Equivalence Operations in Geometry Illustrated by the Pythagorean and Euclidean Theorems

Martin Erik Horn

University of Applied Sciences for Engineering and Economics Berlin
Hochschule für Technik und Wirtschaft (HTW Berlin, Campus Wilhelminenhof)
hornmar@htw-berlin.de / mail@martinerikhorn.de

Englisch Abstract

Surely you know the Pythagorean and Euclidean Theorems. They seem to be a little bit boring if presented with scalars. Please use vectors instead to make these theorems interesting!
The reason to do this is simple: Equivalence operations with scalar variables are a cornerstone concept of modern algebra. It will be shown how equivalence operations with vector variables can be seen as a similar firm cornerstone of geometry. We only have to get rid of the usual restriction to scalars when discussing the Pythagorean, the Euclidean and the many other theorems of geometry.
Vectors, bivector as oriented area elements and other higher-dimensional quantities should be used as variables, too. The Geometric Algebra constructed this way delivers a valuable and convincing conceptual frame for this modern view on geometry.

German Abstract [11]

Äquivalenzumformungen von Gleichungen, deren Variablen zahlenartig (also durch Skalare) belegt sind, stellen einen Grundpfeiler der modernen Algebra dar. Es wird am Beispiel der Satzgruppe des Pythagoras gezeigt, wie Äquivalenzumformungen einen ebenso festen Pfeiler der Geometrie bilden können.
Die Satzgruppe des Pythagoras kenne wir alle. Sie scheint relativ langweilig, wenn sie lediglich mit Skalaren formuliert wird. Machen wir sie also interessant, indem wir einen vektoriellen Blick auf sie werfen!
Dazu wird die Beschränkung auf zahlenartige Größen aufgehoben und eine Belegung von Variablen durch Vektoren, durch Bivektoren als orientierte Flächenstücke oder durch andere geometrische Größen zugelassen. Die so entstehende Geometrische Algebra wird vorgestellt und diskutiert.

1. Mathematicians is a fossilized science

In contrast to the mostly progressive and positive self-image many mathematicians possess, mathematics must be considered as an extremely conservative, backwards looking and extremely retrospective, fossilized and petrified science on the long term, looking for and preserving the old and rejecting the new. It appears that the history of mathematics is characterized by stagnation by far the most time.

So at the beginning of Old-Babylonian mathematics there is a revolutionary start at the times of Hammurabi. According to Derbyshire and Conway [1, p. 32] the mathematics created at this time was transmitted to following generations of mathematicians without significant changes over the following thousands of years.

The same can be seen in Egypt: "We have no grounds for thinking that Egyptian mathematics made any notable progress from the 16th to the 4th century BCE" [1, p. 32].

In a similar conservative, limiting and restricting way we deal today with the revolutionary insights of the group of mathematicians, Diophantus, “the father of algebra” [1, chap. 2] gathered around him. They – or he – invented the notion of variables, transformations of terms and equivalence operations at this special date at the change of millenniums.

Since then equivalence operations constitute the firm cornerstone at the center of modern algebra. Without equivalence operations algebra would not mean anything to us.

But at the same time we refuse to think about equivalence operations in geometry. Even today we still refuse to acknowledge the cornerstone position equivalence operations might have in geometry. It is really disturbing, that we – obsessed by our petrified views about geometry - not even allow simple transformations today.

2. The Pythagorean Theorem

It is really disturbing that we even reject the simple and easy transformation of a sum of two vectors **a** and **b**. Of course this sum

$$\mathbf{a} + \mathbf{b} = \mathbf{c} \qquad \{1\}$$

describes the three side vectors of an arbitrary triangle (see fig, 1).

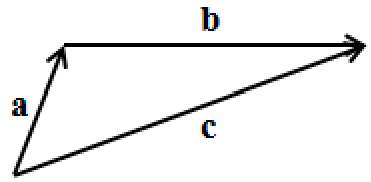

Fig.1: The three vectors of a triangle with arbitrary angles.

