\gamma_x + 8\,\gamma_y)\,(6\,\gamma_t + 3\,\gamma_x - 7\,\gamma_y)$$

$$= -\,30\,\gamma_t^2 - 15\,\gamma_t\gamma_x + 35\,\gamma_t\gamma_y + 24\,\gamma_x\gamma_t + 12\,\gamma_x^2 - 28\,\gamma_x\gamma_y + 48\,\gamma_y\gamma_t + 24\,\gamma_y\gamma_x - 56\,\gamma_y^2$$

$$= -\,30 - 15\,\gamma_t\gamma_x + 35\,\gamma_t\gamma_y - 24\,\gamma_t\gamma_x - 12 - 28\,\gamma_x\gamma_y - 48\,\gamma_t\gamma_y - 24\,\gamma_x\gamma_y + 56$$

$$= 14 - 39\,\gamma_t\gamma_x - 13\,\gamma_t\gamma_y - 52\,\gamma_x\gamma_y \qquad \Rightarrow\ \mathbf{a} \wedge \mathbf{b} = -\,39\,\gamma_t\gamma_x - 13\,\gamma_t\gamma_y - 52\,\gamma_x\gamma_y$$

$$\mathbf{a\,b\,c} = (14 - 39\,\gamma_t\gamma_x - 13\,\gamma_t\gamma_y - 52\,\gamma_x\gamma_y)\,(-\,\gamma_t - 2\,\gamma_x + 9\,\gamma_y)$$

$$= -\,14\,\gamma_t - 28\,\gamma_x + 126\,\gamma_y + 39\,\gamma_t\gamma_x\gamma_t + 78\,\gamma_t\gamma_x\gamma_x - 351\,\gamma_t\gamma_x\gamma_y + 13\,\gamma_t\gamma_y\gamma_t + 26\,\gamma_t\gamma_y\gamma_x - 117\,\gamma_t\gamma_y\gamma_y$$
$$+ 52\,\gamma_x\gamma_y\gamma_t + 104\,\gamma_x\gamma_y\gamma_x - 468\,\gamma_x\gamma_y\gamma_y$$

$$= -\,14\,\gamma_t - 28\,\gamma_x + 126\,\gamma_y - 39\,\gamma_x - 78\,\gamma_t - 351\,\gamma_t\gamma_x\gamma_y - 13\,\gamma_y - 26\,\gamma_t\gamma_x\gamma_y + 117\,\gamma_t$$
$$+ 52\,\gamma_t\gamma_x\gamma_y + 104\,\gamma_y + 468\,\gamma_x$$

$$= 25\,\gamma_t + 401\,\gamma_x + 217\,\gamma_y - 325\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow\ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -\,325\,\gamma_t\gamma_x\gamma_y$$

$$\Rightarrow\ \left|\mathbf{V}\right| = 325$$

$\Rightarrow$ The spacetime volume of the parallelepiped is 377 cm^3.

Check by applying the rule of Sarrus:

$$\begin{aligned} \mathbf{a} &= -\,5\,\gamma_t + 4\,\gamma_x + 8\,\gamma_y \\ \mathbf{b} &= 6\,\gamma_t + 3\,\gamma_x - 7\,\gamma_y \\ \mathbf{c} &= -\,\gamma_t - 2\,\gamma_x + 9\,\gamma_y \end{aligned} \quad \Rightarrow \quad \mathbf{D} = \begin{bmatrix} -5 & 6 & -1 \\ 4 & 3 & -2 \\ 8 & -7 & 9 \end{bmatrix} \quad \Rightarrow \quad \det \mathbf{D} = -\,325$$

Magnitude check: $\mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2 = ((-5)^2 - 4^2 - 8^2)\cdot(6^2 - 3^2 - (-7)^2)\cdot((-1)^2 - (-2)^2 - 9^2) = -101640$

$$(\mathbf{a\,b\,c})\,(\mathbf{a\,b\,c})^{\sim} = 25^2 - 401^2 - 217^2 + 325^2 = -101640 \qquad \surd$$

Problem 2:

a) $3x + 8y = 28 \quad \Rightarrow \quad \mathbf{a} = 3\,\gamma_t + 6\,\gamma_x + 2\,\gamma_y \qquad \mathbf{r} = 28\,\gamma_t + 28\,\gamma_x + 28\,\gamma_y$

$6x + 2y = 28 \qquad \mathbf{b} = 8\,\gamma_t + 2\,\gamma_x + 4\,\gamma_y$

$2x + 4y + 2z = 28 \qquad \mathbf{c} = 2\,\gamma_y$

Outer products:

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = -84\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1} = \frac{1}{-84}\,\gamma_y\gamma_x\gamma_t = \frac{1}{84}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r}\wedge\mathbf{b}\wedge\mathbf{c} = -336\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a}\wedge\mathbf{r}\wedge\mathbf{c} = -168\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{r} = -504\,\gamma_t\gamma_x\gamma_y$$

Solution of the system of linear equations:

$$x = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\mathbf{r}\wedge\mathbf{b}\wedge\mathbf{c}) = \frac{-336}{-84} = 4$$

$$y = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\mathbf{a}\wedge\mathbf{r}\wedge\mathbf{c}) = \frac{-168}{-84} = 2$$

$$z = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{r}) = \frac{-504}{-84} = 6$$

Check: $3\cdot4 + 8\cdot2 = 12 + 16 = 28$

$6\cdot4 + 2\cdot2 = 24 + 4 = 28$

$2\cdot4 + 4\cdot2 + 2\cdot6 = 8 + 8 + 12 = 28$

b) $8x + 5y + 10z = 396 \quad \Rightarrow \quad \mathbf{a} = 8\,\gamma_t + 3\,\gamma_x + 2\,\gamma_y \qquad \mathbf{r} = 396\,\gamma_t + 375\,\gamma_x + 386\,\gamma_y$

$3x + 7y + 12z = 375 \qquad \mathbf{b} = 5\,\gamma_t + 7\,\gamma_x + 6\,\gamma_y$

$2x + 6y + 14z = 386 \qquad \mathbf{c} = 10\,\gamma_t + 12\,\gamma_x + 14\,\gamma_y$

Outer products:

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = 158\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1} = \frac{1}{158}\,\gamma_y\gamma_x\gamma_t = -\frac{1}{158}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r}\wedge\mathbf{b}\wedge\mathbf{c} = 2686\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a}\wedge\mathbf{r}\wedge\mathbf{c} = 1896\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{r} = 3160\,\gamma_t\gamma_x\gamma_y$$

Solution of the system of linear equations:

$$x = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\mathbf{r}\wedge\mathbf{b}\wedge\mathbf{c}) = \frac{2686}{158} = 17$$

$$y = (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^{-1}\,(\mathbf{a}\wedge\mathbf{r}\wedge\mathbf{c}) = \frac{1896}{158} = 12$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) = \frac{3160}{158} = 20$$

Check: $8 \cdot 17 + 5 \cdot 12 + 10 \cdot 20 = 136 + 60 + 200 = 396$
$3 \cdot 17 + 7 \cdot 12 + 12 \cdot 20 = 51 + 84 + 240 = 375$
$2 \cdot 17 + 6 \cdot 12 + 14 \cdot 20 = 34 + 72 + 280 = 386$

c) $\begin{aligned} 3x - 5y + 6z &= 41 \\ -2x + 5y + 8z &= 111 \\ 7x + y + 9z &= 185 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 3\gamma_t - 2\gamma_x + 7\gamma_y \\ \mathbf{b} &= -5\gamma_t + 5\gamma_x + \gamma_y \\ \mathbf{c} &= 6\gamma_t + 8\gamma_x + 9\gamma_y \end{aligned} \qquad \mathbf{r} = 41\gamma_t + 111\gamma_x + 185\gamma_y$

Outer products:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -481\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{-481}\,\gamma_y\gamma_x\gamma_t = \frac{1}{481}\,\gamma_t\gamma_x\gamma_y$$

$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = -5772\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = -5291\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = -4810\,\gamma_t\gamma_x\gamma_y$

Solution of the system of linear equations:

$$x = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-5772}{-481} = 12$$

$$y = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) = \frac{-5291}{-481} = 11$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) = \frac{-4810}{-481} = 10$$

Check: $3 \cdot 12 - 5 \cdot 11 + 6 \cdot 10 = 36 - 55 + 60 = 41$
$-2 \cdot 12 + 5 \cdot 11 + 8 \cdot 10 = -24 + 55 + 80 = 111$
$7 \cdot 12 + 11 + 9 \cdot 10 = 84 + 11 + 90 = 185$

d) $\begin{aligned} {}^2/_5\,x + {}^7/_5\,y + {}^9/_5\,z &= 210 \\ {}^8/_5\,x + {}^1/_5\,y + {}^3/_5\,z &= 138 \\ {}^4/_5\,x + {}^{12}/_5\,y + {}^6/_5\,z &= 282 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= {}^2/_5\,\gamma_t + {}^8/_5\,\gamma_x + {}^4/_5\,\gamma_y \\ \mathbf{b} &= {}^7/_5\,\gamma_t + {}^1/_5\,\gamma_x + {}^{12}/_5\,\gamma_y \\ \mathbf{c} &= {}^9/_5\,\gamma_t + {}^3/_5\,\gamma_x + {}^6/_5\,\gamma_y \end{aligned} \qquad \mathbf{r} = 210\gamma_t + 138\gamma_x + 282\gamma_y$

Outer products:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 4.128\,\gamma_t\gamma_x\gamma_y = \frac{516}{125}\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{4.128}\,\gamma_y\gamma_x\gamma_t = -\frac{125}{516}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = 247.680\,\gamma_t\gamma_x\gamma_y = \frac{30960}{125}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = 309.600\,\gamma_t\gamma_x\gamma_y = \frac{38700}{125}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = 185.760\,\gamma_t\gamma_x\gamma_y = \frac{23220}{125}\,\gamma_t\gamma_x\gamma_y$$

Solution of the system of linear equations:

$$x = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{30960}{516} = 60$$

$$y = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) = \frac{38700}{516} = 75$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) = \frac{23220}{516} = 45$$

Check: $^2/_5 \cdot 60 + {}^7/_5 \cdot 75 + {}^9/_5 \cdot 45 = 24 + 105 + 81 = 210$
$^8/_5 \cdot 60 + {}^1/_5 \cdot 75 + {}^3/_5 \cdot 45 = 96 + 15 + 27 = 138$
$^4/_5 \cdot 60 + {}^{12}/_5 \cdot 75 + {}^6/_5 \cdot 45 = 48 + 180 + 54 = 282$

Problem 3:

$7x + 2y + 5z = 500 \quad \Rightarrow \quad \mathbf{a} = 7\gamma_t + 3\gamma_x + 4\gamma_y \qquad \mathbf{r} = 500\gamma_t + 780\gamma_x + 880\gamma_y$
$3x + 9y + \quad = 780 \qquad \mathbf{b} = 2\gamma_t + 9\gamma_x + 6\gamma_y$
$4x + 6y + 8z = 880 \qquad \mathbf{c} = 5\gamma_t + \quad + 8\gamma_y$

Outer products:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 366\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{366}\,\gamma_y\gamma_x\gamma_t = -\frac{1}{366}\,\gamma_t\gamma_x\gamma_y$$

$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = 7320\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = 29280\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = 14640\,\gamma_t\gamma_x\gamma_y$

Solution of the system of linear equations:

$$x = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{7320}{366} = 20$$

$$y = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) = \frac{29280}{366} = 80$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) = \frac{14640}{366} = 40$$

Check: $7 \cdot 20 + 2 \cdot 80 + 5 \cdot 40 = 140 + 160 + 200 = 500$
$3 \cdot 20 + 9 \cdot 80 = 60 + 720 = 780$
$4 \cdot 20 + 6 \cdot 80 + 8 \cdot 40 = 80 + 480 + 320 = 880$

$\Rightarrow$ If 500 units of the first raw material R_1, 780 units of the second raw material R_2, and 880 units of the third raw material R_3 are consumed in the production process, 20 units of the first final product P_1, 80 units of the second final product P_2, and 40 units of the third final product P_3 will be produced.

Problem 4:

$$\begin{aligned} 12x + 30y + 10z &= 12000 \\ 20x + 15y + 8z &= 13900 \\ 16x + 28y + 25z &= 18300 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{a} &= 12\gamma_t + 20\gamma_x + 16\gamma_y \\ \mathbf{b} &= 30\gamma_t + 15\gamma_x + 28\gamma_y \\ \mathbf{c} &= 10\gamma_t + 8\gamma_x + 25\gamma_y \end{aligned} \qquad \mathbf{r} = 12000\gamma_t + 13900\gamma_x + 18300\gamma_y$$

Outer products:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -6148\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{-6148}\gamma_y\gamma_x\gamma_t = \frac{1}{6148}\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c} = -3074000\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c} = -614800\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r} = -1844400\,\gamma_t\gamma_x\gamma_y$$

Solution of the system of linear equations:

$$x = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}(\mathbf{r} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-3074000}{-6148} = 500$$

$$y = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}(\mathbf{a} \wedge \mathbf{r} \wedge \mathbf{c}) = \frac{-614800}{-6148} = 100$$

$$z = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r}) = \frac{-1844400}{-6148} = 300$$

Check:
$$\begin{aligned} 12 \cdot 500 + 30 \cdot 100 + 10 \cdot 300 &= 6000 + 3000 + 3000 = 12000 \\ 20 \cdot 500 + 15 \cdot 100 + 8 \cdot 300 &= 10000 + 1500 + 2400 = 13900 \\ 16 \cdot 500 + 28 \cdot 100 + 25 \cdot 300 &= 8000 + 2800 + 7500 = 18300 \end{aligned}$$

$\Rightarrow$ If 12000 units of the first raw material R_1, 13900 units of the second raw material R_2, and 18300 units of the third raw material R_3 are consumed in the production process, 500 units of the first final product P_1, 100 units of the second final product P_2, and 300 units of the third final product P_3 will be produced.

Problem 5:

$$\begin{bmatrix} 9 & 3 & 4 \\ 2 & 2 & 3 \\ 7 & 5 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}}_{\mathbf{P}} = \underbrace{\begin{array}{cc} \text{first quarter} & \text{second quarter} \\ \downarrow & \downarrow \\ \left[\begin{matrix} 98 \\ 35 \\ 76 \end{matrix}\right. & \left.\begin{matrix} 61 \\ 30 \\ 59 \end{matrix}\right] \end{array}}_{\mathbf{R}}$$

R .... matrix of quarterly consumption of raw materials (consumption matrix)

P .... matrix of quarterly production (production matrix)

$\Rightarrow$ Two systems of linear equations:

$$\begin{aligned} 9x_1 + 3y_1 + 4z_1 &= 98 \\ 2x_1 + 2y_1 + 3z_1 &= 35 \\ 7x_1 + 5y_1 + 2z_1 &= 76 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 9x_2 + 3y_2 + 4z_2 &= 61 \\ 2x_2 + 2y_2 + 3z_2 &= 30 \\ 7x_2 + 5y_2 + 2z_2 &= 59 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 9\,\gamma_t + 2\,\gamma_x + 7\,\gamma_y$ $\qquad$ $\mathbf{r_1} = 98\,\gamma_t + 35\,\gamma_x + 76\,\gamma_y$

$\mathbf{b} = 3\,\gamma_t + 2\,\gamma_x + 5\,\gamma_y$ $\qquad$ $\mathbf{r_2} = 61\,\gamma_t + 30\,\gamma_x + 59\,\gamma_y$

$\mathbf{c} = 4\,\gamma_t + 3\,\gamma_x + 2\,\gamma_y$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \;-64\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \dfrac{1}{-64}\,\gamma_y\gamma_x\gamma_t = \dfrac{1}{64}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = -512\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = -128\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = -320\,\gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \dfrac{-512}{-64} = 8$

$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \dfrac{-128}{-64} = 2$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \dfrac{-320}{-64} = 5$

Outer products of the second system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \;-64\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \dfrac{1}{-64}\,\gamma_y\gamma_x\gamma_t = \dfrac{1}{64}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = -192\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = -384\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = -256\,\gamma_t\gamma_x\gamma_y$

Solution of the second system of linear equations:

$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \dfrac{-192}{-64} = 3$

$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \dfrac{-384}{-64} = 6$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \dfrac{-256}{-64} = 4$

Check:				
			8	3
			2	6
			5	4
9	3	4	98	61
2	2	3	35	30
7	5	2	76	59

$\Rightarrow$ 8 units of the first final product P_1, 2 units of the second final product P_2, and 5 units of the third final product P_3 will be produced in the first quarter.

3 units of the first final product P_1, 6 units of the second final product P_2, and 4 units of the third final product P_3 will be produced in the second quarter.

Problem 6:

$$\underbrace{\begin{bmatrix} 10 & 15 & 11 \\ 17 & 20 & 16 \\ 12 & 14 & 25 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 964 & 814 \\ 1409 & 1184 \\ 1320 & 1093 \end{bmatrix}}_{\mathbf{D}} \qquad \mathbf{A\,B = D}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$\Rightarrow$ Two systems of linear equations:

$$\begin{aligned} 10\,x_1 + 15\,y_1 + 11\,z_1 &= 964 \\ 17\,x_1 + 20\,y_1 + 16\,z_1 &= 1409 \\ 12\,x_1 + 14\,y_1 + 25\,z_1 &= 1320 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 10\,x_2 + 15\,y_2 + 11\,z_2 &= 814 \\ 17\,x_2 + 20\,y_2 + 16\,z_2 &= 1184 \\ 12\,x_2 + 14\,y_2 + 25\,z_2 &= 1093 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 10\,\gamma_t + 17\,\gamma_x + 12\,\gamma_y$ $\qquad$ $\mathbf{r_1} = 964\,\gamma_t + 1409\,\gamma_x + 1320\,\gamma_y$

$\mathbf{b} = 15\,\gamma_t + 20\,\gamma_x + 14\,\gamma_y$ $\qquad$ $\mathbf{r_2} = 814\,\gamma_t + 1184\,\gamma_x + 1093\,\gamma_y$

$\mathbf{c} = 11\,\gamma_t + 16\,\gamma_x + 25\,\gamma_y$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -757\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \dfrac{1}{-757}\,\gamma_y\gamma_x\gamma_t = \dfrac{1}{757}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = -18925\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = -22710\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = -18168\,\gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-18925}{-757} = 25$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \frac{-22710}{-757} = 30$$

$$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \frac{-18168}{-757} = 24$$

Outer products of the second system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -757\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \dfrac{1}{-757}\,\gamma_y\gamma_x\gamma_t = \dfrac{1}{757}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = -15140\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = -20439\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = -14383\ \gamma_t\gamma_x\gamma_y$

Solution of the second system of linear equations:

$$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\ (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-15140}{-757} = 20$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\ (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \frac{-20439}{-757} = 27$$

$$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\ (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \frac{-14383}{-757} = 19$$

Check:				
			25	20
			30	27
			24	19
10	15	11	964	814
17	20	16	1409	1184
12	14	25	1320	1093

$\Rightarrow$ Demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 25 & 20 \\ 30 & 27 \\ 24 & 19 \end{bmatrix}$

Problem 7:

$$\underbrace{\begin{bmatrix} 8 & 6 & 6 \\ 7 & 5 & 7 \\ 5 & 4 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 228 & 186 & 308 \\ 214 & 166 & 282 \\ 108 & 107 & 160 \end{bmatrix}}_{\mathbf{D}} \qquad \mathbf{A\,B} = \mathbf{D}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$\Rightarrow$ Three systems of linear equations:

$$\begin{aligned} 8x_1 + 6y_1 + 6z_1 &= 228 \\ 7x_1 + 5y_1 + 7z_1 &= 214 \\ 5x_1 + 4y_1 &= 108 \end{aligned} \quad \text{and} \quad \begin{aligned} 8x_2 + 6y_2 + 6z_2 &= 186 \\ 7x_2 + 5y_2 + 7z_2 &= 166 \\ 5x_2 + 4y_2 &= 107 \end{aligned} \quad \text{and} \quad \begin{aligned} 8x_3 + 6y_3 + 6z_3 &= 308 \\ 7x_3 + 5y_3 + 7z_3 &= 282 \\ 5x_3 + 4y_3 &= 160 \end{aligned}$$