And of course squaring this sum of two vectors

$$\begin{aligned}(\mathbf{a} + \mathbf{b})^2 &= (\mathbf{a} + \mathbf{b})\,(\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a}^2 + \mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a} + \mathbf{b}^2 \qquad \{2\} \\ &= \mathbf{c}^2\end{aligned}$$

must result in the square of the third side vector $\mathbf{c}^2$. If $\mathbf{a} + \mathbf{b} = \mathbf{c}$ is valid, $(\mathbf{a} + \mathbf{b})^2 = \mathbf{c}^2$ has to be valid, too – always and forever! If a mathematical system is not capable of reproducing $(\mathbf{a} + \mathbf{b})^2 = \mathbf{c}^2$ correctly, it should be considered as useless. And it should be dismissed immediately [2].

If now the two side vectors **a** and **b** are the orthogonal legs of a right-angled triangle with hypotenuse **c** (see fig. 2), the square of their sum can be found according to eq. {2} again. The conventional formula of the Pythagorean Theorem on the right side of

$$(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2 \qquad \{3\}$$

will then result, if both legs anti-commute:

$$\mathbf{a}\,\mathbf{b} = -\,\mathbf{b}\,\mathbf{a} \qquad \{4\}$$

Eq. {4} is an algebraic relation for orthogonal vectors, which is well-known since the invention of the theory of extensions by Hermann Grassmann [3]. And eq. {4} has been ignored by main-stream mathematicians again and again when trying to preserve the old and to reject the new.

Since more than 175 years traditional school and highschool mathematics (and in an even fiercer way didactics of mathematics) bluntly fight against basing the mathematics of vectors on a modern foundation which includes the impressive findings of Grassmann and Clifford.

Rota [4, pp. 232/233] comments on this grotesque backwardness and retarded mathematical antiquatedness with the words: “The neglect of exterior algebra is the mathematical tragedy of this century. (…) Meanwhile, we have to bear with mathematicians who are exterior algebra-blind.”

3. The Euclidean Theorems

While orthogonal vectors anti-commute, parallel vectors commute and their product is commutative with respect to multiplication.

Therefore the vectors of the two hypotenuse segments **m** and **n** (see fig. 2) multiply according to

$$\mathbf{m}\,\mathbf{n} = \mathbf{n}\,\mathbf{m} \qquad \{5\}$$

Together with the Pythagorean Theorem of the small

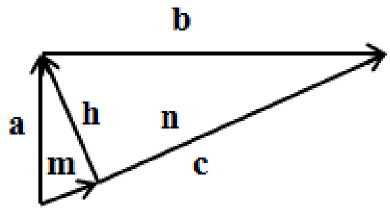

Fig.2: Vectors and vector segments of a right-angled triangle.

right-angled triangles

$$(\mathbf{m} + \mathbf{h})^2 = \mathbf{m}^2 + \mathbf{h}^2 = \mathbf{a}^2 \qquad \{6\}$$

and

$$(\mathbf{n} - \mathbf{h})^2 = \mathbf{n}^2 + \mathbf{h}^2 = \mathbf{b}^2 \qquad \{7\}$$

the square of the altitude vector **h** can be constructed by squaring the hypotenuse **c**

$$\begin{aligned}\mathbf{c}^2 &= (\mathbf{m} + \mathbf{n})^2 \\ &= \mathbf{m}^2 + \mathbf{m}\,\mathbf{n} + \mathbf{n}\,\mathbf{m} + \mathbf{n}^2 \qquad \{8\} \\ &= \mathbf{a}^2 - \mathbf{h}^2 + 2\,\mathbf{m}\,\mathbf{n} + \mathbf{b}^2 - \mathbf{h}^2 \\ &= \mathbf{c}^2 - 2\,\mathbf{h}^2 + 2\,\mathbf{m}\,\mathbf{n}\end{aligned}$$

and then cancelling this hypotenuse square $\mathbf{c}^2$ to get the expected result:

$$\mathbf{h}^2 = \mathbf{m}\,\mathbf{n} \qquad \{9\}$$

The missing Euclidean Theorems are simple modifications of eq. {9} as only the squares of the vectors of the hypotenuse segment $\mathbf{m}^2$