$\Rightarrow$ $\mathbf{a} = 8\gamma_t + 7\gamma_x + 5\gamma_y$ $\qquad \mathbf{r_1} = 228\gamma_t + 214\gamma_x + 108\gamma_y$

$\mathbf{b} = 6\gamma_t + 5\gamma_x + 4\gamma_y$ $\qquad \mathbf{r_2} = 186\gamma_t + 166\gamma_x + 107\gamma_y$

$\mathbf{c} = 6\gamma_t + 7\gamma_x$ $\qquad \mathbf{r_3} = 308\gamma_t + 282\gamma_x + 160\gamma_y$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 4\, \gamma_t\gamma_x\gamma_y \quad\Rightarrow\quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{4}\, \gamma_y\gamma_x\gamma_t = -0.25\, \gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = 48\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = 48\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = 40\, \gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{48}{4} = 12$

$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \frac{48}{4} = 12$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \frac{40}{4} = 10$

Outer products of the second system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 4\, \gamma_t\gamma_x\gamma_y \quad\Rightarrow\quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{4}\, \gamma_y\gamma_x\gamma_t = -0.25\, \gamma_t\gamma_x\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = 60\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = 32\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = 12\, \gamma_t\gamma_x\gamma_y$

Solution of the second system of linear equations:

$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{60}{4} = 15$

$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \frac{32}{4} = 8$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \frac{12}{4} = 3$

Outer products of the third system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 4\, \gamma_t\gamma_x\gamma_y \quad\Rightarrow\quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{4}\, \gamma_y\gamma_x\gamma_t = -0.25\, \gamma_t\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c} = 64\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c} = 80\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3} = 40\, \gamma_t\gamma_x\gamma_y$

Solution of the third system of linear equations:

$x_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{64}{4} = 16$

$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) = \frac{80}{4} = 20$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) = \frac{40}{4} = 10$

Check:

			12	15	16
			12	8	20
			10	3	10
8	6	6	228	186	308
7	5	7	214	166	282
5	4	0	108	107	160

$\Rightarrow$ Demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 12 & 15 & 16 \\ 12 & 8 & 20 \\ 10 & 3 & 10 \end{bmatrix}$

Problem 8:

$$\underbrace{\begin{bmatrix} 82 & 63 & 20 \\ 44 & 19 & 37 \\ 10 & 52 & 92 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 4496 & 5462 & 4815 \\ 2530 & 3482 & 2801 \\ 3224 & 4062 & 4646 \end{bmatrix}}_{\mathbf{D}} \qquad \mathbf{A}\,\mathbf{B} = \mathbf{D}$$

D .... matrix of total demand

B .... demand matrix of the second production step

A .... demand matrix of the first production step

$\Rightarrow$ Three systems of linear equations:

$$\begin{aligned} 82\,x_1 + 63\,y_1 + 20\,z_1 &= 4496 \\ 44\,x_1 + 19\,y_1 + 37\,z_1 &= 2530 \\ 10\,x_1 + 52\,y_1 + 92\,z_1 &= 3224 \end{aligned} \quad \text{and} \quad \begin{aligned} 82\,x_2 + 63\,y_2 + 20\,z_2 &= 5462 \\ 44\,x_2 + 19\,y_2 + 37\,z_2 &= 3482 \\ 10\,x_2 + 52\,y_2 + 92\,z_2 &= 4062 \end{aligned} \quad \text{and} \quad \begin{aligned} 82\,x_3 + 63\,y_3 + 20\,z_3 &= 4815 \\ 44\,x_3 + 19\,y_3 + 37\,z_3 &= 2801 \\ 10\,x_3 + 52\,y_3 + 92\,z_3 &= 4646 \end{aligned}$$

$$\Rightarrow \begin{aligned} \mathbf{a} &= 82\,\gamma_t + 44\,\gamma_x + 10\,\gamma_y \\ \mathbf{b} &= 63\,\gamma_t + 19\,\gamma_x + 52\,\gamma_y \\ \mathbf{c} &= 20\,\gamma_t + 37\,\gamma_x + 92\,\gamma_y \end{aligned} \qquad \begin{aligned} \mathbf{r_1} &= 4496\,\gamma_t + 2530\,\gamma_x + 3224\,\gamma_y \\ \mathbf{r_2} &= 5462\,\gamma_t + 3482\,\gamma_x + 4062\,\gamma_y \\ \mathbf{r_3} &= 4815\,\gamma_t + 2801\,\gamma_x + 4646\,\gamma_y \end{aligned}$$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -204186\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \dfrac{1}{-204186}\,\gamma_y\gamma_x\gamma_t = \dfrac{1}{204186}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = -6533952\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = -4900464\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = -3675348\,\gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-6533952}{-204186} = 32$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \frac{-4900464}{-204186} = 24$$

$$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \frac{-3675348}{-204186} = 18$$

Outer products of the second system of linear equations:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \quad -204\,186\ \gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{-204186}\,\gamma_y\gamma_x\gamma_t = \frac{1}{204186}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = -\,9\,596\,742\ \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = -\,3\,266\,976\ \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = -\,6\,125\,580\ \gamma_t\gamma_x\gamma_y$$

Solution of the second system of linear equations:

$$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-9596742}{-204186} = 47$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \frac{-3266976}{-204186} = 16$$

$$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \frac{-6125580}{-204186} = 30$$

Outer products of the third system of linear equations:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \quad -204\,186\ \gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{-204186}\,\gamma_y\gamma_x\gamma_t = \frac{1}{204186}\,\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c} = -\,5\,104\,650\ \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c} = -\,7\,146\,510\ \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3} = -\,5\,717\,208\ \gamma_t\gamma_x\gamma_y$$

Solution of the third system of linear equations:

$$x_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-5104650}{-204186} = 25$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) = \frac{-7146510}{-204186} = 35$$

$$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) = \frac{-5717208}{-204186} = 28$$

Check:			32	47	25
			24	16	35
			18	30	28
82	63	20	4496	5462	4815
44	19	37	2530	3482	2801
10	52	92	3224	4062	4646

$\Rightarrow$ Demand matrix of the second production step: $\mathbf{B} = \begin{bmatrix} 32 & 47 & 25 \\ 24 & 16 & 35 \\ 18 & 30 & 28 \end{bmatrix}$

Problem 9:

$$\underbrace{\begin{bmatrix} 3 & 5 & 4 \\ 2 & 6 & 3 \\ 8 & 7 & 10 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{A}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I} \text{ .... identity matrix}} \qquad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

$\Rightarrow$ Three systems of linear equations:

$$\begin{aligned} 3x_1 + 5y_1 + 4z_1 &= 1 \\ 2x_1 + 6y_1 + 3z_1 &= 0 \\ 8x_1 + 7y_1 + 10z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 3x_2 + 5y_2 + 4z_2 &= 0 \\ 2x_2 + 6y_2 + 3z_2 &= 1 \\ 8x_2 + 7y_2 + 10z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 3x_3 + 5y_3 + 4z_3 &= 0 \\ 2x_3 + 6y_3 + 3z_3 &= 0 \\ 8x_3 + 7y_3 + 10z_3 &= 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} \mathbf{a} &= 3\,\gamma_t + 2\,\gamma_x + 8\,\gamma_y \\ \mathbf{b} &= 5\,\gamma_t + 6\,\gamma_x + 7\,\gamma_y \\ \mathbf{c} &= 4\,\gamma_t + 3\,\gamma_x + 10\,\gamma_y \end{aligned} \qquad \begin{aligned} \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \\ \mathbf{r_3} &= \gamma_y \end{aligned}$$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 1\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \gamma_y\gamma_x\gamma_t = -\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = 39\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = 4\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = -\,34\,\gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{39}{1} = 39$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \frac{4}{1} = 4$$

$$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \frac{-34}{1} = -\,34$$

$\Rightarrow$ If exactly one unit of the first raw material R_1 had been consumed in the production process, 39 units of the first final product P_1 and 4 units of the second final product P_2 would have been produced and additionally 34 units of the third final product P_3 would have been consumed (and split again completely into the raw materials).

Or more realistic:

If it just happened that one **more** unit of the first raw material R_1 had been delivered accidentally and has had to be consumed in addition in the production process, the output of the first final product P_1 would have been increased by 39 units, the output of the second final product P_2 would have been increased by 4 units, and the output of the third final product P_3 would have been reduced by 34 units.

Outer products of the second system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \quad 1\, \gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \gamma_y\gamma_x\gamma_t = -\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = -\,22\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = \ -\,2\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = \ 19\, \gamma_t\gamma_x\gamma_y$

Solution of the second system of linear equations:

$$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-\,22}{1} = -\,22$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \frac{-\,2}{1} = -\,2$$

$$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \frac{19}{1} = 19$$

$\Rightarrow$ If exactly one unit of the second raw material R_2 had been consumed in the production process, 19 units of the third final product P_3 would have been produced and additionally 22 units of the first final product P_1 and 2 units of the second final product P_2 would have been consumed (and split again completely into the raw materials).

Or more realistic:

If it just happened that one **more** unit of the second raw material R_2 had been delivered accidentally and has had to be consumed in addition in the production process, the output of the first final product P_1 would have been reduced by 22 units, the output of the second final product P_2 would have been reduced by 2 units, and the output of the third final product P_3 would have been increased by 19 units.

Outer products of the third system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \quad 1\, \gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \gamma_y\gamma_x\gamma_t = -\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c} = -\,9\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c} = -\,1\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3} = \ 8\, \gamma_t\gamma_x\gamma_y$

Solution of the third system of linear equations:

$$x_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-\,9}{1} = -\,9$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) = \frac{-1}{1} = -\,1$$

$$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) = \frac{8}{1} = 8$$

$\Rightarrow$ If exactly one unit of the third raw material R_3 had been consumed in the production process, 8 units of the third final product P_3 would have been produced and additionally 9 units of the first final product P_1 and one unit of the second final product P_2 would have been consumed (and split again completely into the raw materials).

Or more realistic:

If it just happened that one **more** unit of the third raw material R_3 had been delivered accidentally and has had to be consumed in addition in the production process, the output of the first final product P_1 would have been reduced by 9 units, the output of the second final product P_2 would have been reduced by one unit, and the output of the third final product P_3 would have been increased by 8 units.

Check:

$$\begin{array}{ccc|ccc} & & & 39 & -22 & -9 \\ & & & 4 & -2 & -1 \\ & & & -34 & 19 & 8 \\ \hline 3 & 5 & 4 & 1 & 0 & 0 \\ 2 & 6 & 3 & 0 & 1 & 0 \\ 8 & 7 & 10 & 0 & 0 & 1 \end{array}$$

$\Rightarrow$ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 39 & -22 & -9 \\ 4 & -2 & -1 \\ -34 & 19 & 8 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 6 & 3 \\ 8 & 7 & 10 \end{bmatrix}$.

Problem 10:

a) $$\underbrace{\begin{bmatrix} 1 & 4 & 9 \\ 7 & 2 & 6 \\ 6 & 3 & 8 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{A}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I} \text{ .... identity matrix}} \qquad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

$\Rightarrow$ Three systems of linear equations:

$$\begin{array}{rcrcr} x_1 + 4\,y_1 + 9\,z_1 = 1 & & x_2 + 4\,y_2 + 9\,z_2 = 0 & & x_3 + 4\,y_3 + 9\,z_3 = 0 \\ 7\,x_1 + 2\,y_1 + 6\,z_1 = 0 & \text{and} & 7\,x_2 + 2\,y_2 + 6\,z_2 = 1 & \text{and} & 7\,x_3 + 2\,y_3 + 6\,z_3 = 0 \\ 6\,x_1 + 3\,y_1 + 8\,z_1 = 0 & & 6\,x_2 + 3\,y_2 + 8\,z_2 = 0 & & 6\,x_3 + 3\,y_3 + 8\,z_3 = 1 \end{array}$$

$$\begin{array}{ll} \Rightarrow \mathbf{a} = \gamma_t + 7\,\gamma_x + 6\,\gamma_y & \mathbf{r_1} = \gamma_t \\ \mathbf{b} = 4\,\gamma_t + 2\,\gamma_x + 3\,\gamma_y & \mathbf{r_2} = \gamma_x \\ \mathbf{c} = 9\,\gamma_t + 6\,\gamma_x + 8\,\gamma_y & \mathbf{r_3} = \gamma_y \end{array}$$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -1\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = -\gamma_y\gamma_x\gamma_t = \gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = -2\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = -20\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = \quad 9\, \gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = -\,(-\,2) = 2$

$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = -\,(-\,20) = 20$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = -\,9$

Outer products of the second system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \quad -\,1\, \gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = -\,\gamma_y\gamma_x\gamma_t = \gamma_t\gamma_x\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = \quad -\,5\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = -\,46\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = \quad 21\, \gamma_t\gamma_x\gamma_y$

Solution of the second system of linear equations:

$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = -\,(-\,5) = 5$

$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = -\,(-\,46) = 46$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = -\,21$

Outer products of the third system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \quad -\,1\, \gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = -\,\gamma_y\gamma_x\gamma_t = \gamma_t\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c} = 6\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c} = 57\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3} = -\,26\, \gamma_t\gamma_x\gamma_y$

Solution of the third system of linear equations:

$x_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) = -\,6$

$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) = -\,57$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) = -\,(-\,26) = 26$

Check:				2	5	– 6
				20	46	– 57
				– 9	– 21	26
1	4	9		1	0	0
7	2	6		0	1	0
6	3	8		0	0	1

$\Rightarrow$ The resulting matrix $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 5 & -6 \\ 20 & 46 & -57 \\ -9 & -21 & 26 \end{bmatrix}$ is the inverse of matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 & 9 \\ 7 & 2 & 6 \\ 6 & 3 & 8 \end{bmatrix}$.

b) $$\underbrace{\begin{bmatrix} 0 & 4 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{B}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I} \text{ .... identity matrix}} \qquad \mathbf{B}\,\mathbf{B}^{-1} = \mathbf{I}$$

$\Rightarrow$ Three systems of linear equations:

$$\begin{aligned} 4\,y_1 + 7\,z_1 &= 1 \\ 4\,x_1 + 5\,y_1 + 8\,z_1 &= 0 \\ 3\,x_1 + 6\,y_1 + 9\,z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 4\,y_2 + 7\,z_2 &= 0 \\ 4\,x_2 + 5\,y_2 + 8\,z_2 &= 1 \\ 3\,x_2 + 6\,y_2 + 9\,z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 4\,y_3 + 7\,z_3 &= 0 \\ 4\,x_3 + 5\,y_3 + 8\,z_3 &= 0 \\ 3\,x_3 + 6\,y_3 + 9\,z_3 &= 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} \mathbf{a} &= 4\,\gamma_x + 3\,\gamma_y \\ \mathbf{b} &= 4\,\gamma_t + 5\,\gamma_x + 6\,\gamma_y \\ \mathbf{c} &= 7\,\gamma_t + 8\,\gamma_x + 9\,\gamma_y \end{aligned} \qquad \begin{aligned} \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \\ \mathbf{r_3} &= \gamma_y \end{aligned}$$

Outer products of the first system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 15\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{15}\,\gamma_y\gamma_x\gamma_t = -\frac{1}{15}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = -3\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = -12\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = 9\,\gamma_t\gamma_x\gamma_y$

Solution of the first system of linear equations:

$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-3}{15} = -\frac{1}{5} = -0.2$

$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \frac{-12}{15} = -\frac{4}{5} = -0.8$

$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \frac{9}{15} = \frac{3}{5} = 0.6$

Outer products of the second system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 15\,\gamma_t\gamma_x\gamma_y \qquad \Rightarrow \qquad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{15}\,\gamma_y\gamma_x\gamma_t = -\frac{1}{15}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = 6\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = -21\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = 12\,\gamma_t\gamma_x\gamma_y$

Solution of the second system of linear equations:

$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \dfrac{6}{15} = \dfrac{2}{5} = 0.4$

$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \dfrac{-21}{15} = -\dfrac{7}{5} = -1.4$

$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \dfrac{12}{15} = \dfrac{4}{5} = 0.8$

Outer products of the third system of linear equations:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 15\,\gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \dfrac{1}{15}\,\gamma_y\gamma_x\gamma_t = -\dfrac{1}{15}\,\gamma_t\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c} = -3\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c} = 28\,\gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3} = -16\,\gamma_t\gamma_x\gamma_y$

Solution of the third system of linear equations:

$x_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) = \dfrac{-3}{15} = -\dfrac{1}{5} = -0.2$

$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) = \dfrac{28}{15} = 1.8\overline{6}.. \approx 1.867$

$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) = \dfrac{-16}{15} = -1.0\overline{6}.. \approx -1.067$

Check:					
			– 0.2	0.4	– 0.2
			– 0.8	– 1.4	$1.8\overline{6}..$
			0.6	0.8	$-1.0\overline{6}..$
0	4	7	1	0	0
4	5	8	0	1	0
3	6	9	0	0	1

Alternative check:					
			– 3	6	– 3
			– 12	– 21	28
			9	12	– 16
0	4	7	15	0	0
4	5	8	0	15	0
3	6	9	0	0	15

$\Rightarrow$ The resulting matrix $\mathbf{B}^{-1} = \begin{bmatrix} -0.2 & 0.4 & -0.2 \\ -0.8 & -1.4 & 1.8\overline{6}.. \\ 0.6 & 0.8 & -1.0\overline{6}.. \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 & 6 & -3 \\ -12 & -21 & 28 \\ 9 & 12 & -16 \end{bmatrix}$ is

the inverse of matrix $\mathbf{B} = \begin{bmatrix} 0 & 4 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

c) $$\underbrace{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{C}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I} \text{ .... identity matrix}} \qquad \mathbf{C}\,\mathbf{C}^{-1} = \mathbf{I}$$

$\Rightarrow$ Three systems of linear equations:

$$\begin{aligned} x_1 + 4\,y_1 + 7\,z_1 &= 1 \\ 2\,x_1 + 5\,y_1 + 8\,z_1 &= 0 \\ 3\,x_1 + 6\,y_1 + 9\,z_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_2 + 4\,y_2 + 7\,z_2 &= 0 \\ 2\,x_2 + 5\,y_2 + 8\,z_2 &= 1 \\ 3\,x_2 + 6\,y_2 + 9\,z_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_3 + 4\,y_3 + 7\,z_3 &= 0 \\ 2\,x_3 + 5\,y_3 + 8\,z_3 &= 0 \\ 3\,x_3 + 6\,y_3 + 9\,z_3 &= 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} \mathbf{a} &= \gamma_t + 2\,\gamma_x + 3\,\gamma_y \\ \mathbf{b} &= 4\,\gamma_t + 5\,\gamma_x + 6\,\gamma_y \\ \mathbf{c} &= 7\,\gamma_t + 8\,\gamma_x + 9\,\gamma_y \end{aligned} \qquad \begin{aligned} \mathbf{r_1} &= \gamma_t \\ \mathbf{r_2} &= \gamma_x \\ \mathbf{r_3} &= \gamma_y \end{aligned}$$

Outer products of the coefficient vectors:

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0$ $\Rightarrow$ $(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = ??$

$\Rightarrow$ As this outer product of the coefficient vectors is zero, the reciprocal value $^1/_0$ (a division by zero) is not defined. Therefore elements of an inverse matrix cannot be found.

$\Rightarrow$ Problem 14 c) is insoluble.

$\Rightarrow$ The inverse $\mathbf{C}^{-1}$ is not defined.

$\Rightarrow$ An inverse of matrix $\mathbf{C} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ does not exist.

d) $$\underbrace{\begin{bmatrix} 3 & 4 & 8 \\ 10 & 5 & 10 \\ 10 & 20 & 15 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{\mathbf{D}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I} \text{ .... identity matrix}} \qquad \mathbf{D}\,\mathbf{D}^{-1} = \mathbf{I}$$

$\Rightarrow$ Three systems of linear equations:

$$\begin{matrix} 3x_1 + 4y_1 + 8z_1 = 1 \\ 10x_1 + 5y_1 + 10z_1 = 0 \\ 10x_1 + 20y_1 + 15z_1 = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} 3x_2 + 4y_2 + 8z_2 = 0 \\ 10x_2 + 5y_2 + 10z_2 = 1 \\ 10x_2 + 20y_2 + 15z_2 = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} 3x_3 + 4y_3 + 8z_3 = 0 \\ 10x_3 + 5y_3 + 10z_3 = 0 \\ 10x_3 + 20y_3 + 15z_3 = 1 \end{matrix}$$

$$\Rightarrow \begin{matrix} \mathbf{a} = 3\gamma_t + 10\gamma_x + 10\gamma_y \\ \mathbf{b} = 4\gamma_t + 5\gamma_x + 20\gamma_y \\ \mathbf{c} = 8\gamma_t + 10\gamma_x + 15\gamma_y \end{matrix} \qquad \begin{matrix} \mathbf{r_1} = \gamma_t \\ \mathbf{r_2} = \gamma_x \\ \mathbf{r_3} = \gamma_y \end{matrix}$$

Outer products of the first system of linear equations:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 625\, \gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{625}\gamma_y\gamma_x\gamma_t = -\frac{1}{625}\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c} = -125\, \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c} = -50\, \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1} = 150\, \gamma_t\gamma_x\gamma_y$$

Solution of the first system of linear equations:

$$x_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_1} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{-125}{625} = -\frac{1}{5} = -0.2$$

$$y_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_1} \wedge \mathbf{c}) = \frac{-50}{625} = -\frac{2}{25} = -0.08$$

$$z_1 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_1}) = \frac{150}{625} = \frac{6}{25} = 0.24$$

Outer products of the second system of linear equations:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 625\, \gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{625}\gamma_y\gamma_x\gamma_t = -\frac{1}{625}\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c} = 100\, \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c} = -35\, \gamma_t\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2} = -20\, \gamma_t\gamma_x\gamma_y$$

Solution of the second system of linear equations:

$$x_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{r_2} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{100}{625} = \frac{4}{25} = 0.16$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{r_2} \wedge \mathbf{c}) = \frac{-35}{625} = -\frac{7}{125} = -0.056$$

$$z_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\, (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_2}) = \frac{-20}{625} = -\frac{4}{125} = -0.032$$

Outer products of the third system of linear equations:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 625\, \gamma_t\gamma_x\gamma_y \quad \Rightarrow \quad (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1} = \frac{1}{625}\gamma_y\gamma_x\gamma_t = -\frac{1}{625}\gamma_t\gamma_x\gamma_y$$

$$\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c} = 0$$

$\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c} = \quad 50\, \gamma_t\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3} = -\,25\, \gamma_t\gamma_x\gamma_y$

Solution of the third system of linear equations:

$$x_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b} \wedge \mathbf{c}) = \frac{0}{625} = 0$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3} \wedge \mathbf{c}) = \frac{50}{625} = \frac{2}{25} = 0.08$$

$$z_3 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^{-1}\,(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{r_3}) = \frac{-25}{625} = -\frac{1}{25} = -\,0.04$$

Check:					
			– 0.20	0.160	0
			– 0.08	– 0.056	0.08
			0.24	– 0.032	– 0.04
3	4	8	1	0	0
10	5	10	0	1	0
10	20	15	0	0	1

Alternative check:					
			– 25	20	0
			– 10	– 7	10
			30	– 4	– 5
3	4	8	125	0	0
10	5	10	0	125	0
10	20	15	0	0	125

$\Rightarrow$ The resulting matrix $\mathbf{D}^{-1} = \begin{bmatrix} -0.20 & 0.160 & 0 \\ -0.08 & -0.056 & 0.08 \\ 0.24 & -0.032 & -0.04 \end{bmatrix} = \frac{1}{125}\begin{bmatrix} -25 & 20 & 0 \\ -10 & -7 & 10 \\ 30 & -4 & -5 \end{bmatrix}$ is

the inverse of matrix $\mathbf{D} = \begin{bmatrix} 3 & 4 & 8 \\ 10 & 5 & 10 \\ 10 & 20 & 15 \end{bmatrix}$.

Problem 11:

a) $5\,x + 0\,y = 125 \quad\Rightarrow\quad \mathbf{a} = 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y$

$4\,x + 0\,y = 100 \qquad\quad \mathbf{b} = 2\,\gamma_y$

$3\,x + 2\,y = 145 \qquad\quad \mathbf{r} = 125\,\gamma_t + 100\,\gamma_x + 145\,\gamma_y$

Outer products:

$$\mathbf{a}\,\mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(2\,\gamma_y)$$
$$= 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y + 6\,\gamma_y^2$$
$$= -6 + 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y \quad\Rightarrow\quad \mathbf{a} \wedge \mathbf{b} = 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y$$

$\cdot\,25$

$$\mathbf{r}\,\mathbf{b} = (125\,\gamma_t + 100\,\gamma_x + 145\,\gamma_y)\,(2\,\gamma_y)$$
$$= 250\,\gamma_t\gamma_y + 200\,\gamma_x\gamma_y + 290\,\gamma_y^2$$
$$= -290 + 250\,\gamma_t\gamma_y + 200\,\gamma_x\gamma_y \quad\Rightarrow\quad \mathbf{r} \wedge \mathbf{b} = 250\,\gamma_t\gamma_y + 200\,\gamma_x\gamma_y$$

$\cdot\,35$

$$\mathbf{a}\,\mathbf{r} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(125\,\gamma_t + 100\,\gamma_x + 145\,\gamma_y)$$
$$= 625\,\gamma_t^2 + 500\,\gamma_t\gamma_x + 725\,\gamma_t\gamma_y + 500\,\gamma_x\gamma_t + 400\,\gamma_x^2 + 580\,\gamma_x\gamma_y + 375\,\gamma_y\gamma_t + 300\,\gamma_y\gamma_x + 435\,\gamma_y^2$$
$$= 625 + 500\,\gamma_t\gamma_x + 725\,\gamma_t\gamma_y - 500\,\gamma_t\gamma_x - 400 + 580\,\gamma_x\gamma_y - 375\,\gamma_t\gamma_y - 300\,\gamma_x\gamma_y - 435$$
$$= -310 + 0\,\gamma_t\gamma_x + 350\,\gamma_t\gamma_y + 280\,\gamma_x\gamma_y$$
$$\Rightarrow\quad \mathbf{a} \wedge \mathbf{r} = 350\,\gamma_t\gamma_y + 280\,\gamma_x\gamma_y$$

Solution of the over-constrained system of linear equations:

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = 25$$

$$y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = 35$$

Check: $5 \cdot 25 + 0 \cdot 35 = 125 + 0 = 125$

$4 \cdot 25 + 0 \cdot 35 = 100 + 0 = 100$

$3 \cdot 25 + 2 \cdot 35 = 75 + 70 = 145$

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

b) $5\,x + 6\,y = 380 \quad\Rightarrow\quad \mathbf{a} = 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y$

$4\,x + 7\,y = 370 \qquad\quad \mathbf{b} = 6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y$

$3\,x + 8\,y = 360 \qquad\quad \mathbf{r} = 380\,\gamma_t + 370\,\gamma_x + 360\,\gamma_y$

Outer products:

$$\mathbf{a}\,\mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y)$$
$$= 30\,\gamma_t^2 + 35\,\gamma_t\gamma_x + 40\,\gamma_t\gamma_y + 24\,\gamma_x\gamma_t + 28\,\gamma_x^2 + 32\,\gamma_x\gamma_y + 18\,\gamma_y\gamma_t + 21\,\gamma_y\gamma_x + 24\,\gamma_y^2$$
$$= 30 + 35\,\gamma_t\gamma_x + 40\,\gamma_t\gamma_y - 24\,\gamma_t\gamma_x - 28 + 32\,\gamma_x\gamma_y - 18\,\gamma_t\gamma_y - 21\,\gamma_x\gamma_y - 24$$
$$= -22 + 11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y$$

$$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 11\, \gamma_t\gamma_x + 22\, \gamma_t\gamma_y + 11\, \gamma_x\gamma_y$$

$$\mathbf{r}\,\mathbf{b} = (380\, \gamma_t + 370\, \gamma_x + 360\, \gamma_y)\,(6\, \gamma_t + 7\, \gamma_x + 8\, \gamma_y)$$

$$= 2280\, \gamma_t^2 + 2660\, \gamma_t\gamma_x + 3040\, \gamma_t\gamma_y + 2220\, \gamma_x\gamma_t + 2590\, \gamma_x^2 + 2960\, \gamma_x\gamma_y + 2160\, \gamma_y\gamma_t + 2520\, \gamma_y\gamma_x + 2880\, \gamma_y^2$$

$\cdot\, 40$

$$= 2280 + 2660\, \gamma_t\gamma_x + 3040\, \gamma_t\gamma_y - 2220\, \gamma_t\gamma_x - 2590 + 2960\, \gamma_x\gamma_y - 2160\, \gamma_t\gamma_y - 2520\, \gamma_x\gamma_y - 2880$$

$$= -\,3190 + 440\, \gamma_t\gamma_x + 880\, \gamma_t\gamma_y + 440\, \gamma_x\gamma_y$$

$\cdot\, 30$

$$\Rightarrow\ \mathbf{r} \wedge \mathbf{b} = 440\, \gamma_t\gamma_x + 880\, \gamma_t\gamma_y + 440\, \gamma_x\gamma_y$$

$$\mathbf{a}\,\mathbf{r} = (5\, \gamma_t + 4\, \gamma_x + 3\, \gamma_y)\,(380\, \gamma_t + 370\, \gamma_x + 360\, \gamma_y)$$

$$= 1900\, \gamma_t^2 + 1850\, \gamma_t\gamma_x + 1800\, \gamma_t\gamma_y + 1520\, \gamma_x\gamma_t + 1480\, \gamma_x^2 + 1440\, \gamma_x\gamma_y + 1140\, \gamma_y\gamma_t + 1110\, \gamma_y\gamma_x + 1080\, \gamma_y^2$$

$$= 1900 + 1850\, \gamma_t\gamma_x + 1800\, \gamma_t\gamma_y - 1520\, \gamma_t\gamma_x - 1480 + 1440\, \gamma_x\gamma_y - 1140\, \gamma_t\gamma_y - 1110\, \gamma_x\gamma_y - 1080$$

$$= -\,660 + 330\, \gamma_t\gamma_x + 660\, \gamma_t\gamma_y + 330\, \gamma_x\gamma_y$$

$$\Rightarrow\ \mathbf{a} \wedge \mathbf{r} = 330\, \gamma_t\gamma_x + 660\, \gamma_t\gamma_y + 330\, \gamma_x\gamma_y$$

Solution of the over-constrained system of linear equations:

$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = 40$

$y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = 30$

Check: $5 \cdot 40 + 6 \cdot 30 = 200 + 180 = 380$
$4 \cdot 40 + 7 \cdot 30 = 160 + 210 = 370$
$3 \cdot 40 + 8 \cdot 30 = 120 + 240 = 360$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

iubh Internationale Hochschule, Winter 2020/2021

Advanced Mathematics (MQM110)

Worksheet 11 – Answers

Problem 1:

One hundred years ago Eliakim Hastings Moore published the paper

> Eliakim H. Moore: On the Reciprocal of the General Algebraic Matrix. In: Bulletin of the American Math. Society, 26 (1920), pp. 394 – 395,

in which he described Moore-Penrose matrix inverses for the first time. 35 years later, Roger Penrose wrote his paper

> Roger Penrose: A Generalized Inverse for Matrices. In: Proceedings of the Cambridge Philosophical Society, 51 (1955) pp. 406 – 413,

elaborating the ideas of Moore and the mathematical foundations of generalized matrix inverses in a broader way. Meanwhile the mathematics of Moore-Penrose matrix inverses is even taught in introductory math courses at some universities in Germany, see e.g.

> Karsten Schmidt, Götz Trenkler: Einführung in die Moderne Matrix-Algebra. Mit Anwendungen in der Statistik. 3rd edition, Springer/Gabler, Berlin, Heidelberg 2015.

The authors of this book write: "Der vermittelte Stoff soll aktuell und modern sein. Deshalb bedienen wir uns der in letzter Zeit immer populärer gewordenen Hilfsmittel wie verallgemeinerte Inversen und Moore-Penrose-Inverse von Matrizen und ihrer Anwendung zur Lösung linearer Gleichungssysteme." Thus they claim:

- Generalized matrix inverses and Moore-Penrose matrix inverses are relevant.
- Generalized matrix inverses and Moore-Penrose matrix inverses are of topical interest.
- Generalized matrix inverses and Moore-Penrose matrix inverses are modern.
- Generalized matrix inverses and Moore-Penrose matrix inverses are fashionable.
- Generalized matrix inverses and Moore-Penrose matrix inverses can be used to solve systems of linear equations in an easy and accessible way.

Especially the last point is of some interest for non-mathematicians who want to apply generalized matrix inverses quickly without caring too much about the mathematical background: "Leser dieses Buchs sollen schnell und unmittelbar an den Umgang mit Matrizen herangeführt werden. Aus diesem Grund verzichten wir bewusst auf die Darstellung der abstrakten Theorie der linearen Algebra." It is not necessary to understand the complete underlying abstract theory to use generalized matrix inverses in practice.

But if you are interested in the mathematical foundations of generalized matrix inverses, the following book will be helpful:

> Adi Ben-Israel, Thomas N.E. Greville: Generalized Inverses. Theory and Applications. 2nd edition (Canadian Mathematical Society/CMS books in mathematics), Springer-Verlag, New York, Berlin, Heidelberg 2003.

Problem 2:

Repetition of the conventional solution strategy already discussed at school (using substitution or elimination and which is a little bit boring):

$$\text{a)}\ \left.\begin{array}{lcl} 5\,x + 0\,y = 125 & \Rightarrow & x = 25 \\ 4\,x + 0\,y = 100 & \Rightarrow & x = 25 \\ 3\,x + 2\,y = 145 & & \end{array}\right\}\ \begin{array}{l} 3 \cdot 25 + 8\,y = 75 + 2\,y = 145 \\ \Rightarrow \quad 2\,y = \ \ 70 \quad \Rightarrow \quad x = 35 \end{array}$$

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

$$\text{b)}\ \left.\begin{array}{l} \left.\begin{array}{l} 5\,x + 6\,y = 380 \\ 4\,x + 7\,y = 370 \end{array}\right\}\ 9\,x + 13\,y = \ 750 \\ 3\,x + 8\,y = 360 \ \Rightarrow \ 9\,x + 24\,y = 1080 \end{array}\right\}\ 11\,y = 330 \ \Rightarrow \ y = \frac{330}{11} = 30$$

$$\Rightarrow \ 5\,x + 180 = 380 \ \Rightarrow \ 5\,x = 200 \ \Downarrow \ x = \frac{200}{5} = 40$$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Repetition of the solution strategy using Pauli algebra:

$$\text{a)}\ \begin{array}{lcl} 5\,x + 0\,y = 125 & \Rightarrow & \mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z \\ 4\,x + 0\,y = 100 & & \mathbf{b} = \qquad\qquad\quad 2\,\sigma_z \\ 3\,x + 2\,y = 145 & & \mathbf{r} = 125\,\sigma_x + 100\,\sigma_y + 145\,\sigma_z \end{array}$$

Outer products:

$$\begin{aligned} \mathbf{a}\,\mathbf{b} &= (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(2\,\sigma_z) \\ &= 10\,\sigma_x\sigma_z + 8\,\sigma_y\sigma_z + 6\,\sigma_z^2 \\ &= 6 + 8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x \qquad \Rightarrow \qquad \mathbf{a} \wedge \mathbf{b} = 8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x \end{aligned}$$

$\cdot 25$

$$\begin{aligned} \mathbf{r}\,\mathbf{b} &= (125\,\sigma_x + 100\,\sigma_y + 145\,\sigma_z)\,(2\,\sigma_z) \\ &= 250\,\sigma_x\sigma_z + 200\,\sigma_y\sigma_z + 290\,\sigma_z^2 \\ &= 290 + 200\,\sigma_y\sigma_z - 250\,\sigma_z\sigma_x \qquad \Rightarrow \qquad \mathbf{r} \wedge \mathbf{b} = 200\,\sigma_y\sigma_z - 250\,\sigma_z\sigma_x \end{aligned}$$

$\cdot 35$

$$\begin{aligned} \mathbf{a}\,\mathbf{r} &= (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(125\,\sigma_x + 100\,\sigma_y + 145\,\sigma_z) \\ &= 625\,\sigma_x^2 + 500\,\sigma_x\sigma_y + 725\,\sigma_x\sigma_z + 500\,\sigma_y\sigma_x + 400\,\sigma_y^2 + 580\,\sigma_y\sigma_z + 375\,\sigma_z\sigma_x + 300\,\sigma_z\sigma_y + 435\,\sigma_z^2 \\ &= 625 + 500\,\sigma_x\sigma_y - 725\,\sigma_z\sigma_x - 500\,\sigma_x\sigma_y + 400 + 580\,\sigma_y\sigma_z + 375\,\sigma_z\sigma_x - 300\,\sigma_y\sigma_z + 435 \\ &= 1460 + 0\,\sigma_x\sigma_y + 280\,\sigma_y\sigma_z - 350\,\sigma_z\sigma_x \end{aligned}$$

$$\Rightarrow \ \mathbf{a} \wedge \mathbf{r} = 280\,\sigma_y\sigma_z - 350\,\sigma_z\sigma_x$$

Solution of the over-constrained system of linear equations:

$$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = 25$$

$y = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r}) = 35$

Check: $5 \cdot 25 + 0 \cdot 35 = 125 + 0 = 125$
$4 \cdot 25 + 0 \cdot 35 = 100 + 0 = 100$
$3 \cdot 25 + 2 \cdot 35 = 75 + 70 = 145$

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

b) $5\,x + 6\,y = 380$ $\quad\Rightarrow\quad$ $\mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z$
$4\,x + 7\,y = 370$ $\quad\quad$ $\mathbf{b} = 6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z$
$3\,x + 8\,y = 360$ $\quad\quad$ $\mathbf{r} = 380\,\sigma_x + 370\,\sigma_y + 360\,\sigma_z$

Outer products:

$$\mathbf{a}\,\mathbf{b} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z)$$
$$= 30\,\sigma_x^2 + 35\,\sigma_x\sigma_y + 40\,\sigma_x\sigma_z + 24\,\sigma_y\sigma_x + 28\,\sigma_y^2 + 32\,\sigma_y\sigma_z + 18\,\sigma_z\sigma_x + 21\,\sigma_z\sigma_y + 24\,\sigma_z^2$$
$$= 30 + 35\,\sigma_x\sigma_y - 40\,\sigma_z\sigma_x - 24\,\sigma_x\sigma_y + 28 + 32\,\sigma_y\sigma_z + 18\,\sigma_z\sigma_x - 21\,\sigma_y\sigma_z + 24$$
$$= 82 + 11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x$$
$$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x$$

$$\mathbf{r}\,\mathbf{b} = (380\,\sigma_x + 370\,\sigma_y + 360\,\sigma_z)\,(6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z)$$
$$= 2280\,\sigma_x^2 + 2660\,\sigma_x\sigma_y + 3040\,\sigma_x\sigma_z + 2220\,\sigma_y\sigma_x + 2590\,\sigma_y^2 + 2960\,\sigma_y\sigma_z + 2160\,\sigma_z\sigma_x + 2520\,\sigma_z\sigma_y + 2880\,\sigma_z^2$$

$\cdot\,40$

$$= 2280 + 2660\,\sigma_x\sigma_y - 3040\,\sigma_z\sigma_x - 2220\,\sigma_x\sigma_y + 2590 + 2960\,\sigma_y\sigma_z + 2160\,\sigma_z\sigma_x - 2520\,\sigma_y\sigma_z + 2880$$
$$= 7750 + 440\,\sigma_x\sigma_y + 440\,\sigma_y\sigma_z - 880\,\sigma_z\sigma_x$$
$$\Rightarrow\ \mathbf{r} \wedge \mathbf{b} = 440\,\sigma_x\sigma_y + 440\,\sigma_y\sigma_z - 880\,\sigma_z\sigma_x$$