$$\begin{aligned}\mathbf{a}^2 &= \mathbf{h}^2 + \mathbf{m}^2 \\ &= \mathbf{m}\,\mathbf{n} + \mathbf{m}^2 \qquad \{10\} \\ &= \mathbf{m}\,(\mathbf{n} + \mathbf{m}) \\ &= \mathbf{m}\,\mathbf{c}\end{aligned}$$

or of the hypotenuse segment $\mathbf{n}^2$

$$\begin{aligned}\mathbf{b}^2 &= \mathbf{h}^2 + \mathbf{n}^2 \\ &= \mathbf{m}\,\mathbf{n} + \mathbf{n}^2 \qquad \{11\} \\ &= (\mathbf{m} + \mathbf{n})\,\mathbf{n} \\ &= \mathbf{n}\,\mathbf{c}\end{aligned}$$

should be added.

All these squares of vectors and all theses products of two parallel vectors are scalars. Consequently they are completely different quantities compared to products of two orthogonal vectors, which always are oriented area elements and thus bivectors.

The Pythagorean and Euclidean Theorems should therefore be supplemented and completed by an area theorem

$$\mathbf{a}\,\mathbf{b} = \mathbf{h}\,\mathbf{c} \qquad \{12\}$$

which can be motivated by comparing the oriented area of an right-angled triangle.

4. More equivalence operations

While in conventional presentations of the Pythagorean and Euclidean Theorems only scalars a, b, c, m, n, h are taken into account, we are now able to work with oriented line elements or vectors **a**, **b**, **c**, **m**, **n**, **h** (printed in boldface) and thus get a better picture of the geometrical situation.

All these eqs. {9} – {12} can be transformed by equivalence operations. And it is obvious that it is even possible to divide by vectors [5, eq. 23]. To do this it is only necessary to construct inverse vectors. e.g. the inverse of the hypotenuse **c**:

$$\mathbf{c}^{-1} = \frac{\mathbf{c}}{\mathbf{c}^2} \qquad \{13\}$$

The inverse vectors can then either be pre-multiplied from the left or post-multiplied from the right. In this way the Euclidean Theorems can be solved for unknown vectors, e.g.

$$\mathbf{m} = \mathbf{a}^2\,\mathbf{c}^{-1} = \frac{\mathbf{a}^2\,\mathbf{c}}{\mathbf{c}^2} \qquad \{14\}$$

$$\mathbf{n} = \mathbf{b}^2\,\mathbf{c}^{-1} = \frac{\mathbf{b}^2\,\mathbf{c}}{\mathbf{c}^2} \qquad \{15\}$$

$$\mathbf{h} = \mathbf{a}\,\mathbf{b}\,\mathbf{c}^{-1} = \frac{\mathbf{a}\,\mathbf{b}\,\mathbf{c}}{\mathbf{c}^2} \qquad \{16\}$$

As the square of a vector is identical to a scalar, the division by a vector is hidden behind the multiplication by a vector and the division by a scalar.

5. First example problem

As an example for this strategy a right-angled triangle is given by the following three points A(20; 15), B(0; 0), and C(0; 15) in [6].

After finding the Geometric Algebra vectors of hypotenuse **c** and legs **a** and **b**

$$\mathbf{a} = 15\,\sigma_y$$
$$\mathbf{b} = 20\,\sigma_x \qquad \{17\}$$
$$\mathbf{c} = 20\,\sigma_x + 15\,\sigma_y$$

and theirs squares

$$\mathbf{a}^2 = (15\,\sigma_y)^2 = 225$$
$$\mathbf{b}^2 = (20\,\sigma_x)^2 = 400 \qquad \{18\}$$
$$\mathbf{c}^2 = (20\,\sigma_x + 15\,\sigma_y)^2 = 625$$

the Pythagorean Theorem can be checked by

$$(\mathbf{a} + \mathbf{b})^2 = (15\,\sigma_y + 20\,\sigma_x)^2$$
$$= \mathbf{a}^2 + \mathbf{b}^2 = 225 + 400 \qquad \{19\}$$
$$= \mathbf{c}^2 = 625$$