$\cdot\,30$

$$\mathbf{a}\,\mathbf{r} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(380\,\sigma_x + 370\,\sigma_y + 360\,\sigma_z)$$
$$= 1900\,\sigma_x^2 + 1850\,\sigma_x\sigma_y + 1800\,\sigma_x\sigma_z + 1520\,\sigma_y\sigma_x + 1480\,\sigma_y^2 + 1440\,\sigma_y\sigma_z + 1140\,\sigma_z\sigma_x + 1110\,\sigma_z\sigma_y + 1080\,\sigma_z^2$$
$$= 1900 + 1850\,\sigma_x\sigma_y - 1800\,\sigma_z\sigma_x - 1520\,\sigma_x\sigma_y + 1480 + 1440\,\sigma_y\sigma_z + 1140\,\sigma_z\sigma_x - 1110\,\sigma_y\sigma_z + 1080$$
$$= 4460 + 330\,\sigma_x\sigma_y + 330\,\sigma_y\sigma_z - 660\,\sigma_z\sigma_x$$
$$\Rightarrow\ \mathbf{a} \wedge \mathbf{r} = 330\,\sigma_x\sigma_y + 330\,\sigma_y\sigma_z - 660\,\sigma_z\sigma_x$$

Solution of the over-constrained system of linear equations:

$x = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r} \wedge \mathbf{b}) = 40$

$y = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r}) = 30$

Check: $5 \cdot 40 + 6 \cdot 30 = 200 + 180 = 380$
$4 \cdot 40 + 7 \cdot 30 = 160 + 210 = 370$
$3 \cdot 40 + 8 \cdot 30 = 120 + 240 = 360$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Repetition of the solution strategy using Dirac algebra:

a)
$$\begin{aligned} 5x + 0y &= 125 \qquad\Rightarrow\qquad & \mathbf{a} &= 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y \\ 4x + 0y &= 100 & \mathbf{b} &= 2\,\gamma_y \\ 3x + 2y &= 145 & \mathbf{r} &= 125\,\gamma_t + 100\,\gamma_x + 145\,\gamma_y \end{aligned}$$

Outer products:

$$\begin{aligned} \mathbf{a}\,\mathbf{b} &= (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(2\,\gamma_y) \\ &= 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y + 6\,\gamma_y^2 \\ &= -6 + 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y \qquad\Rightarrow\qquad \mathbf{a}\wedge\mathbf{b} = 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y \end{aligned}$$

$\cdot\,25$

$$\begin{aligned} \mathbf{r}\,\mathbf{b} &= (125\,\gamma_t + 100\,\gamma_x + 145\,\gamma_y)\,(2\,\gamma_y) \\ &= 250\,\gamma_t\gamma_y + 200\,\gamma_x\gamma_y + 290\,\gamma_y^2 \\ &= -290 + 250\,\gamma_t\gamma_y + 200\,\gamma_x\gamma_y \qquad\Rightarrow\qquad \mathbf{r}\wedge\mathbf{b} = 250\,\gamma_t\gamma_y + 200\,\gamma_x\gamma_y \end{aligned}$$

$\cdot\,35$

$$\begin{aligned} \mathbf{a}\,\mathbf{r} &= (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(125\,\gamma_t + 100\,\gamma_x + 145\,\gamma_y) \\ &= 625\,\gamma_t^2 + 500\,\gamma_t\gamma_x + 725\,\gamma_t\gamma_y + 500\,\gamma_x\gamma_t + 400\,\gamma_x^2 + 580\,\gamma_x\gamma_y + 375\,\gamma_y\gamma_t + 300\,\gamma_y\gamma_x + 435\,\gamma_y^2 \\ &= 625 + 500\,\gamma_t\gamma_x + 725\,\gamma_t\gamma_y - 500\,\gamma_t\gamma_x - 400 + 580\,\gamma_x\gamma_y - 375\,\gamma_t\gamma_y - 300\,\gamma_x\gamma_y - 435 \\ &= -310 + 0\,\gamma_t\gamma_x + 350\,\gamma_t\gamma_y + 280\,\gamma_x\gamma_y \end{aligned}$$

$$\Rightarrow\quad \mathbf{a}\wedge\mathbf{r} = 350\,\gamma_t\gamma_y + 280\,\gamma_x\gamma_y$$

Solution of the over-constrained system of linear equations:

$$x = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{r}\wedge\mathbf{b}) = 25$$

$$y = (\mathbf{a}\wedge\mathbf{b})^{-1}\,(\mathbf{a}\wedge\mathbf{r}) = 35$$

Check:
$$\begin{aligned} 5\cdot 25 + 0\cdot 35 &= 125 + 0 = 125 \\ 4\cdot 25 + 0\cdot 35 &= 100 + 0 = 100 \\ 3\cdot 25 + 2\cdot 35 &= 75 + 70 = 145 \end{aligned}$$

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

b)
$$\begin{aligned} 5x + 6y &= 380 \qquad\Rightarrow\qquad & \mathbf{a} &= 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y \\ 4x + 7y &= 370 & \mathbf{b} &= 6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y \\ 3x + 8y &= 360 & \mathbf{r} &= 380\,\gamma_t + 370\,\gamma_x + 360\,\gamma_y \end{aligned}$$

Outer products:

$\mathbf{a}\,\mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y)$

$= 30\,\gamma_t^2 + 35\,\gamma_t\gamma_x + 40\,\gamma_t\gamma_y + 24\,\gamma_x\gamma_t + 28\,\gamma_x^2 + 32\,\gamma_x\gamma_y + 18\,\gamma_y\gamma_t + 21\,\gamma_y\gamma_x + 24\,\gamma_y^2$

$= 30 + 35\,\gamma_t\gamma_x + 40\,\gamma_t\gamma_y - 24\,\gamma_t\gamma_x - 28 + 32\,\gamma_x\gamma_y - 18\,\gamma_t\gamma_y - 21\,\gamma_x\gamma_y - 24$

$= -\,22 + 11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y$

$\Rightarrow\ \mathbf{a} \wedge \mathbf{b} = 11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y$

$\mathbf{r}\,\mathbf{b} = (380\,\gamma_t + 370\,\gamma_x + 360\,\gamma_y)\,(6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y)$

$= 2280\,\gamma_t^2 + 2660\,\gamma_t\gamma_x + 3040\,\gamma_t\gamma_y + 2220\,\gamma_x\gamma_t + 2590\,\gamma_x^2 + 2960\,\gamma_x\gamma_y$
$\quad + 2160\,\gamma_y\gamma_t + 2520\,\gamma_y\gamma_x + 2880\,\gamma_y^2$ $\cdot 40$

$= 2280 + 2660\,\gamma_t\gamma_x + 3040\,\gamma_t\gamma_y - 2220\,\gamma_t\gamma_x - 2590 + 2960\,\gamma_x\gamma_y$
$\quad - 2160\,\gamma_t\gamma_y - 2520\,\gamma_x\gamma_y - 2880$

$= -\,3190 + 440\,\gamma_t\gamma_x + 880\,\gamma_t\gamma_y + 440\,\gamma_x\gamma_y$

$\Rightarrow\ \mathbf{r} \wedge \mathbf{b} = 440\,\gamma_t\gamma_x + 880\,\gamma_t\gamma_y + 440\,\gamma_x\gamma_y$ $\cdot 30$

$\mathbf{a}\,\mathbf{r} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y)\,(380\,\gamma_t + 370\,\gamma_x + 360\,\gamma_y)$

$= 1900\,\gamma_t^2 + 1850\,\gamma_t\gamma_x + 1800\,\gamma_t\gamma_y + 1520\,\gamma_x\gamma_t + 1480\,\gamma_x^2 + 1440\,\gamma_x\gamma_y$
$\quad + 1140\,\gamma_y\gamma_t + 1110\,\gamma_y\gamma_x + 1080\,\gamma_y^2$

$= 1900 + 1850\,\gamma_t\gamma_x + 1800\,\gamma_t\gamma_y - 1520\,\gamma_t\gamma_x - 1480 + 1440\,\gamma_x\gamma_y$
$\quad - 1140\,\gamma_t\gamma_y - 1110\,\gamma_x\gamma_y - 1080$

$= -\,660 + 330\,\gamma_t\gamma_x + 660\,\gamma_t\gamma_y + 330\,\gamma_x\gamma_y$

$\Rightarrow\ \mathbf{a} \wedge \mathbf{r} = 330\,\gamma_t\gamma_x + 660\,\gamma_t\gamma_y + 330\,\gamma_x\gamma_y$

Solution of the over-constrained system of linear equations:

$x = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r} \wedge \mathbf{b}) = 40$

$y = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r}) = 30$

Check: $5 \cdot 40 + 6 \cdot 30 = 200 + 180 = 380$
$4 \cdot 40 + 7 \cdot 30 = 160 + 210 = 370$
$3 \cdot 40 + 8 \cdot 30 = 120 + 240 = 360$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Problem 3:

2 a) $5\,x + 0\,y = 125$
$4\,x + 0\,y = 100$
$3\,x + 2\,y = 145$

$$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$$

$\mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z$
$\mathbf{b} = 2\,\sigma_z$
$\mathbf{r_1} = \sigma_x \qquad \mathbf{r_2} = \sigma_y \qquad \mathbf{r_3} = \sigma_z$

$$\Rightarrow \quad \mathbf{D}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{D}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$\mathbf{a} \wedge \mathbf{b} = (5\ \sigma_x + 4\ \sigma_y + 3\ \sigma_z) \wedge (2\ \sigma_z) = 8\ \sigma_y\sigma_z - 10\ \sigma_z\sigma_x$

$$(\mathbf{a} \wedge \mathbf{b})^2 = (8\ \sigma_y\sigma_z - 10\ \sigma_z\sigma_x)^2$$
$$= (8\ \sigma_y\sigma_z - 10\ \sigma_z\sigma_x)\ (8\ \sigma_y\sigma_z - 10\ \sigma_z\sigma_x)$$
$$= 64\ \sigma_y\sigma_z\sigma_y\sigma_z - 80\ \sigma_y\sigma_z\sigma_z\sigma_x - 80\ \sigma_z\sigma_x\sigma_y\sigma_z + 100\ \sigma_z\sigma_x\sigma_z\sigma_x$$
$$= -64 + 80\ \sigma_x\sigma_y - 80\ \sigma_x\sigma_y - 100$$
$$= -164$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\ (\mathbf{a} \wedge \mathbf{b}) = -\frac{1}{164}\ (8\ \sigma_y\sigma_z - 10\ \sigma_z\sigma_x) = -\frac{1}{82}\ (4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x)$$

Outer products of resulting vectors and coefficient vectors:

$\mathbf{r_1} \wedge \mathbf{b} = \sigma_x \wedge (2\ \sigma_z) = -2\ \sigma_z\sigma_x$

$\mathbf{a} \wedge \mathbf{r_1} = (5\ \sigma_x + 4\ \sigma_y + 3\ \sigma_z) \wedge \sigma_x = -4\ \sigma_x\sigma_y + 3\ \sigma_z\sigma_x$

$\mathbf{r_2} \wedge \mathbf{b} = \sigma_y \wedge (2\ \sigma_z) = 2\ \sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{r_2} = (5\ \sigma_x + 4\ \sigma_y + 3\ \sigma_z) \wedge \sigma_y = 5\ \sigma_x\sigma_y - 3\ \sigma_y\sigma_z$

$\mathbf{r_3} \wedge \mathbf{b} = \sigma_z \wedge (2\ \sigma_z) = 0$

$\mathbf{a} \wedge \mathbf{r_3} = (5\ \sigma_x + 4\ \sigma_y + 3\ \sigma_z) \wedge \sigma_z = 4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x$

Elements of the Pauli algebra generalized matrix inverse $\mathbf{D}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is pre-multiplied from the left:

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r_1} \wedge \mathbf{b})$$
$$= -\frac{1}{82}\ (4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x)\ (-2\ \sigma_z\sigma_x)$$
$$= -\frac{1}{82}\ (-8\ \sigma_y\sigma_z\sigma_z\sigma_x + 10\ \sigma_z\sigma_x\sigma_z\sigma_x)$$
$$= -\frac{1}{82}\ (8\ \sigma_x\sigma_y - 10)$$
$$= \frac{5}{41} - \frac{4}{41}\ \sigma_x\sigma_y$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r_2} \wedge \mathbf{b})$$
$$= -\frac{1}{82}\ (4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x)\ (2\ \sigma_y\sigma_z)$$
$$= -\frac{1}{82}\ (8\ \sigma_y\sigma_z\sigma_y\sigma_z - 10\ \sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= -\frac{1}{82}\ (-8 - 10\ \sigma_x\sigma_y)$$

$$= \frac{4}{41} + \frac{5}{41}\ \sigma_x\sigma_y$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r_3} \wedge \mathbf{b})$$

$$= -\frac{1}{82}\ (4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x) \cdot 0$$

$$= 0$$

Intermediate check of the first row of the Pauli algebra matrix inverse:

			5	0
			4	0
			3	2
$\frac{5}{41} - \frac{4}{41}\ \sigma_x\sigma_y$	$\frac{4}{41} + \frac{5}{41}\ \sigma_x\sigma_y$	0	$\frac{41}{41}$	0
y_1	y_2	y_3	?	?

$$y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{a} \wedge \mathbf{r_1})$$

$$= -\frac{1}{82}\ (4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x)\ (-4\ \sigma_x\sigma_y + 3\ \sigma_z\sigma_x)$$

$$= -\frac{1}{82}\ (-16\ \sigma_y\sigma_z\sigma_x\sigma_y + 20\ \sigma_z\sigma_x\sigma_x\sigma_y + 12\ \sigma_y\sigma_z\sigma_z\sigma_x - 15\ \sigma_z\sigma_x\sigma_z\sigma_x)$$

$$= -\frac{1}{82}\ (-16\ \sigma_z\sigma_x - 20\ \sigma_y\sigma_z - 12\ \sigma_x\sigma_y + 15)$$

$$= -\frac{15}{82} + \frac{6}{41}\ \sigma_x\sigma_y + \frac{10}{41}\ \sigma_y\sigma_z + \frac{8}{41}\ \sigma_z\sigma_x$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{a} \wedge \mathbf{r_2})$$

$$= -\frac{1}{82}\ (4\ \sigma_y\sigma_z - 5\ \sigma_z\sigma_x)\ (5\ \sigma_x\sigma_y - 3\ \sigma_y\sigma_z)$$

$$= -\frac{1}{82}\ (20\ \sigma_y\sigma_z\sigma_x\sigma_y - 25\ \sigma_z\sigma_x\sigma_x\sigma_y - 12\ \sigma_y\sigma_z\sigma_y\sigma_z + 15\ \sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= -\frac{1}{82}\ (20\ \sigma_z\sigma_x + 25\ \sigma_y\sigma_z + 12 + 15\ \sigma_x\sigma_y)$$

$$= -\frac{6}{41} - \frac{15}{82}\ \sigma_x\sigma_y - \frac{25}{82}\ \sigma_y\sigma_z - \frac{10}{41}\ \sigma_z\sigma_x$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1} \, (\mathbf{a} \wedge \mathbf{r_3})$$

$$= -\frac{1}{82} \, (4\, \sigma_y\sigma_z - 5\, \sigma_z\sigma_x)\, (4\, \sigma_y\sigma_z - 5\, \sigma_z\sigma_x)$$

$$= -\frac{1}{82} \, (16\, \sigma_y\sigma_z\sigma_y\sigma_z - 20\, \sigma_z\sigma_x\sigma_y\sigma_z - 20\, \sigma_y\sigma_z\sigma_z\sigma_x + 25\, \sigma_z\sigma_x\sigma_z\sigma_x)$$

$$= -\frac{1}{82} \, (-16 - 20\, \sigma_x\sigma_y + 20\, \sigma_x\sigma_y - 25)$$

$$= \frac{1}{2}$$

$$\Rightarrow \; \mathbf{D}^{-1} = \begin{bmatrix} \frac{5}{41} - \frac{4}{41}\sigma_x\sigma_y & \frac{4}{41} + \frac{5}{41}\sigma_x\sigma_y & 0 \\ -\frac{15}{82} + \frac{6}{41}\sigma_x\sigma_y + \frac{10}{41}\sigma_y\sigma_z + \frac{8}{41}\sigma_z\sigma_x & -\frac{6}{41} - \frac{15}{82}\sigma_x\sigma_y - \frac{25}{82}\sigma_y\sigma_z - \frac{10}{41}\sigma_z\sigma_x & \frac{1}{2} \end{bmatrix}$$

Check of Pauli algebra matrix inverse: $\mathbf{D}^{-1}\, \mathbf{D} = \mathbf{I}$

			5	0
			4	0
			3	2
$\frac{5}{41} - \frac{4}{41}\sigma_x\sigma_y$	$\frac{4}{41} + \frac{5}{41}\sigma_x\sigma_y$	0	1	0
$-\frac{15}{82} + \frac{6}{41}\sigma_x\sigma_y + \frac{10}{41}\sigma_y\sigma_z + \frac{8}{41}\sigma_z\sigma_x$	$-\frac{6}{41} - \frac{15}{82}\sigma_x\sigma_y - \frac{25}{82}\sigma_y\sigma_z - \frac{10}{41}\sigma_z\sigma_x$	$\frac{1}{2}$	0	1

Quantities of final products, which will be produced:

			125
			100
			145
$\frac{5}{41} - \frac{4}{41}\sigma_x\sigma_y$	$\frac{4}{41} + \frac{5}{41}\sigma_x\sigma_y$	0	25
$-\frac{15}{82} + \frac{6}{41}\sigma_x\sigma_y + \frac{10}{41}\sigma_y\sigma_z + \frac{8}{41}\sigma_z\sigma_x$	$-\frac{6}{41} - \frac{15}{82}\sigma_x\sigma_y - \frac{25}{82}\sigma_y\sigma_z - \frac{10}{41}\sigma_z\sigma_x$	$\frac{1}{2}$	35

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

2 b) $5\,x + 6\,y = 380$

$4\,x + 7\,y = 370$

$3\,x + 8\,y = 360$

$$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 3 & 8 \end{bmatrix}$$

$\mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z$

$\mathbf{b} = 6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z$

$\mathbf{r_1} = \sigma_x \qquad \mathbf{r_2} = \sigma_y \qquad \mathbf{r_3} = \sigma_z$

$$\Rightarrow \qquad \mathbf{D}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad \Leftrightarrow \qquad \mathbf{D}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$\mathbf{a} \wedge \mathbf{b} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z)$

$\qquad = (35 - 24)\,\sigma_x\sigma_y + (32 - 21)\,\sigma_y\sigma_z + (18 - 40)\,\sigma_z\sigma_x$

$\qquad = 11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x$

$(\mathbf{a} \wedge \mathbf{b})^2 = (11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)^2$

$\qquad = (11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)\,(11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)$

$\qquad = 121\,\sigma_x\sigma_y\sigma_x\sigma_y + 121\,\sigma_x\sigma_y\sigma_y\sigma_z - 242\,\sigma_x\sigma_y\sigma_z\sigma_x + 121\,\sigma_y\sigma_z\sigma_x\sigma_y + 121\,\sigma_y\sigma_z\sigma_y\sigma_z$
$\qquad\qquad - 242\,\sigma_y\sigma_z\sigma_z\sigma_x - 242\,\sigma_z\sigma_x\sigma_x\sigma_y - 242\,\sigma_z\sigma_x\sigma_y\sigma_z + 484\,\sigma_z\sigma_x\sigma_z\sigma_x$

$\qquad = -\,121 - 121\,\sigma_z\sigma_x - 242\,\sigma_y\sigma_z + 121\,\sigma_z\sigma_x - 121 + 242\,\sigma_x\sigma_y + 242\,\sigma_y\sigma_z - 242\,\sigma_x\sigma_y - 484$

$\qquad = -\,121 - 121 - 484$

$\qquad = -\,726$

$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = -\dfrac{1}{726}\,(11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)$

$\qquad = \dfrac{1}{66}\,(-\,\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$

Outer products of resulting vectors and coefficient vectors:

$\mathbf{r_1} \wedge \mathbf{b} = \sigma_x \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = 7\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x$

$\mathbf{a} \wedge \mathbf{r_1} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_x = -\,4\,\sigma_x\sigma_y + 3\,\sigma_z\sigma_x$

$\mathbf{r_2} \wedge \mathbf{b} = \sigma_y \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = -\,6\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{r_2} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_y = 5\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z$

$\mathbf{r_3} \wedge \mathbf{b} = \sigma_z \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = 6\,\sigma_z\sigma_x - 7\,\sigma_y\sigma_z$

$\mathbf{a} \wedge \mathbf{r_3} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_z = 4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x$

Elements of the Pauli algebra generalized matrix inverse $\mathbf{D}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is pre-multiplied from the left:

$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b})$

$\quad = \dfrac{1}{66}\,(-\,\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)\,(7\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x)$

$\quad = \dfrac{1}{66}\,(-\,7\,\sigma_x\sigma_y\sigma_x\sigma_y + 8\,\sigma_x\sigma_y\sigma_z\sigma_x - 7\,\sigma_y\sigma_z\sigma_x\sigma_y + 8\,\sigma_y\sigma_z\sigma_z\sigma_x + 14\,\sigma_z\sigma_x\sigma_x\sigma_y - 16\,\sigma_z\sigma_x\sigma_z\sigma_x)$

$$= \frac{1}{66}\ (7 + 8\ \sigma_y\sigma_z - 7\ \sigma_z\sigma_x - 8\ \sigma_x\sigma_y - 14\ \sigma_y\sigma_z + 16)$$

$$= \frac{1}{66}\ (23 - 8\ \sigma_x\sigma_y - 6\ \sigma_y\sigma_z - 7\ \sigma_z\sigma_x)$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r_2} \wedge \mathbf{b})$$

$$= \frac{1}{66}\ (-\ \sigma_x\sigma_y - \sigma_y\sigma_z + 2\ \sigma_z\sigma_x)\ (-\ 6\ \sigma_x\sigma_y + 8\ \sigma_y\sigma_z)$$

$$= \frac{1}{66}\ (6\,\sigma_x\sigma_y\sigma_x\sigma_y - 8\,\sigma_x\sigma_y\sigma_y\sigma_z + 6\,\sigma_y\sigma_z\sigma_x\sigma_y - 8\,\sigma_y\sigma_z\sigma_y\sigma_z - 12\,\sigma_z\sigma_x\sigma_x\sigma_y + 16\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= \frac{1}{66}\ (-\ 6 + 8\,\sigma_z\sigma_x + 6\ \sigma_z\sigma_x + 8 + 12\ \sigma_y\sigma_z + 16\ \sigma_x\sigma_y)$$

$$= \frac{1}{66}\ (2 + 16\ \sigma_x\sigma_y + 12\ \sigma_y\sigma_z + 14\ \sigma_z\sigma_x)$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{r_3} \wedge \mathbf{b})$$

$$= \frac{1}{66}\ (-\ \sigma_x\sigma_y - \sigma_y\sigma_z + 2\ \sigma_z\sigma_x)\ (6\ \sigma_z\sigma_x - 7\ \sigma_y\sigma_z)$$

$$= \frac{1}{66}\ (-\ 6\,\sigma_x\sigma_y\sigma_z\sigma_x + 7\,\sigma_x\sigma_y\sigma_y\sigma_z - 6\,\sigma_y\sigma_z\sigma_z\sigma_x + 7\,\sigma_y\sigma_z\sigma_y\sigma_z + 12\,\sigma_z\sigma_x\sigma_z\sigma_x - 14\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= \frac{1}{66}\ (-\ 6\ \sigma_y\sigma_z - 7\ \sigma_z\sigma_x + 6\,\sigma_x\sigma_y - 7 - 12 - 14\ \sigma_x\sigma_y)$$

$$= \frac{1}{66}\ (-\ 19 - 8\ \sigma_x\sigma_y - 6\ \sigma_y\sigma_z - 7\ \sigma_z\sigma_x)$$

Intermediate check of first row of the Pauli algebra matrix inverse:

			5	6
			4	7
			3	8
$23 - 8\,\sigma_x\sigma_y - 6\,\sigma_y\sigma_z - 7\,\sigma_z\sigma_x$	$2 + 16\,\sigma_x\sigma_y + 12\,\sigma_y\sigma_z + 14\,\sigma_z\sigma_x$	$-19 - 8\,\sigma_x\sigma_y - 6\,\sigma_y\sigma_z - 7\,\sigma_z\sigma_x$	66	0
y_1	y_2	y_3	?	?

$$y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{a} \wedge \mathbf{r_1})$$

$$= \frac{1}{66}\ (-\ \sigma_x\sigma_y - \sigma_y\sigma_z + 2\ \sigma_z\sigma_x)\ (-\ 4\ \sigma_x\sigma_y + 3\ \sigma_z\sigma_x)$$

$$= \frac{1}{66}\ (4\,\sigma_x\sigma_y\sigma_x\sigma_y - 3\,\sigma_x\sigma_y\sigma_z\sigma_x + 4\,\sigma_y\sigma_z\sigma_x\sigma_y - 3\,\sigma_y\sigma_z\sigma_z\sigma_x - 8\,\sigma_z\sigma_x\sigma_x\sigma_y + 6\,\sigma_z\sigma_x\sigma_z\sigma_x)$$

$$= \frac{1}{66}\ (-\ 4 - 3\ \sigma_y\sigma_z + 4\ \sigma_z\sigma_x + 3\ \sigma_x\sigma_y + 8\ \sigma_y\sigma_z - 6)$$

$$= \frac{1}{66}\ (-\ 10 + 3\ \sigma_x\sigma_y + 5\ \sigma_y\sigma_z + 4\ \sigma_z\sigma_x)$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2})$$

$$= \frac{1}{66}\,(-\,\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)\,(5\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z)$$

$$= \frac{1}{66}\,(-\,5\,\sigma_x\sigma_y\sigma_x\sigma_y + 3\,\sigma_x\sigma_y\sigma_y\sigma_z - 5\,\sigma_y\sigma_z\sigma_x\sigma_y + 3\,\sigma_y\sigma_z\sigma_y\sigma_z + 10\,\sigma_z\sigma_x\sigma_x\sigma_y - 6\,\sigma_z\sigma_x\sigma_y\sigma_z)$$

$$= \frac{1}{66}\,(5 - 3\,\sigma_z\sigma_x - 5\,\sigma_z\sigma_x - 3 - 10\,\sigma_y\sigma_z - 6\,\sigma_x\sigma_y)$$

$$= \frac{1}{66}\,(2 - 6\,\sigma_x\sigma_y - 10\,\sigma_y\sigma_z - 8\,\sigma_z\sigma_x)$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_3})$$

$$= \frac{1}{66}\,(-\,\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$

$$= \frac{1}{66}\,(-\,4\,\sigma_x\sigma_y\sigma_y\sigma_z + 5\,\sigma_x\sigma_y\sigma_z\sigma_x - 4\,\sigma_y\sigma_z\sigma_y\sigma_z + 5\,\sigma_y\sigma_z\sigma_z\sigma_x + 8\,\sigma_z\sigma_x\sigma_y\sigma_z - 10\,\sigma_z\sigma_x\sigma_z\sigma_x)$$

$$= \frac{1}{66}\,(4\,\sigma_z\sigma_x + 5\,\sigma_y\sigma_z + 4 - 5\,\sigma_x\sigma_y + 8\,\sigma_x\sigma_y + 10)$$

$$= \frac{1}{66}\,(14 + 3\,\sigma_x\sigma_y + 5\,\sigma_y\sigma_z + 4\,\sigma_z\sigma_x)$$

$$\Rightarrow\ \mathbf{D}^{-1} = \frac{1}{66}\begin{bmatrix} 23-8\sigma_x\sigma_y-6\sigma_y\sigma_z-7\sigma_z\sigma_x & 2+16\sigma_x\sigma_y+12\sigma_y\sigma_z+14\sigma_z\sigma_x & -19-8\sigma_x\sigma_y-6\sigma_y\sigma_z-7\sigma_z\sigma_x \\ -10+3\sigma_x\sigma_y+5\sigma_y\sigma_z+4\sigma_z\sigma_x & 2-6\sigma_x\sigma_y-10\sigma_y\sigma_z-8\sigma_z\sigma_x & 14+3\sigma_x\sigma_y+5\sigma_y\sigma_z+4\sigma_z\sigma_x \end{bmatrix}$$

Check of Pauli algebra matrix inverse: $\mathbf{D}^{-1}\,\mathbf{D} = \mathbf{I}$

$$\begin{array}{ccc|cc} & & & 5 & 6 \\ & & & 4 & 7 \\ & & & 3 & 8 \\ \hline 23-8\sigma_x\sigma_y-6\sigma_y\sigma_z-7\sigma_z\sigma_x & 2+16\sigma_x\sigma_y+12\sigma_y\sigma_z+14\sigma_z\sigma_x & -19-8\sigma_x\sigma_y-6\sigma_y\sigma_z-7\sigma_z\sigma_x & 66 & 0 \\ -10+3\sigma_x\sigma_y+5\sigma_y\sigma_z+4\sigma_z\sigma_x & 2-6\sigma_x\sigma_y-10\sigma_y\sigma_z-8\sigma_z\sigma_x & 14+3\sigma_x\sigma_y+5\sigma_y\sigma_z+4\sigma_z\sigma_x & 0 & 66 \end{array}$$

Quantities of final products, which will be produced:

$$\begin{array}{ccc|c} & & & 380 \\ & & & 370 \\ & & & 360 \\ \hline 23-8\sigma_x\sigma_y-6\sigma_y\sigma_z-7\sigma_z\sigma_x & 2+16\sigma_x\sigma_y+12\sigma_y\sigma_z+14\sigma_z\sigma_x & -19-8\sigma_x\sigma_y-6\sigma_y\sigma_z-7\sigma_z\sigma_x & 2640 \\ -10+3\sigma_x\sigma_y+5\sigma_y\sigma_z+4\sigma_z\sigma_x & 2-6\sigma_x\sigma_y-10\sigma_y\sigma_z-8\sigma_z\sigma_x & 14+3\sigma_x\sigma_y+5\sigma_y\sigma_z+4\sigma_z\sigma_x & 1980 \end{array}$$

Completing the result: $x = \dfrac{2640}{66} = 40 \qquad y = \dfrac{1980}{66} = 30$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Problem 4:

Nearly the complete solution of problem 3 can be copied. Only the bivector parts of the demand matrices are reversed, because now the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is post-multiplied from the right.

2 a) $5x + 0y = 125$, $4x + 0y = 100$, $3x + 2y = 145$ $\Rightarrow$ $\mathbf{D} = \begin{bmatrix} 5 & 0 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$

$\mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z$

$\mathbf{b} = 2\,\sigma_z$

$\mathbf{r_1} = \sigma_x \qquad \mathbf{r_2} = \sigma_y \qquad \mathbf{r_3} = \sigma_z$

$$\Rightarrow \quad \underline{\mathbf{D}}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$$\mathbf{a} \wedge \mathbf{b} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge (2\,\sigma_z) = 8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x$$

$$(\mathbf{a} \wedge \mathbf{b})^2 = (8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x)^2 = -164$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = -\frac{1}{164}\,(8\,\sigma_y\sigma_z - 10\,\sigma_z\sigma_x) = -\frac{1}{82}\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$

Outer products of resulting vectors and coefficient vectors:

$$\mathbf{r_1} \wedge \mathbf{b} = \sigma_x \wedge (2\,\sigma_z) = -2\,\sigma_z\sigma_x$$

$$\mathbf{a} \wedge \mathbf{r_1} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_x = -4\,\sigma_x\sigma_y + 3\,\sigma_z\sigma_x$$

$$\mathbf{r_2} \wedge \mathbf{b} = \sigma_y \wedge (2\,\sigma_z) = 2\,\sigma_y\sigma_z$$

$$\mathbf{a} \wedge \mathbf{r_2} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_y = 5\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z$$

$$\mathbf{r_3} \wedge \mathbf{b} = \sigma_z \wedge (2\,\sigma_z) = 0$$

$$\mathbf{a} \wedge \mathbf{r_3} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_z = 4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x$$

Elements of the Pauli algebra generalized matrix inverse $\underline{\mathbf{D}}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is post-multiplied from the right:

$$x_1 = (\mathbf{r_1} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{82}\,(-2\,\sigma_z\sigma_x)\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x) = \frac{5}{41} + \frac{4}{41}\,\sigma_x\sigma_y$$

$$x_2 = (\mathbf{r_2} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{82}\,(2\,\sigma_y\sigma_z)\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x) = \frac{4}{41} - \frac{5}{41}\,\sigma_x\sigma_y$$

$$x_3 = (\mathbf{r_3} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{82} \cdot 0 \cdot (4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x) = 0$$

$$y_1 = (\mathbf{a} \wedge \mathbf{r_1})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{82}\,(-4\,\sigma_x\sigma_y + 3\,\sigma_z\sigma_x)\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$

$$= -\frac{15}{82} - \frac{6}{41}\,\sigma_x\sigma_y - \frac{10}{41}\,\sigma_y\sigma_z - \frac{8}{41}\,\sigma_z\sigma_x$$

$$y_2 = (\mathbf{a} \wedge \mathbf{r_2})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{82}\,(5\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z)\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$

$$= -\frac{6}{41} + \frac{15}{82}\,\sigma_x\sigma_y + \frac{25}{82}\,\sigma_y\sigma_z + \frac{10}{41}\,\sigma_z\sigma_x$$

$$y_3 = (\mathbf{a} \wedge \mathbf{r_3})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = -\frac{1}{82}\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)$$

$$= \frac{1}{2}$$

$$\Rightarrow\ \underline{\mathbf{D}}^{-1} = \begin{bmatrix} \frac{5}{41} + \frac{4}{41}\sigma_x\sigma_y & \frac{4}{41} - \frac{5}{41}\sigma_x\sigma_y & 0 \\ -\frac{15}{82} - \frac{6}{41}\sigma_x\sigma_y - \frac{10}{41}\sigma_y\sigma_z - \frac{8}{41}\sigma_z\sigma_x & -\frac{6}{41} + \frac{15}{82}\sigma_x\sigma_y + \frac{25}{82}\sigma_y\sigma_z + \frac{10}{41}\sigma_z\sigma_x & \frac{1}{2} \end{bmatrix}$$

Check of Pauli algebra matrix inverse: $\underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$

			5	0
			4	0
			3	2
$\frac{5}{41} + \frac{4}{41}\,\sigma_x\sigma_y$	$\frac{4}{41} - \frac{5}{41}\,\sigma_x\sigma_y$	0	1	0
$-\frac{15}{82} - \frac{6}{41}\,\sigma_x\sigma_y - \frac{10}{41}\,\sigma_y\sigma_z - \frac{8}{41}\,\sigma_z\sigma_x$	$-\frac{6}{41} + \frac{15}{82}\,\sigma_x\sigma_y + \frac{25}{82}\,\sigma_y\sigma_z + \frac{10}{41}\,\sigma_z\sigma_x$	$\frac{1}{2}$	0	1

Quantities of final products, which will be produced:

			125
			100
			145
$\frac{5}{41} + \frac{4}{41}\,\sigma_x\sigma_y$	$\frac{4}{41} - \frac{5}{41}\,\sigma_x\sigma_y$	0	25
$-\frac{15}{82} - \frac{6}{41}\,\sigma_x\sigma_y - \frac{10}{41}\,\sigma_y\sigma_z - \frac{8}{41}\,\sigma_z\sigma_x$	$-\frac{6}{41} + \frac{15}{82}\,\sigma_x\sigma_y + \frac{25}{82}\,\sigma_y\sigma_z + \frac{10}{41}\,\sigma_z\sigma_x$	$\frac{1}{2}$	35

⇒ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

2 b) $5\,x + 6\,y = 380$
$4\,x + 7\,y = 370$
$3\,x + 8\,y = 360$

$$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 3 & 8 \end{bmatrix} \qquad \mathbf{a} = 5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z, \quad \mathbf{b} = 6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z, \quad \mathbf{r_1} = \sigma_x \quad \mathbf{r_2} = \sigma_y \quad \mathbf{r_3} = \sigma_z$$

$$\Rightarrow \quad \underline{\mathbf{D}}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$$\mathbf{a} \wedge \mathbf{b} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = 11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x$$

$$(\mathbf{a} \wedge \mathbf{b})^2 = (11\,\sigma_x\sigma_y + 11\,\sigma_y\sigma_z - 22\,\sigma_z\sigma_x)^2 = -726$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = \frac{1}{66}\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$

Outer products of resulting vectors and coefficient vectors:

$$\mathbf{r_1} \wedge \mathbf{b} = \sigma_x \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = 7\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x$$

$$\mathbf{a} \wedge \mathbf{r_1} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_x = -4\,\sigma_x\sigma_y + 3\,\sigma_z\sigma_x$$

$$\mathbf{r_2} \wedge \mathbf{b} = \sigma_y \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = -6\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z$$

$$\mathbf{a} \wedge \mathbf{r_2} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_y = 5\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z$$

$$\mathbf{r_3} \wedge \mathbf{b} = \sigma_z \wedge (6\,\sigma_x + 7\,\sigma_y + 8\,\sigma_z) = 6\,\sigma_z\sigma_x - 7\,\sigma_y\sigma_z$$

$$\mathbf{a} \wedge \mathbf{r_3} = (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z) \wedge \sigma_z = 4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x$$

Elements of the Pauli algebra generalized matrix inverse $\underline{\mathbf{D}}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is post-multiplied from the right:

$$x_1 = (\mathbf{r_1} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{66}\,(7\,\sigma_x\sigma_y - 8\,\sigma_z\sigma_x)\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$
$$= \frac{1}{66}\,(23 + 8\,\sigma_x\sigma_y + 6\,\sigma_y\sigma_z + 7\,\sigma_z\sigma_x)$$

$$x_2 = (\mathbf{r_2} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{66}\,(-6\,\sigma_x\sigma_y + 8\,\sigma_y\sigma_z)\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$
$$= \frac{1}{66}\,(2 - 16\,\sigma_x\sigma_y - 12\,\sigma_y\sigma_z - 14\,\sigma_z\sigma_x)$$

$$x_3 = (\mathbf{r_3} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{66}\,(6\,\sigma_z\sigma_x - 7\,\sigma_y\sigma_z)\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$
$$= \frac{1}{66}\,(-19 + 8\,\sigma_x\sigma_y + 6\,\sigma_y\sigma_z + 7\,\sigma_z\sigma_x)$$

$$y_1 = (\mathbf{a} \wedge \mathbf{r_1})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{66}\,(-4\,\sigma_x\sigma_y + 3\,\sigma_z\sigma_x)\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$

$$= \frac{1}{66}\,(-10 - 3\,\sigma_x\sigma_y - 5\,\sigma_y\sigma_z - 4\,\sigma_z\sigma_x)$$

$$y_2 = (\mathbf{a} \wedge \mathbf{r_2})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{66}\,(5\,\sigma_x\sigma_y - 3\,\sigma_y\sigma_z)\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$

$$= \frac{1}{66}\,(2 + 6\,\sigma_x\sigma_y + 10\,\sigma_y\sigma_z + 8\,\sigma_z\sigma_x)$$

$$y_3 = (\mathbf{a} \wedge \mathbf{r_3})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{66}\,(4\,\sigma_y\sigma_z - 5\,\sigma_z\sigma_x)\,(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\,\sigma_z\sigma_x)$$

$$= \frac{1}{66}\,(14 - 3\,\sigma_x\sigma_y - 5\,\sigma_y\sigma_z - 4\,\sigma_z\sigma_x)$$

$$\Rightarrow\ \underline{\mathbf{D}}^{-1} = \frac{1}{66}\begin{bmatrix} 23+8\sigma_x\sigma_y+6\sigma_y\sigma_z+7\sigma_z\sigma_x & 2-16\sigma_x\sigma_y-12\sigma_y\sigma_z-14\sigma_z\sigma_x & -19+8\sigma_x\sigma_y+6\sigma_y\sigma_z+7\sigma_z\sigma_x \\ -10-3\sigma_x\sigma_y-5\sigma_y\sigma_z-4\sigma_z\sigma_x & 2+6\sigma_x\sigma_y+10\sigma_y\sigma_z+8\sigma_z\sigma_x & 14-3\sigma_x\sigma_y-5\sigma_y\sigma_z-4\sigma_z\sigma_x \end{bmatrix}$$

Check of Pauli algebra matrix inverse: $\underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$

			5	6
			4	7
			3	8
$23+8\sigma_x\sigma_y+6\sigma_y\sigma_z+7\sigma_z\sigma_x$	$2-16\sigma_x\sigma_y-12\sigma_y\sigma_z-14\sigma_z\sigma_x$	$-19+8\sigma_x\sigma_y+6\sigma_y\sigma_z+7\sigma_z\sigma_x$	66	0
$-10-3\sigma_x\sigma_y-5\sigma_y\sigma_z-4\sigma_z\sigma_x$	$2+6\sigma_x\sigma_y+10\sigma_y\sigma_z+8\sigma_z\sigma_x$	$14-3\sigma_x\sigma_y-5\sigma_y\sigma_z-4\sigma_z\sigma_x$	0	66

Quantities of final products, which will be produced:

			380
			370
			360
$23+8\sigma_x\sigma_y+6\sigma_y\sigma_z+7\sigma_z\sigma_x$	$2-16\sigma_x\sigma_y-12\sigma_y\sigma_z-14\sigma_z\sigma_x$	$-19+8\sigma_x\sigma_y+6\sigma_y\sigma_z+7\sigma_z\sigma_x$	2640
$-10-3\sigma_x\sigma_y-5\sigma_y\sigma_z-4\sigma_z\sigma_x$	$2+6\sigma_x\sigma_y+10\sigma_y\sigma_z+8\sigma_z\sigma_x$	$14-3\sigma_x\sigma_y-5\sigma_y\sigma_z-4\sigma_z\sigma_x$	1980

As this second Pauli algebra generalized matrix inverse $\underline{\mathbf{D}}^{-1}$ also is pre-multiplied from the left to the consumption vector of raw materials (which has to be post-multiplied from the right), this second generalized matrix inverse again is a left-sided matrix inverse.