The vector of hypotenuse segment **m** can be found by applying eq. {14}:

$$\mathbf{m} = \frac{\mathbf{a}^2}{\mathbf{c}^2}\,\mathbf{c} = \frac{225}{625}\,(20\,\sigma_x + 15\,\sigma_y) \qquad \{20\}$$
$$= 7.2\,\sigma_x + 5.4\,\sigma_y$$

And the vector of the second hypotenuse segment **n**

$$\mathbf{n} = \mathbf{c} - \mathbf{m} = 12.8\,\sigma_x + 9.6\,\sigma_y \qquad \{21\}$$

can be found by applying eq. {15}:

$$\mathbf{n} = \frac{\mathbf{b}^2}{\mathbf{c}^2}\,\mathbf{c} = \frac{400}{625}\,(20\,\sigma_x + 15\,\sigma_y) \qquad \{22\}$$
$$= 12.8\,\sigma_x + 9.6\,\sigma_y$$

Finally the vector of the altitude **h**

$$\mathbf{h} = \mathbf{a} - \mathbf{m} = -\,7.2\,\sigma_x + 9.6\,\sigma_y \qquad \{23\}$$

or

$$\mathbf{h} = \mathbf{n} - \mathbf{b} = -\,7.2\,\sigma_x + 9.6\,\sigma_y \qquad \{24\}$$

can be found by applying eq. {16}:

$$\mathbf{h} = \frac{\mathbf{a}\,\mathbf{b}}{\mathbf{c}^2}\,\mathbf{c} = \frac{-300\,\sigma_x\sigma_y}{625}\,(20\,\sigma_x + 15\,\sigma_y)$$
$$= -\,7.2\,\sigma_x + 9.6\,\sigma_y \qquad \{25\}$$

These example calculations hopefully show, that the first two sentences of Snygg in the introduction of his book [7, p. XIII] are completely correct: “Much of Clifford algebra is quite simple minded. If this fact were generally recognized, Clifford algebra would be more widely used as a computational tool.”

And the only problem then is: Geometric Algebra is “so simple only a child can do it” [8, p. 1177].

6. Generalized versions of the Pythagorean and Euclidean Theorems

Similar calculations at general or oblique triangles, which do not possess orthogonal sides, can be done again by squaring the sum of vectors **c** = **a** + **b**:

$$\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})^2 = (\mathbf{a} + \mathbf{b})\,(\mathbf{a} + \mathbf{b}) \qquad \{26\}$$
$$= \mathbf{a}^2 + \mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a} + \mathbf{b}^2$$

But this time the generalized Pythagorean Theorem can only be found if the definition of the inner product of two vectors **a** and **b** with Geometric Algebra scalar lengths of $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$ (see [5, p. 107] or [8, p. 1178])

$$\mathbf{a} \bullet \mathbf{b} = \frac{1}{2}\,(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a}) \qquad \{27\}$$
$$= 2\,a\,b\cos\gamma = 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\gamma$$

is taken into account:

$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\,\mathbf{a} \bullet \mathbf{b} \qquad \{28\}$$

This result again is a logical consequence of transforming geometrically motivated variables. We do not square scalars, we are squaring vectors now.

Eq. {28} is the generalized Pythagorean Theorem. It can be split into two parts

$$(\mathbf{m} + \mathbf{n})\,\mathbf{c} = \mathbf{a}^2 + \mathbf{a} \bullet \mathbf{b} + \mathbf{b}^2 + \mathbf{a} \bullet \mathbf{b} \qquad \{29\}$$

which constitute the generalized Euclidean Theorems

$$\mathbf{m}\,\mathbf{c} = \mathbf{a}^2 + \mathbf{a} \bullet \mathbf{b} \qquad \{30\}$$

and

$$\mathbf{n}\,\mathbf{c} = \mathbf{b}^2 + \mathbf{a} \bullet \mathbf{b} \qquad \{31\}$$