Completing the result: $x = \dfrac{2640}{66} = 40 \qquad y = \dfrac{1980}{66} = 30$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Problem 5:

2 a) $5\,x + 0\,y = 125$ $\quad$ $4\,x + 0\,y = 100$ $\quad$ $3\,x + 2\,y = 145$

$$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 4 & 0 \\ 3 & 2 \end{bmatrix} \qquad \begin{array}{l} \mathbf{a} = 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y \\ \mathbf{b} = 2\,\gamma_y \\ \mathbf{r_1} = \gamma_t \qquad \mathbf{r_2} = \gamma_x \qquad \mathbf{r_3} = \gamma_y \end{array}$$

$$\Rightarrow \quad \mathbf{D}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{D}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$\mathbf{a} \wedge \mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge (2\,\gamma_y) = 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y$

$$\begin{aligned}(\mathbf{a} \wedge \mathbf{b})^2 &= (10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y)^2 \\ &= (10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y)\,(10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y) \\ &= 100\,\gamma_t\gamma_y\gamma_t\gamma_y + 80\,\gamma_t\gamma_y\gamma_x\gamma_y + 80\,\gamma_x\gamma_y\gamma_t\gamma_y + 64\,\gamma_x\gamma_y\gamma_x\gamma_y \\ &= 100 + 80\,\gamma_t\gamma_x - 80\,\gamma_t\gamma_x - 64 \\ &= 36\end{aligned}$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = \frac{1}{36}\,(10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y) = \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)$$

Outer products of resulting vectors and coefficient vectors:

$\mathbf{r_1} \wedge \mathbf{b} = \gamma_t \wedge (2\,\gamma_y) = 2\,\gamma_t\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_t = -\,4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} = \gamma_x \wedge (2\,\gamma_y) = 2\,\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_x = 5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} = \gamma_y \wedge (2\,\gamma_y) = 0$

$\mathbf{a} \wedge \mathbf{r_3} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_y = 5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y$

Elements of the Dirac algebra generalized matrix inverse $\mathbf{D}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is pre-multiplied from the left:

$$\begin{aligned}x_1 &= (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_1} \wedge \mathbf{b}) \\ &= \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)\,(2\,\gamma_t\gamma_y) \\ &= \frac{1}{18}\,(10\,\gamma_t\gamma_y\gamma_t\gamma_y + 8\,\gamma_x\gamma_y\gamma_t\gamma_y) \\ &= \frac{1}{18}\,(10 - 8\,\gamma_t\gamma_x)\end{aligned}$$

$$= \frac{5}{9} - \frac{4}{9}\,\gamma_t\gamma_x$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_2} \wedge \mathbf{b})$$

$$= \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)\,(2\,\gamma_x\gamma_y)$$

$$= \frac{1}{18}\,(10\,\gamma_t\gamma_y\gamma_x\gamma_y + 8\,\gamma_x\gamma_y\gamma_x\gamma_y)$$

$$= \frac{1}{18}\,(10\,\gamma_t\gamma_x - 8)$$

$$= -\frac{4}{9} + \frac{5}{9}\,\gamma_t\gamma_x$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{r_3} \wedge \mathbf{b})$$

$$= \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y) \cdot 0$$

$$= 0$$

Intermediate check of the first row of the Dirac algebra matrix inverse:

			5	0
			4	0
			3	2
$\frac{5}{9} - \frac{4}{9}\,\gamma_t\gamma_x$	$-\frac{4}{9} + \frac{5}{9}\,\gamma_t\gamma_x$	0	$\frac{9}{9}$	0
y_1	y_2	y_3	?	?

$$y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_1})$$

$$= \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)\,(-4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y)$$

$$= \frac{1}{18}\,(-20\,\gamma_t\gamma_y\gamma_t\gamma_x - 15\,\gamma_t\gamma_y\gamma_t\gamma_y - 16\,\gamma_x\gamma_y\gamma_t\gamma_x - 12\,\gamma_x\gamma_y\gamma_t\gamma_y)$$

$$= \frac{1}{18}\,(-20\,\gamma_x\gamma_y - 15 - 16\,\gamma_t\gamma_y + 12\,\gamma_t\gamma_x)$$

$$= -\frac{15}{18} + \frac{12}{18}\,\gamma_t\gamma_x - \frac{16}{18}\,\gamma_t\gamma_y - \frac{20}{18}\,\gamma_x\gamma_y$$

$$y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1}\,(\mathbf{a} \wedge \mathbf{r_2})$$

$$= \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)\,(5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y)$$

$$= \frac{1}{18}\ (25\ \gamma_t\gamma_y\gamma_t\gamma_x - 15\ \gamma_t\gamma_y\gamma_x\gamma_y + 20\ \gamma_x\gamma_y\gamma_t\gamma_x - 12\ \gamma_x\gamma_y\gamma_x\gamma_y)$$

$$= \frac{1}{18}\ (25\ \gamma_x\gamma_y - 15\ \gamma_t\gamma_x + 20\ \gamma_t\gamma_y + 12)$$

$$= \frac{12}{18} - \frac{15}{18}\ \gamma_t\gamma_x + \frac{20}{18}\ \gamma_t\gamma_y + \frac{25}{18}\ \gamma_x\gamma_y$$

$$y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1}\ (\mathbf{a} \wedge \mathbf{r_3})$$

$$= \frac{1}{18}\ (5\ \gamma_t\gamma_y + 4\ \gamma_x\gamma_y)\ (5\ \gamma_t\gamma_y + 4\ \gamma_x\gamma_y)$$

$$= \frac{1}{18}\ (25\ \gamma_t\gamma_y\gamma_t\gamma_y + 20\ \gamma_t\gamma_y\gamma_x\gamma_y + 20\ \gamma_x\gamma_y\gamma_t\gamma_y + 16\ \gamma_x\gamma_y\gamma_x\gamma_y)$$

$$= \frac{1}{18}\ (25 + 20\ \gamma_t\gamma_x - 20\ \gamma_t\gamma_x - 16)$$

$$= \frac{1}{2}$$

$$\Rightarrow\ \mathbf{D}^{-1} = \begin{bmatrix} \frac{5}{9} - \frac{4}{9}\gamma_t\gamma_x & -\frac{4}{9} + \frac{5}{9}\gamma_t\gamma_x & 0 \\ -\frac{15}{18} + \frac{12}{18}\gamma_t\gamma_x - \frac{16}{18}\gamma_t\gamma_y - \frac{20}{18}\gamma_x\gamma_y & \frac{12}{18} - \frac{15}{18}\gamma_t\gamma_x + \frac{20}{18}\gamma_t\gamma_y + \frac{25}{18}\gamma_x\gamma_y & \frac{1}{2} \end{bmatrix}$$

Check of Dirac algebra matrix inverse: $\mathbf{D}^{-1}\ \mathbf{D} = \mathbf{I}$				
			5	0
			4	0
			3	2
$\frac{5}{9} - \frac{4}{9}\ \gamma_t\gamma_x$	$-\frac{4}{9} + \frac{5}{9}\ \gamma_t\gamma_x$	0	1	0
$-\frac{15}{18} + \frac{12}{18}\ \gamma_t\gamma_x - \frac{16}{18}\ \gamma_t\gamma_y - \frac{20}{18}\ \gamma_x\gamma_y$	$\frac{12}{18} - \frac{15}{18}\ \gamma_t\gamma_x + \frac{20}{18}\ \gamma_t\gamma_y + \frac{25}{18}\ \gamma_x\gamma_y$	$\frac{1}{2}$	0	1

Quantities of final products, which will be produced:			
			125
			100
			145
$\frac{5}{9} - \frac{4}{9}\ \gamma_t\gamma_x$	$-\frac{4}{9} + \frac{5}{9}\ \gamma_t\gamma_x$	0	25
$-\frac{15}{18} + \frac{12}{18}\ \gamma_t\gamma_x - \frac{16}{18}\ \gamma_t\gamma_y - \frac{20}{18}\ \gamma_x\gamma_y$	$\frac{12}{18} - \frac{15}{18}\ \gamma_t\gamma_x + \frac{20}{18}\ \gamma_t\gamma_y + \frac{25}{18}\ \gamma_x\gamma_y$	$\frac{1}{2}$	35

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

2 b) $5\,x + 6\,y = 380$
$4\,x + 7\,y = 370$
$3\,x + 8\,y = 360$

$$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 3 & 8 \end{bmatrix} \qquad \mathbf{a} = 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y \quad \mathbf{b} = 6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y \quad \mathbf{r_1} = \gamma_t \quad \mathbf{r_2} = \gamma_x \quad \mathbf{r_3} = \gamma_y$$

$$\Rightarrow \quad \mathbf{D}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{D}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$$\mathbf{a} \wedge \mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y)$$
$$= (35 - 24)\,\gamma_t\gamma_x + (40 - 18)\,\gamma_t\gamma_y + (32 - 21)\,\gamma_x\gamma_y$$
$$= 11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y$$

$$(\mathbf{a} \wedge \mathbf{b})^2 = (11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y)^2$$
$$= (11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y)\,(11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y)$$
$$= 121\,\gamma_t\gamma_x\gamma_t\gamma_x + 242\,\gamma_t\gamma_x\gamma_t\gamma_y + 121\,\gamma_t\gamma_x\gamma_x\gamma_y + 242\,\gamma_t\gamma_y\gamma_t\gamma_x + 484\,\gamma_t\gamma_y\gamma_t\gamma_y$$
$$+ 242\,\gamma_t\gamma_y\gamma_x\gamma_y + 121\,\gamma_x\gamma_y\gamma_t\gamma_x + 242\,\gamma_x\gamma_y\gamma_t\gamma_y + 121\,\gamma_x\gamma_y\gamma_x\gamma_y$$
$$= 121 - 242\,\gamma_x\gamma_y - 121\,\gamma_t\gamma_y + 242\,\gamma_x\gamma_y + 484 + 242\,\gamma_t\gamma_x + 121\,\gamma_t\gamma_y - 242\,\gamma_t\gamma_x - 121$$
$$= 121 + 484 - 121$$
$$= 484$$

$$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = \frac{1}{484}\,(11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y)$$
$$= \frac{1}{44}\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

Outer products of resulting vectors and coefficient vectors:

$$\mathbf{r_1} \wedge \mathbf{b} = \gamma_t \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = 7\,\gamma_t\gamma_x + 8\,\gamma_t\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_1} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_t = -\,4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y$$

$$\mathbf{r_2} \wedge \mathbf{b} = \gamma_x \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = -\,6\,\gamma_t\gamma_x + 8\,\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_2} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_x = 5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y$$

$$\mathbf{r_3} \wedge \mathbf{b} = \gamma_y \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = -\,6\,\gamma_t\gamma_y - 7\,\gamma_x\gamma_y$$

$$\mathbf{a} \wedge \mathbf{r_3} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_y = 5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y$$

Elements of the Dirac algebra generalized matrix inverse $\mathbf{D}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is pre-multiplied from the left:

$$x_1 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_1} \wedge \mathbf{b})$$

$$= \frac{1}{44} (\gamma_t\gamma_x + 2\, \gamma_t\gamma_y + \gamma_x\gamma_y)\, (7\, \gamma_t\gamma_x + 8\, \gamma_t\gamma_y)$$

$$= \frac{1}{44} (7\, \gamma_t\gamma_x\gamma_t\gamma_x + 8\, \gamma_t\gamma_x\gamma_t\gamma_y + 14\, \gamma_t\gamma_y\gamma_t\gamma_x + 16\, \gamma_t\gamma_y\gamma_t\gamma_y + 7\, \gamma_x\gamma_y\gamma_t\gamma_x + 8\, \gamma_x\gamma_y\gamma_t\gamma_y)$$

$$= \frac{1}{44} (7 - 8\, \gamma_x\gamma_y + 14\, \gamma_x\gamma_y + 16 + 7\, \gamma_t\gamma_y - 8\, \gamma_t\gamma_x)$$

$$= \frac{1}{44} (23 - 8\, \gamma_t\gamma_x + 7\, \gamma_t\gamma_y + 6\, \gamma_x\gamma_y)$$

$$x_2 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_2} \wedge \mathbf{b})$$

$$= \frac{1}{44} (\gamma_t\gamma_x + 2\, \gamma_t\gamma_y + \gamma_x\gamma_y)\, (-\, 6\, \gamma_t\gamma_x + 8\, \gamma_x\gamma_y)$$

$$= \frac{1}{44} (-\, 6\, \gamma_t\gamma_x\gamma_t\gamma_x + 8\, \gamma_t\gamma_x\gamma_x\gamma_y - 12\, \gamma_t\gamma_y\gamma_t\gamma_x + 16\, \gamma_t\gamma_y\gamma_x\gamma_y - 6\, \gamma_x\gamma_y\gamma_t\gamma_x + 8\, \gamma_x\gamma_y\gamma_x\gamma_y)$$

$$= \frac{1}{44} (-\, 6 - 8\, \gamma_t\gamma_y - 12\, \gamma_x\gamma_y + 16\, \gamma_t\gamma_x - 6\, \gamma_t\gamma_y - 8)$$

$$= \frac{1}{44} (-\, 14 + 16\, \gamma_t\gamma_x - 14\, \gamma_t\gamma_y - 12\, \gamma_x\gamma_y)$$

$$x_3 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{r_3} \wedge \mathbf{b})$$

$$= \frac{1}{44} (\gamma_t\gamma_x + 2\, \gamma_t\gamma_y + \gamma_x\gamma_y)\, (-\, 6\, \gamma_t\gamma_y - 7\, \gamma_x\gamma_y)$$

$$= \frac{1}{44} (-\, 6\, \gamma_t\gamma_x\gamma_t\gamma_y - 7\, \gamma_t\gamma_x\gamma_x\gamma_y - 12\, \gamma_t\gamma_y\gamma_t\gamma_y - 14\, \gamma_t\gamma_y\gamma_x\gamma_y - 6\, \gamma_x\gamma_y\gamma_t\gamma_y - 7\, \gamma_x\gamma_y\gamma_x\gamma_y)$$

$$= \frac{1}{44} (6\, \gamma_x\gamma_y + 7\, \gamma_t\gamma_y - 12 - 14\, \gamma_t\gamma_x + 6\, \gamma_t\gamma_x + 7)$$

$$= \frac{1}{44} (-\, 5 - 8\, \gamma_t\gamma_x + 7\, \gamma_t\gamma_y + 6\, \gamma_x\gamma_y)$$

Intermediate check of first row of the Dirac algebra matrix inverse:

			5	6
			4	7
			3	8
$23 - 8\, \gamma_t\gamma_x + 7\, \gamma_t\gamma_y + 6\, \gamma_x\gamma_y$	$-\, 14 + 16\, \gamma_t\gamma_x - 14\, \gamma_t\gamma_y - 12\, \gamma_x\gamma_y$	$-\, 5 - 8\, \gamma_t\gamma_x + 7\, \gamma_t\gamma_y + 6\, \gamma_x\gamma_y$	44	0
y_1	y_2	y_3	?	?

$y_1 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_1})$

$= \frac{1}{44} (\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)\,(-4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y)$

$= \frac{1}{44} (-4\,\gamma_t\gamma_x\gamma_t\gamma_x - 3\,\gamma_t\gamma_x\gamma_t\gamma_y - 8\,\gamma_t\gamma_y\gamma_t\gamma_x - 6\,\gamma_t\gamma_y\gamma_t\gamma_y - 4\,\gamma_x\gamma_y\gamma_t\gamma_x - 3\,\gamma_x\gamma_y\gamma_t\gamma_y)$

$= \frac{1}{44} (-4 + 3\,\gamma_x\gamma_y - 8\,\gamma_x\gamma_y - 6 - 4\,\gamma_t\gamma_y + 3\,\gamma_t\gamma_x)$

$= \frac{1}{44} (-10 + 3\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_y - 5\,\gamma_x\gamma_y)$

$y_2 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_2})$

$= \frac{1}{44} (\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)\,(5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y)$

$= \frac{1}{44} (5\,\gamma_t\gamma_x\gamma_t\gamma_x - 3\,\gamma_t\gamma_x\gamma_x\gamma_y + 10\,\gamma_t\gamma_y\gamma_t\gamma_x - 6\,\gamma_t\gamma_y\gamma_x\gamma_y + 5\,\gamma_x\gamma_y\gamma_t\gamma_x - 3\,\gamma_x\gamma_y\gamma_x\gamma_y)$

$= \frac{1}{44} (5 + 3\,\gamma_t\gamma_y + 10\,\gamma_x\gamma_y - 6\,\gamma_t\gamma_x + 5\,\gamma_t\gamma_y + 3)$

$= \frac{1}{44} (8 - 6\,\gamma_t\gamma_x + 8\,\gamma_t\gamma_y + 10\,\gamma_x\gamma_y)$

$y_3 = (\mathbf{a} \wedge \mathbf{b})^{-1} (\mathbf{a} \wedge \mathbf{r_3})$

$= \frac{1}{44} (\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)$

$= \frac{1}{44} (5\,\gamma_t\gamma_x\gamma_t\gamma_y + 4\,\gamma_t\gamma_x\gamma_x\gamma_y + 10\,\gamma_t\gamma_y\gamma_t\gamma_y + 8\,\gamma_t\gamma_y\gamma_x\gamma_y + 5\,\gamma_x\gamma_y\gamma_t\gamma_y + 4\,\gamma_x\gamma_y\gamma_x\gamma_y)$

$= \frac{1}{44} (-5\,\gamma_x\gamma_y - 4\,\gamma_t\gamma_y + 10 + 8\,\gamma_t\gamma_x - 5\,\gamma_t\gamma_x - 4)$

$= \frac{1}{44} (6 + 3\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_y - 5\,\gamma_x\gamma_y)$

$$\Rightarrow\ \mathbf{D}^{-1} = \frac{1}{44}\begin{bmatrix} 23-8\gamma_t\gamma_x+7\gamma_t\gamma_y+6\gamma_x\gamma_y & -14+16\gamma_t\gamma_x-14\gamma_t\gamma_y-12\gamma_x\gamma_y & -5-8\gamma_t\gamma_x+7\gamma_t\gamma_y+6\gamma_x\gamma_y \\ -10+3\gamma_t\gamma_x-4\gamma_t\gamma_y-5\gamma_x\gamma_y & 8-6\gamma_t\gamma_x+8\gamma_t\gamma_y+10\gamma_x\gamma_y & 6+3\gamma_t\gamma_x-4\gamma_t\gamma_y-5\gamma_x\gamma_y \end{bmatrix}$$

Check of Dirac algebra matrix inverse: $\mathbf{D}^{-1}\,\mathbf{D} = \mathbf{I}$			5	6
			4	7
			3	8
$23 - 8\,\gamma_t\gamma_x + 7\,\gamma_t\gamma_y + 6\,\gamma_x\gamma_y$	$-14 + 16\,\gamma_t\gamma_x - 14\,\gamma_t\gamma_y - 12\,\gamma_x\gamma_y$	$-5 - 8\,\gamma_t\gamma_x + 7\,\gamma_t\gamma_y + 6\,\gamma_x\gamma_y$	44	0
$-10 + 3\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_y - 5\,\gamma_x\gamma_y$	$8 - 6\,\gamma_t\gamma_x + 8\,\gamma_t\gamma_y + 10\,\gamma_x\gamma_y$	$6 + 3\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_y - 5\,\gamma_x\gamma_y$	0	44

Quantities of final products, which will be produced:

			380
			370
			360
$23 - 8\,\gamma_t\gamma_x + 7\,\gamma_t\gamma_y + 6\,\gamma_x\gamma_y$	$-14 + 16\,\gamma_t\gamma_x - 14\,\gamma_t\gamma_y - 12\,\gamma_x\gamma_y$	$-5 - 8\,\gamma_t\gamma_x + 7\,\gamma_t\gamma_y + 6\,\gamma_x\gamma_y$	1760
$-10 + 3\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_y - 5\,\gamma_x\gamma_y$	$8 - 6\,\gamma_t\gamma_x + 8\,\gamma_t\gamma_y + 10\,\gamma_x\gamma_y$	$6 + 3\,\gamma_t\gamma_x - 4\,\gamma_t\gamma_y - 5\,\gamma_x\gamma_y$	1320

Completing the result: $x = \dfrac{1760}{44} = 40 \qquad y = \dfrac{1320}{44} = 30$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Problem 6:

2 a)
$$\begin{aligned} 5\,x + 0\,y &= 125 \\ 4\,x + 0\,y &= 100 \\ 3\,x + 2\,y &= 145 \end{aligned} \quad \Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$$

$\mathbf{a} = 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y$

$\mathbf{b} = 2\,\gamma_y$

$\mathbf{r_1} = \gamma_t \qquad \mathbf{r_2} = \gamma_x \qquad \mathbf{r_3} = \gamma_y$

$$\Rightarrow \quad \underline{\mathbf{D}}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$\mathbf{a} \wedge \mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge (2\,\gamma_y) = 10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y$

$(\mathbf{a} \wedge \mathbf{b})^2 = (10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y)^2 = 36$

$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = \dfrac{1}{36}\,(10\,\gamma_t\gamma_y + 8\,\gamma_x\gamma_y) = \dfrac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)$

Outer products of resulting vectors and coefficient vectors:

$\mathbf{r_1} \wedge \mathbf{b} = \gamma_t \wedge (2\,\gamma_y) = 2\,\gamma_t\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_t = -\,4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} = \gamma_x \wedge (2\,\gamma_y) = 2\,\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_x = 5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} = \gamma_y \wedge (2\,\gamma_y) = 0$

$\mathbf{a} \wedge \mathbf{r_3} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_y = 5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y$

Elements of the Dirac algebra generalized matrix inverse $\underline{\mathbf{D}}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is post-multiplied from the right:

$$x_1 = (\mathbf{r_1} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{18}\,(2\,\gamma_t\gamma_y)\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y) = \frac{5}{9} + \frac{4}{9}\,\gamma_t\gamma_x$$

$$x_2 = (\mathbf{r_2} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{18}\,(2\,\gamma_x\gamma_y)\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y) = -\frac{4}{9} - \frac{5}{9}\,\gamma_t\gamma_x$$

$$x_3 = (\mathbf{r_3} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{18}\cdot 0 \cdot (5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y) = 0$$

$$y_1 = (\mathbf{a} \wedge \mathbf{r_1})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{18}\,(-4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y)\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)$$

$$= -\frac{15}{18} - \frac{12}{18}\,\gamma_t\gamma_x + \frac{16}{18}\,\gamma_t\gamma_y + \frac{20}{18}\,\gamma_x\gamma_y$$

$$y_2 = (\mathbf{a} \wedge \mathbf{r_2})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{18}\,(5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y)\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)$$

$$= \frac{12}{18} + \frac{15}{18}\,\gamma_t\gamma_x - \frac{20}{18}\,\gamma_t\gamma_y - \frac{25}{18}\,\gamma_x\gamma_y$$

$$y_3 = (\mathbf{a} \wedge \mathbf{r_3})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{18}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)$$

$$= \frac{1}{2}$$

$$\Rightarrow\ \underline{\mathbf{D}}^{-1} = \begin{bmatrix} \frac{5}{9}+\frac{4}{9}\gamma_t\gamma_x & -\frac{4}{9}-\frac{5}{9}\gamma_t\gamma_x & 0 \\ -\frac{15}{18}-\frac{12}{18}\gamma_t\gamma_x+\frac{16}{18}\gamma_t\gamma_y+\frac{20}{18}\gamma_x\gamma_y & \frac{12}{18}+\frac{15}{18}\gamma_t\gamma_x-\frac{20}{18}\gamma_t\gamma_y-\frac{25}{18}\gamma_x\gamma_y & \frac{1}{2} \end{bmatrix}$$

Check of Dirac algebra matrix inverse: $\underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$

$$\begin{array}{ccc|cc} & & & 5 & 0 \\ & & & 4 & 0 \\ & & & 3 & 2 \\ \hline \frac{5}{9} + \frac{4}{9}\,\gamma_t\gamma_x & -\frac{4}{9} - \frac{5}{9}\,\gamma_t\gamma_x & 0 & 1 & 0 \\ -\frac{15}{18} - \frac{12}{18}\,\gamma_t\gamma_x + \frac{16}{18}\,\gamma_t\gamma_y + \frac{20}{18}\,\gamma_x\gamma_y & \frac{12}{18} + \frac{15}{18}\,\gamma_t\gamma_x - \frac{20}{18}\,\gamma_t\gamma_y - \frac{25}{18}\,\gamma_x\gamma_y & \frac{1}{2} & 0 & 1 \end{array}$$

Quantities of final products, which will be produced:

			125 100 145
$\frac{5}{9} + \frac{4}{9}\,\gamma_t\gamma_x$	$-\frac{4}{9} - \frac{5}{9}\,\gamma_t\gamma_x$	0	25
$-\frac{15}{18} - \frac{12}{18}\,\gamma_t\gamma_x + \frac{16}{18}\,\gamma_t\gamma_y + \frac{20}{18}\,\gamma_x\gamma_y$	$\frac{12}{18} + \frac{15}{18}\,\gamma_t\gamma_x - \frac{20}{18}\,\gamma_t\gamma_y$ ä $- \frac{25}{18}\,\gamma_x\gamma_y$	$\frac{1}{2}$	35

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

2 b) $5\,x + 6\,y = 380$

$4\,x + 7\,y = 370$

$3\,x + 8\,y = 360$

$$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 3 & 8 \end{bmatrix}$$

$\mathbf{a} = 5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y$

$\mathbf{b} = 6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y$

$\mathbf{r_1} = \gamma_t \qquad \mathbf{r_2} = \gamma_x \qquad \mathbf{r_3} = \gamma_y$

$$\Rightarrow \quad \underline{\mathbf{D}}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \Leftrightarrow \quad \underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$$

Outer product and inverse of the two coefficient vectors:

$\mathbf{a} \wedge \mathbf{b} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = 11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y$

$(\mathbf{a} \wedge \mathbf{b})^2 = (11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y)^2 = 484$

$(\mathbf{a} \wedge \mathbf{b})^{-1} = (\mathbf{a} \wedge \mathbf{b})^{-2}\,(\mathbf{a} \wedge \mathbf{b}) = \frac{1}{484}\,(11\,\gamma_t\gamma_x + 22\,\gamma_t\gamma_y + 11\,\gamma_x\gamma_y) = \frac{1}{44}\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$

Outer products of resulting vectors and coefficient vectors:

$\mathbf{r_1} \wedge \mathbf{b} = \gamma_t \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = 7\,\gamma_t\gamma_x + 8\,\gamma_t\gamma_y$

$\mathbf{a} \wedge \mathbf{r_1} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_t = -\,4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y$

$\mathbf{r_2} \wedge \mathbf{b} = \gamma_x \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = -\,6\,\gamma_t\gamma_x + 8\,\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_2} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_x = 5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y$

$\mathbf{r_3} \wedge \mathbf{b} = \gamma_y \wedge (6\,\gamma_t + 7\,\gamma_x + 8\,\gamma_y) = -\,6\,\gamma_t\gamma_y - 7\,\gamma_x\gamma_y$

$\mathbf{a} \wedge \mathbf{r_3} = (5\,\gamma_t + 4\,\gamma_x + 3\,\gamma_y) \wedge \gamma_y = 5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y$

Elements of the Dirac algebra generalized matrix inverse $\underline{\mathbf{D}}^{-1}$, if the inverse of the outer product of the coefficient vectors $(\mathbf{a} \wedge \mathbf{b})^{-1}$ is post-multiplied from the right:

$$x_1 = (\mathbf{r_1} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{44}\,(7\,\gamma_t\gamma_x + 8\,\gamma_t\gamma_y)\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

$$= \frac{1}{44}\,(23 + 8\,\gamma_t\gamma_x - 7\,\gamma_t\gamma_y - 6\,\gamma_x\gamma_y)$$

$$x_2 = (\mathbf{r_2} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{44}\,(-6\,\gamma_t\gamma_x + 8\,\gamma_x\gamma_y)\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

$$= \frac{1}{44}\,(-14 - 16\,\gamma_t\gamma_x + 14\,\gamma_t\gamma_y + 12\,\gamma_x\gamma_y)$$

$$x_3 = (\mathbf{r_3} \wedge \mathbf{b})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{44}\,(-6\,\gamma_t\gamma_y - 7\,\gamma_x\gamma_y)\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

$$= \frac{1}{44}\,(-5 + 8\,\gamma_t\gamma_x - 7\,\gamma_t\gamma_y - 6\,\gamma_x\gamma_y)$$

$$y_1 = (\mathbf{a} \wedge \mathbf{r_1})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{44}\,(-4\,\gamma_t\gamma_x - 3\,\gamma_t\gamma_y)\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

$$= \frac{1}{44}\,(-10 - 3\,\gamma_t\gamma_x + 4\,\gamma_t\gamma_y + 5\,\gamma_x\gamma_y)$$

$$y_2 = (\mathbf{a} \wedge \mathbf{r_2})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{44}\,(5\,\gamma_t\gamma_x - 3\,\gamma_x\gamma_y)\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

$$= \frac{1}{44}\,(8 + 6\,\gamma_t\gamma_x - 8\,\gamma_t\gamma_y - 10\,\gamma_x\gamma_y)$$

$$y_3 = (\mathbf{a} \wedge \mathbf{r_3})\,(\mathbf{a} \wedge \mathbf{b})^{-1} = \frac{1}{44}\,(5\,\gamma_t\gamma_y + 4\,\gamma_x\gamma_y)\,(\gamma_t\gamma_x + 2\,\gamma_t\gamma_y + \gamma_x\gamma_y)$$

$$= \frac{1}{44}\,(6 - 3\,\gamma_t\gamma_x + 4\,\gamma_t\gamma_y + 5\,\gamma_x\gamma_y)$$

$$\Rightarrow\ \underline{\mathbf{D}}^{-1} = \frac{1}{44}\begin{bmatrix} 23+8\gamma_t\gamma_x-7\gamma_t\gamma_y-6\gamma_x\gamma_y & -14-16\gamma_t\gamma_x+14\gamma_t\gamma_y+12\gamma_x\gamma_y & -5+8\gamma_t\gamma_x-7\gamma_t\gamma_y-6\gamma_x\gamma_y \\ -10-3\gamma_t\gamma_x+4\gamma_t\gamma_y+5\gamma_x\gamma_y & 8+6\gamma_t\gamma_x-8\gamma_t\gamma_y-10\gamma_x\gamma_y & 6-3\gamma_t\gamma_x+4\gamma_t\gamma_y+5\gamma_x\gamma_y \end{bmatrix}$$

Check of Dirac algebra matrix inverse: $\underline{\mathbf{D}}^{-1}\,\mathbf{D} = \mathbf{I}$

			5	6
			4	7
			3	8
$23 + 8\,\gamma_t\gamma_x - 7\,\gamma_t\gamma_y - 6\,\gamma_x\gamma_y$	$-14 - 16\,\gamma_t\gamma_x + 14\,\gamma_t\gamma_y + 12\,\gamma_x\gamma_y$	$-5 + 8\,\gamma_t\gamma_x - 7\,\gamma_t\gamma_y - 6\,\gamma_x\gamma_y$	44	0
$-10 - 3\,\gamma_t\gamma_x + 4\,\gamma_t\gamma_y + 5\,\gamma_x\gamma_y$	$8 + 6\,\gamma_t\gamma_x - 8\,\gamma_t\gamma_y - 10\,\gamma_x\gamma_y$	$6 - 3\,\gamma_t\gamma_x + 4\,\gamma_t\gamma_y + 5\,\gamma_x\gamma_y$	0	44

Quantities of final products, which will be produced:

$$\begin{array}{ccc|c} & & & 380 \\ & & & 370 \\ & & & 360 \\ \hline 23 + 8\,\gamma_t\gamma_x - 7\,\gamma_t\gamma_y - 6\,\gamma_x\gamma_y & -14 - 16\,\gamma_t\gamma_x + 14\,\gamma_t\gamma_y + 12\,\gamma_x\gamma_y & -5 + 8\,\gamma_t\gamma_x - 7\,\gamma_t\gamma_y - 6\,\gamma_x\gamma_y & 1760 \\ -10 - 3\,\gamma_t\gamma_x + 4\,\gamma_t\gamma_y + 5\,\gamma_x\gamma_y & 8 + 6\,\gamma_t\gamma_x - 8\,\gamma_t\gamma_y - 10\,\gamma_x\gamma_y & 6 - 3\,\gamma_t\gamma_x + 4\,\gamma_t\gamma_y + 5\,\gamma_x\gamma_y & 1320 \end{array}$$

Completing the result: $x = \dfrac{1760}{44} = 40 \qquad y = \dfrac{1320}{44} = 30$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Problem 7:

The scalar terms of the elements of the Pauli algebra generalized matrix inverses $\mathbf{D}^{-1}$ of problem 3 and $\underline{\mathbf{D}}^{-1}$ of problem 4 are the elements of the Moore-Penrose matrix inverses:

$$\mathbf{D}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{D}^{+} = \begin{bmatrix} \langle x_1\rangle_0 & \langle x_2\rangle_0 & \langle x_3\rangle_0 \\ \langle y_1\rangle_0 & \langle y_2\rangle_0 & \langle y_3\rangle_0 \end{bmatrix}$$

or

$$\underline{\mathbf{D}}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{D}^{+} = \begin{bmatrix} \langle x_1\rangle_0 & \langle x_2\rangle_0 & \langle x_3\rangle_0 \\ \langle y_1\rangle_0 & \langle y_2\rangle_0 & \langle y_3\rangle_0 \end{bmatrix}$$

As all the bivector terms have opposite signs, they will cancel when added:

$$\mathbf{D}^{+} = \frac{1}{2}\,(\mathbf{D}^{-1} + \underline{\mathbf{D}}^{-1})$$

2 a)
$$\begin{aligned} 5\,x + 0\,y &= 125 \\ 4\,x + 0\,y &= 100 \\ 3\,x + 2\,y &= 145 \end{aligned} \qquad \Rightarrow \qquad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{5}{41} - \frac{4}{41}\sigma_x\sigma_y & \frac{4}{41} + \frac{5}{41}\sigma_x\sigma_y & 0 \\ -\frac{15}{82} + \frac{6}{41}\sigma_x\sigma_y + \frac{10}{41}\sigma_y\sigma_z + \frac{8}{41}\sigma_z\sigma_x & -\frac{6}{41} - \frac{15}{82}\sigma_x\sigma_y - \frac{25}{82}\sigma_y\sigma_z - \frac{10}{41}\sigma_z\sigma_x & \frac{1}{2} \end{bmatrix}$$

$$\underline{\mathbf{D}}^{-1} = \begin{bmatrix} \frac{5}{41} + \frac{4}{41}\sigma_x\sigma_y & \frac{4}{41} - \frac{5}{41}\sigma_x\sigma_y & 0 \\ -\frac{15}{82} - \frac{6}{41}\sigma_x\sigma_y - \frac{10}{41}\sigma_y\sigma_z - \frac{8}{41}\sigma_z\sigma_x & -\frac{6}{41} + \frac{15}{82}\sigma_x\sigma_y + \frac{25}{82}\sigma_y\sigma_z + \frac{10}{41}\sigma_z\sigma_x & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{D}^+ = \frac{1}{2}\left(\mathbf{D}^{-1} + \underline{\mathbf{D}}^{-1}\right) = \begin{bmatrix} \frac{5}{41} & \frac{4}{41} & 0 \\ -\frac{15}{82} & -\frac{6}{41} & \frac{1}{2} \end{bmatrix} = \frac{1}{82}\begin{bmatrix} 10 & 8 & 0 \\ -15 & -12 & 41 \end{bmatrix}$$

Moore-Penrose conditions:

I. $\mathbf{D}\,\mathbf{D}^+\,\mathbf{D} = \mathbf{D}$

II. $\mathbf{D}^+\,\mathbf{D}\,\mathbf{D}^+ = \mathbf{D}^+$

III. $\mathbf{D}\,\mathbf{D}^+ = \left(\mathbf{D}\,\mathbf{D}^+\right)^T$

IV. $\mathbf{D}^+\,\mathbf{D} = \left(\mathbf{D}^+\,\mathbf{D}\right)^T$

Check of the first and third Moore-Penrose condition: $\mathbf{D}\,\mathbf{D}^+\,\mathbf{D} = \left(\mathbf{D}\,\mathbf{D}^+\right)^T \mathbf{D} = \mathbf{D}$

$$\begin{array}{cc|ccc|cc}
 & & & & & 5 & 0 \\
 & & \frac{5}{41} & \frac{4}{41} & 0 & 4 & 0 \\
 & & -\frac{15}{82} & -\frac{6}{41} & \frac{1}{2} & 3 & 2 \\ \hline
5 & 0 & \frac{25}{41} & \frac{20}{41} & 0 & 5 & 0 \\
4 & 0 & \frac{20}{41} & \frac{16}{41} & 0 & 4 & 0 \\
3 & 2 & 0 & 0 & 1 & 3 & 2
\end{array}
\quad \mathbf{D} = \mathbf{D}\,\mathbf{D}^+\,\mathbf{D}$$

$$\mathbf{D}\,\mathbf{D}^+ = \left(\mathbf{D}\,\mathbf{D}^+\right)^T$$

Check of the second and fourth Moore-Penrose condition: $\mathbf{D}^+\,\mathbf{D}\,\mathbf{D}^+ = \left(\mathbf{D}^+\,\mathbf{D}\right)^T \mathbf{D}^+ = \mathbf{D}^+$

$$\begin{array}{ccc|cc|ccc}
 & & & 5 & 0 & & & \\
 & & & 4 & 0 & \frac{5}{41} & \frac{4}{41} & 0 \\
 & & & 3 & 2 & -\frac{15}{82} & -\frac{6}{41} & \frac{1}{2} \\ \hline
\frac{5}{41} & \frac{4}{41} & 0 & 1 & 0 & \frac{5}{41} & \frac{4}{41} & 0 \\
-\frac{15}{82} & -\frac{6}{41} & \frac{1}{2} & 0 & 1 & -\frac{15}{82} & -\frac{6}{41} & \frac{1}{2}
\end{array}
\quad \mathbf{D}^+ = \mathbf{D}^+\,\mathbf{D}\,\mathbf{D}^+$$

$$\mathbf{D}^+\,\mathbf{D} = \left(\mathbf{D}^+\,\mathbf{D}\right)^T$$

Quantities of final products, which will be produced:

			125
			100
			145
$\frac{5}{41}$	$\frac{4}{41}$	0	25
$-\frac{15}{82}$	$-\frac{6}{41}$	$\frac{1}{2}$	35

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

2 b) $5\,x + 6\,y = 380$
$4\,x + 7\,y = 370$
$3\,x + 8\,y = 360$
$\Rightarrow \quad \mathbf{D} = \begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 3 & 8 \end{bmatrix}$

$$\mathbf{D}^{-1} = \frac{1}{66}\begin{bmatrix} 23 - 8\sigma_x\sigma_y - 6\sigma_y\sigma_z - 7\sigma_z\sigma_x & 2 + 16\sigma_x\sigma_y + 12\sigma_y\sigma_z + 14\sigma_z\sigma_x & -19 - 8\sigma_x\sigma_y - 6\sigma_y\sigma_z - 7\sigma_z\sigma_x \\ -10 + 3\sigma_x\sigma_y + 5\sigma_y\sigma_z + 4\sigma_z\sigma_x & 2 - 6\sigma_x\sigma_y - 10\sigma_y\sigma_z - 8\sigma_z\sigma_x & 14 + 3\sigma_x\sigma_y + 5\sigma_y\sigma_z + 4\sigma_z\sigma_x \end{bmatrix}$$

$$\underline{\mathbf{D}}^{-1} = \frac{1}{66}\begin{bmatrix} 23 + 8\sigma_x\sigma_y + 6\sigma_y\sigma_z + 7\sigma_z\sigma_x & 2 - 16\sigma_x\sigma_y - 12\sigma_y\sigma_z - 14\sigma_z\sigma_x & -19 + 8\sigma_x\sigma_y + 6\sigma_y\sigma_z + 7\sigma_z\sigma_x \\ -10 - 3\sigma_x\sigma_y - 5\sigma_y\sigma_z - 4\sigma_z\sigma_x & 2 + 6\sigma_x\sigma_y + 10\sigma_y\sigma_z + 8\sigma_z\sigma_x & 14 - 3\sigma_x\sigma_y - 5\sigma_y\sigma_z - 4\sigma_z\sigma_x \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{D}^{+} = \frac{1}{2}\left(\mathbf{D}^{-1} + \underline{\mathbf{D}}^{-1}\right) = \begin{bmatrix} \frac{23}{66} & \frac{1}{33} & -\frac{19}{66} \\ -\frac{5}{33} & \frac{1}{33} & \frac{7}{33} \end{bmatrix} = \frac{1}{66}\begin{bmatrix} 23 & 2 & -19 \\ -10 & 2 & 14 \end{bmatrix}$$

Moore-Penrose conditions:

I. $\mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D} = \mathbf{D}$

II. $\mathbf{D}^{+}\,\mathbf{D}\,\mathbf{D}^{+} = \mathbf{D}^{+}$

III. $\mathbf{D}\,\mathbf{D}^{+} = \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}$

IV. $\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}^{+}\,\mathbf{D}\right)^{T}$

Check of the first and third Moore-Penrose condition: $\mathbf{D\,D^+\,D} = \left(\mathbf{D\,D^+}\right)^{\mathrm{T}}\mathbf{D} = \mathbf{D}$

$$
\begin{array}{cc|ccc|cc}
 & & & & & 5 & 6\\
 & & \frac{23}{66} & \frac{1}{33} & -\frac{19}{66} & 4 & 7\\
 & & -\frac{5}{33} & \frac{1}{33} & \frac{7}{33} & 3 & 8\\
\hline
5 & 6 & \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} & 5 & 6\\
4 & 7 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 4 & 7\\
3 & 8 & -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} & 3 & 8
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\}\ \mathbf{D} = \mathbf{D\,D^+\,D}
$$

$$\mathbf{D\,D^+} = \left(\mathbf{D\,D^+}\right)^{\mathrm{T}}$$

Check of the second and fourth Moore-Penrose condition: $\mathbf{D^+\,D\,D^+} = \left(\mathbf{D^+\,D}\right)^{\mathrm{T}}\mathbf{D^+} = \mathbf{D^+}$

$$
\begin{array}{ccc|cc|ccc}
 & & & 5 & 6 & & & \\
 & & & 4 & 7 & \frac{23}{66} & \frac{1}{33} & -\frac{19}{66}\\
 & & & 3 & 8 & -\frac{5}{33} & \frac{1}{33} & \frac{7}{33}\\
\hline
\frac{23}{66} & \frac{1}{33} & -\frac{19}{66} & 1 & 0 & \frac{23}{66} & \frac{1}{33} & -\frac{19}{66}\\
-\frac{5}{33} & \frac{1}{33} & \frac{7}{33} & 0 & 1 & -\frac{5}{33} & \frac{1}{33} & \frac{7}{33}
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}\ \mathbf{D^+} = \mathbf{D^+\,D\,D^+}
$$

$$\mathbf{D^+\,D} = \left(\mathbf{D^+\,D}\right)^{\mathrm{T}}$$

Quantities of final products, which will be produced:

$$
\begin{array}{ccc|c}
 & & & 380 \\
 & & & 370 \\
 & & & 360 \\
\hline
\frac{23}{66} & \frac{1}{33} & -\frac{19}{66} & 40 \\
-\frac{5}{33} & \frac{1}{33} & \frac{7}{33} & 30
\end{array}
$$

$\Rightarrow$ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.

Problem 8*:*

The scalar terms of the elements of the Dirac algebra generalized matrix inverses $\mathbf{D}^{-1}$ of problem 5 and $\underline{\mathbf{D}}^{-1}$ of problem 6 are the elements of the spacetime matrix inverses:

$$\mathbf{D}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{D}^{+} = \begin{bmatrix} \langle x_1\rangle_0 & \langle x_2\rangle_0 & \langle x_3\rangle_0 \\ \langle y_1\rangle_0 & \langle y_2\rangle_0 & \langle y_3\rangle_0 \end{bmatrix}$$

or

$$\underline{\mathbf{D}}^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{D}^{+} = \begin{bmatrix} \langle x_1\rangle_0 & \langle x_2\rangle_0 & \langle x_3\rangle_0 \\ \langle y_1\rangle_0 & \langle y_2\rangle_0 & \langle y_3\rangle_0 \end{bmatrix}$$

As all the bivector terms have opposite signs, they will cancel when added:

$$\mathbf{D}^{+} = \frac{1}{2}\,(\mathbf{D}^{-1} + \underline{\mathbf{D}}^{-1})$$

2 a)

$$\begin{aligned} 5\,x + 0\,y &= 125 \\ 4\,x + 0\,y &= 100 \\ 3\,x + 2\,y &= 145 \end{aligned} \qquad \Rightarrow \qquad \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{5}{9} - \frac{4}{9}\gamma_t\gamma_x & -\frac{4}{9} + \frac{5}{9}\gamma_t\gamma_x & 0 \\ -\frac{15}{18} + \frac{12}{18}\gamma_t\gamma_x - \frac{16}{18}\gamma_t\gamma_y - \frac{20}{18}\gamma_x\gamma_y & \frac{12}{18} - \frac{15}{18}\gamma_t\gamma_x + \frac{20}{18}\gamma_t\gamma_y + \frac{25}{18}\gamma_x\gamma_y & \frac{1}{2} \end{bmatrix}$$

$$\underline{\mathbf{D}}^{-1} = \begin{bmatrix} \frac{5}{9} + \frac{4}{9}\gamma_t\gamma_x & -\frac{4}{9} - \frac{5}{9}\gamma_t\gamma_x & 0 \\ -\frac{15}{18} - \frac{12}{18}\gamma_t\gamma_x + \frac{16}{18}\gamma_t\gamma_y + \frac{20}{18}\gamma_x\gamma_y & \frac{12}{18} + \frac{15}{18}\gamma_t\gamma_x - \frac{20}{18}\gamma_t\gamma_y - \frac{25}{18}\gamma_x\gamma_y & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{D}^{+} = \frac{1}{2}\left(\mathbf{D}^{-1} + \underline{\mathbf{D}}^{-1}\right) = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & 0 \\ -\frac{15}{18} & \frac{12}{18} & \frac{1}{2} \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 10 & -8 & 0 \\ -15 & 12 & 41 \end{bmatrix}$$

Moore-Penrose conditions: I. $\mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D} = \mathbf{D}$ III. $\mathbf{D}\,\mathbf{D}^{+} = \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}$

II. $\mathbf{D}^{+}\,\mathbf{D}\,\mathbf{D}^{+} = \mathbf{D}^{+}$ IV. $\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}^{+}\,\mathbf{D}\right)^{T}$

Check of the first and third Moore-Penrose condition: $\mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}\mathbf{D} = \mathbf{D}$

$$\begin{array}{cc|ccc|cc}
 & & & & & 5 & 0 \\
 & & \frac{10}{18} & -\frac{8}{18} & 0 & 4 & 0 \\
 & & -\frac{15}{18} & \frac{12}{18} & \frac{1}{2} & 3 & 2 \\
\hline
5 & 0 & \frac{50}{18} & -\frac{40}{18} & 0 & 5 & 0 \\
4 & 0 & \frac{40}{18} & -\frac{32}{18} & 0 & 4 & 0 \\
3 & 2 & 0 & 0 & 1 & 3 & 2
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} \mathbf{D} = \mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D}$$

$$\underbrace{\begin{array}{ccc} \frac{50}{18} & -\frac{40}{18} & 0 \\ \frac{40}{18} & -\frac{32}{18} & 0 \\ 0 & 0 & 1 \end{array}}_{\mathbf{D}\,\mathbf{D}^{+} \neq \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}}$$

$\Rightarrow$ The first Moore-Penrose condition is true for this spacetime generalized matrix inverse.

$\Rightarrow$ The third Moore-Penrose condition is not true for this spacetime generalized matrix inverse.

Check of the second and fourth Moore-Penrose condition: $\mathbf{D}^{+}\,\mathbf{D}\,\mathbf{D}^{+} = \left(\mathbf{D}^{+}\,\mathbf{D}\right)^{T}\mathbf{D}^{+} = \mathbf{D}^{+}$

$$\begin{array}{ccc|cc|ccc}
 & & & 5 & 0 & & & \\
 & & & 4 & 0 & \frac{10}{18} & -\frac{8}{18} & 0 \\
 & & & 3 & 2 & -\frac{15}{18} & \frac{12}{18} & \frac{1}{2} \\
\hline
\frac{10}{18} & -\frac{8}{18} & 0 & 1 & 0 & \frac{10}{18} & -\frac{8}{18} & 0 \\
-\frac{15}{18} & \frac{12}{18} & \frac{1}{2} & 0 & 1 & -\frac{15}{18} & \frac{12}{18} & \frac{1}{2}
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} \mathbf{D}^{+} = \mathbf{D}^{+}\,\mathbf{D}\,\mathbf{D}^{+}$$

$$\underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}^{+}\,\mathbf{D}\right)^{T}}$$

$\Rightarrow$ The second and the fourth Moore-Penrose condition are true for this spacetime generalized matrix inverse.

Finding a new third (and fourth) condition:

As the first, second, and fourth Moore-Penrose conditions hold, we only have to modify the third condition. To do this we remember the discussion of the first lesson: Mathematics can be considered as a free invention of the human mind – something, which does not exist outside our brains, because it is not part of nature (Vince: “The universe does not need any of these mathematical ideas to run its machinery.” See: Mathematics for Computer Graphics. 5th ed., sec., 1.4, Springer 2017). Mathematics is constructed and thus it is invented by us.

So let us invent a spacetime matrix transposition with the following definition for 2×2 or 3×3 matrices:

If $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the spacetime transposed matrix will be: $\mathbf{A}^{\text{stT}} = \begin{bmatrix} a_{11} & -a_{21} \\ -a_{12} & a_{22} \end{bmatrix}$.

If $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the spacetime transposed matrix will be: $\mathbf{A}^{\text{stT}} = \begin{bmatrix} a_{11} & -a_{21} & -a_{31} \\ -a_{12} & a_{22} & a_{32} \\ -a_{13} & a_{23} & a_{33} \end{bmatrix}$.

The non-diagonal elements of the first row and first column will change their signs when spacetimely transposed.

The modified conditions can then be stated with this spacetime transposition:

I. $\mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D} = \mathbf{D}$

II. $\mathbf{D}^{+}\,\mathbf{D}\,\mathbf{D}^{+} = \mathbf{D}^{+}$

III. $\mathbf{D}\,\mathbf{D}^{+} = \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{\text{stT}}$

IV. $\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}^{+}\,\mathbf{D}\right)^{\text{stT}}$

Quantities of final products, which will be produced:

			125
			100
			145
$\frac{10}{18}$	$-\frac{8}{18}$	0	25
$-\frac{15}{18}$	$\frac{12}{18}$	$\frac{1}{2}$	35

$\Rightarrow$ If 125 units of the first raw material R_1, 100 units of the second raw material R_2, and 145 units of the third raw material R_3 are consumed in the production process, 25 units of the first final product P_1 and 35 units of the second final product P_2 will be produced.

2 b) $5\,x + 6\,y = 380$

$4\,x + 7\,y = 370 \qquad \Rightarrow \qquad \mathbf{D} = \begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 3 & 8 \end{bmatrix}$

$3\,x + 8\,y = 360$

$$\mathbf{D}^{-1} = \frac{1}{44}\begin{bmatrix} 23-8\gamma_t\gamma_x+7\gamma_t\gamma_y+6\gamma_x\gamma_y & -14+16\gamma_t\gamma_x-14\gamma_t\gamma_y-12\gamma_x\gamma_y & -5-8\gamma_t\gamma_x+7\gamma_t\gamma_y+6\gamma_x\gamma_y \\ -10+3\gamma_t\gamma_x-4\gamma_t\gamma_y-5\gamma_x\gamma_y & 8-6\gamma_t\gamma_x+8\gamma_t\gamma_y+10\gamma_x\gamma_y & 6+3\gamma_t\gamma_x-4\gamma_t\gamma_y-5\gamma_x\gamma_y \end{bmatrix}$$

$$\underline{\mathbf{D}}^{-1} = \frac{1}{44}\begin{bmatrix} 23+8\gamma_t\gamma_x-7\gamma_t\gamma_y-6\gamma_x\gamma_y & -14-16\gamma_t\gamma_x+14\gamma_t\gamma_y+12\gamma_x\gamma_y & -5+8\gamma_t\gamma_x-7\gamma_t\gamma_y-6\gamma_x\gamma_y \\ -10-3\gamma_t\gamma_x+4\gamma_t\gamma_y+5\gamma_x\gamma_y & 8+6\gamma_t\gamma_x-8\gamma_t\gamma_y-10\gamma_x\gamma_y & 6-3\gamma_t\gamma_x+4\gamma_t\gamma_y+5\gamma_x\gamma_y \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{D}^{+} = \frac{1}{2}\left(\mathbf{D}^{-1}+\underline{\mathbf{D}}^{-1}\right) = \begin{bmatrix} \frac{23}{44} & -\frac{14}{44} & -\frac{5}{44} \\ -\frac{10}{44} & \frac{8}{44} & \frac{6}{44} \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 23 & -14 & -5 \\ -10 & 8 & 6 \end{bmatrix}$$

Moore-Penrose conditions: I. $\mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D} = \mathbf{D}$ III. $\mathbf{D}\,\mathbf{D}^{+} = \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}$

II. $\mathbf{D}^{+}\,\mathbf{D}\,\mathbf{D}^{+} = \mathbf{D}^{+}$ IV. $\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}^{+}\,\mathbf{D}\right)^{T}$

Check of the first and third Moore-Penrose condition: $\mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D} = \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}\mathbf{D} = \mathbf{D}$

$$\begin{array}{cc|ccc|cc}
 & & & & & 5 & 6 \\
 & & \frac{23}{44} & -\frac{14}{44} & -\frac{5}{44} & 4 & 7 \\
 & & -\frac{10}{44} & \frac{8}{44} & \frac{6}{44} & 3 & 8 \\
\hline
5 & 6 & \frac{55}{44} & -\frac{22}{44} & \frac{11}{44} & 5 & 6 \\
4 & 7 & \frac{22}{44} & \frac{0}{44} & \frac{22}{44} & 4 & 7 \\
3 & 8 & -\frac{11}{44} & \frac{22}{44} & \frac{33}{44} & 3 & 8
\end{array}$$

$$\mathbf{D} = \mathbf{D}\,\mathbf{D}^{+}\,\mathbf{D}$$

$$\mathbf{D}\,\mathbf{D}^{+} \neq \left(\mathbf{D}\,\mathbf{D}^{+}\right)^{T}$$

$\Rightarrow$ The first Moore-Penrose condition is true for this spacetime generalized matrix inverse.

$\Rightarrow$ The third Moore-Penrose condition is not true for this spacetime generalized matrix inverse.

Check of the second and fourth Moore-Penrose condition: $\mathbf{D}^+\,\mathbf{D}\,\mathbf{D}^+ = \left(\mathbf{D}^+\,\mathbf{D}\right)^{\mathrm{T}}\,\mathbf{D}^+ = \mathbf{D}^+$

$$
\begin{array}{ccc|cc|ccc}
 & & & 5 & 6 & & & \\
 & & & 4 & 7 & \frac{23}{44} & -\frac{14}{44} & -\frac{5}{44} \\
 & & & 3 & 8 & -\frac{10}{44} & \frac{8}{44} & \frac{6}{44} \\
\hline
\frac{23}{44} & -\frac{14}{44} & -\frac{5}{44} & 1 & 0 & \frac{23}{44} & -\frac{14}{44} & -\frac{5}{44} \\
-\frac{10}{44} & \frac{8}{44} & \frac{6}{44} & 0 & 1 & -\frac{10}{44} & \frac{8}{44} & \frac{6}{44}
\end{array}
\quad \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} \; \mathbf{D}^+ = \mathbf{D}^+\,\mathbf{D}\,\mathbf{D}^+
$$

$$\underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{\mathbf{D}^+\,\mathbf{D} = \left(\mathbf{D}^+\,\mathbf{D}\right)^{\mathrm{T}}}$$

⇒ The second and the fourth Moore-Penrose condition are true for this spacetime generalized matrix inverse.

Finding a new third (and fourth) condition:

As the first, second, and fourth Moore-Penrose conditions hold, we only have to modify the third condition. But again it is possible to change the third and the fourth condition as well into:

I. $\mathbf{D}\,\mathbf{D}^+\,\mathbf{D} = \mathbf{D}$

II. $\mathbf{D}^+\,\mathbf{D}\,\mathbf{D}^+ = \mathbf{D}^+$

III. $\mathbf{D}\,\mathbf{D}^+ = \left(\mathbf{D}\,\mathbf{D}^+\right)^{\mathrm{stT}}$

IV. $\mathbf{D}^+\,\mathbf{D} = \left(\mathbf{D}^+\,\mathbf{D}\right)^{\mathrm{stT}}$

Quantities of final products, which will be produced:

$$
\begin{array}{ccc|c}
 & & & 380 \\
 & & & 370 \\
 & & & 360 \\
\hline
\frac{23}{44} & -\frac{14}{44} & -\frac{5}{44} & 40 \\
-\frac{10}{44} & \frac{8}{44} & \frac{6}{44} & 30
\end{array}
$$

⇒ If 380 units of the first raw material R_1, 370 units of the second raw material R_2, and 360 units of the third raw material R_3 are consumed in the production process, 40 units of the first final product P_1 and 30 units of the second final product P_2 will be produced.