And it is shown in [6] that squaring the vector **c** as sum of the two parallel segments **c** = **m** + **n**:

$$\mathbf{c}^2 = (\mathbf{m} + \mathbf{n})^2 = \mathbf{m}^2 + 2\,\mathbf{m}\,\mathbf{n} + \mathbf{n}^2 \qquad \{32\}$$

and solving for **m n** = **n m** results in the last generalized Euclidean Theorem

$$\mathbf{m}\,\mathbf{n} = \mathbf{h}^2 + \mathbf{a} \bullet \mathbf{b} \qquad \{33\}$$

As the outer product of the two side vectors **a** and **b** (see [5, p. 107] or [8, p. 1178])

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}\,(\mathbf{a}\,\mathbf{b} - \mathbf{b}\,\mathbf{a}) = \mathbf{h}\,\mathbf{c} \qquad \{34\}$$

is identical to the oriented area **h c** of the oblique triangle the canonical decomposition of the Geometric Product (see [5, p. 107] or [8, p. 1179])

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \qquad \{35\}$$

can be identified straightforwardly with the generalized area theorem

$$\mathbf{a}\,\mathbf{b} = \mathbf{h}\,\mathbf{c} + \mathbf{a} \bullet \mathbf{b} \qquad \{36\}$$

The standard versions and generalized versions of these theorems are summarized in fig. 3. The structural lesson is clear: Just add the inner product of eq. {27} and everything will be o.k.

$$\mathbf{a} = (1 - 6)\,\sigma_x + (4 - 2)\,\sigma_y = -5\,\sigma_x + 2\,\sigma_y$$
$$\mathbf{b} = (0 - 1)\,\sigma_x + (0 - 4)\,\sigma_y = -\sigma_x - 4\,\sigma_y \qquad \{37\}$$
$$\mathbf{c} = (0 - 6)\,\sigma_x + (0 - 2)\,\sigma_y = -6\,\sigma_x - 2\,\sigma_y$$

They square to

$$\mathbf{a}^2 = (-5\,\sigma_x + 2\,\sigma_y)^2 = 29$$
$$\mathbf{b}^2 = (-\sigma_x - 4\,\sigma_y)^2 = 17 \qquad \{38\}$$
$$\mathbf{c}^2 = (-6\,\sigma_x - 2\,\sigma_y)^2 = 40$$

and have the following inner product of vectors **a** and **b**:

$$\mathbf{a} \bullet \mathbf{b} = 5 - 8 = -3 \qquad \{39\}$$

Thus the Pythagorean Theorem can be checked by

$$(\mathbf{a} + \mathbf{b})^2 = (-6\,\sigma_x - 2\,\sigma_y)^2$$
$$= \mathbf{a}^2 + \mathbf{b}^2 + 2\,\mathbf{a} \bullet \mathbf{b} = 29 + 17 - 6$$
$$= \mathbf{c}^2 = 40 \qquad \{40\}$$

The vector segment **m** can be found by solving eq. {30} for **m**:

$$\mathbf{m} = \frac{\mathbf{a}^2 + \mathbf{a} \bullet \mathbf{b}}{\mathbf{c}^2}\,\mathbf{c} = \frac{29 - 3}{40}\,(-6\,\sigma_x - 2\,\sigma_y)$$
$$= -3.9\,\sigma_x - 1.3\,\sigma_y \qquad \{41\}$$

And the second vector segment **n**

$$\mathbf{n} = \mathbf{c} - \mathbf{m} = -2.1\,\sigma_x - 0.7\,\sigma_y \qquad \{42\}$$

can be found by solving eq. {31} for **n**:

$$\mathbf{n} = \frac{\mathbf{b}^2 + \mathbf{a} \bullet \mathbf{b}}{\mathbf{c}^2}\,\mathbf{c} = \frac{17 - 3}{40}\,(-6\,\sigma_x - 2\,\sigma_y)$$
$$= -2.1\,\sigma_x - 0.7\,\sigma_y \qquad \{43\}$$

The vector of the altitude **h**

$$\mathbf{h} = \mathbf{a} - \mathbf{m} = -1.1\,\sigma_x + 3.3\,\sigma_y \qquad \{44\}$$

or

$$\mathbf{h} = \mathbf{n} - \mathbf{b} = -1.1\,\sigma_x + 3.3\,\sigma_y \qquad \{45\}$$

right triangles		general triangles	
$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2$		$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\,\mathbf{a} \bullet \mathbf{b}$	
$\mathbf{m}\,\mathbf{c} = \mathbf{a}^2$	$\mathbf{n}\,\mathbf{c} = \mathbf{b}^2$	$\mathbf{m}\,\mathbf{c} = \mathbf{a}^2 + \mathbf{a} \bullet \mathbf{b}$	$\mathbf{n}\,\mathbf{c} = \mathbf{b}^2 + \mathbf{a} \bullet \mathbf{b}$
$\mathbf{m}\,\mathbf{n} = \mathbf{h}^2$	$\mathbf{a}\,\mathbf{b} = \mathbf{h}\,\mathbf{c}$	$\mathbf{m}\,\mathbf{n} = \mathbf{h}^2 + \mathbf{a} \bullet \mathbf{b}$	$\mathbf{a}\,\mathbf{b} = \mathbf{h}\,\mathbf{c} + \mathbf{a} \bullet \mathbf{b}$

Fig.3: Overview of right-angled and generalized triangle theorems of Pythagoras and Euclid.

7. Second example problem

As an “homework problem” example for applying the generalized theorems a general triangle was given by the following three points A(0; 0), B(6; 2), and C(1; 4) in [6, slide 77].

If vector **c** points from point B to point A (and consequently vector **a** points from point B to point C, vector **b** points from point C to point A) in contrast to [6, slide 78]) the Geometric Algebra vectors **a**, **b**, and **c** of the three sides are given by

can be found by solving eq. {36} for **h**:

$$\mathbf{h} = \frac{\mathbf{a}\,\mathbf{b} - \mathbf{a} \bullet \mathbf{b}}{\mathbf{c}^2}\,\mathbf{c} \qquad \{46\}$$
$$= \frac{22\,\sigma_x\sigma_y - 3 + 3}{40}\,(-6\,\sigma_x - 2\,\sigma_y)$$
$$= -1.1\,\sigma_x + 3.3\,\sigma_y$$

And finally eq. {33} can be checked:

$$\mathbf{m}\,\mathbf{n} = 9.1 = 12.1 - 3 = \mathbf{h}^2 + \mathbf{a} \bullet \mathbf{b} \qquad \{47\}$$

All this is simple mathematics. It is extremely simple And it is much simpler compared to many other attempts to write down generalizations of the Pythagorean and Euclidean Theorems.

Therefore we should recall he conclusion Parra Serra has drawn in [9, p. 820]: "Clifford algebra has attained such a degree of completeness that can be claimed *it can be explained to the first person you meet in the street*. There is no reasonable nor solid argumentation for denying its inclusion in the high school curriculum where it can play the important role of introducing and relating many otherwise unconnected fields."

8. Outlook: Complex conjugation means betraying symmetry

In previous sections we did a lot of squaring. But of course it is not forbidden to compute more powers $\mathbf{c}^n$ of higher orders n of the hypotenuse **c**

$$\mathbf{c} = \mathbf{m} + \mathbf{n} = \mathbf{a} + \mathbf{b} \qquad \{48\}$$

of a right-angled triangle.

$$\begin{array}{ccccccccccccccc} &&&&&&&\mathbf{1}&&&&&&& \\ &&&&&&\mathbf{1}&&\mathbf{1}&&&&&& \\ &&&&&\mathbf{1}&&\mathbf{2}&&\mathbf{1}&&&&& \\ &&&&\mathbf{1}&&\mathbf{3}&&\mathbf{3}&&\mathbf{1}&&&& \\ &&&\mathbf{1}&&\mathbf{4}&&\mathbf{6}&&\mathbf{4}&&\mathbf{1}&&& \\ &&\mathbf{1}&&\mathbf{5}&&\mathbf{10}&&\mathbf{10}&&\mathbf{5}&&\mathbf{1}&& \\ &\mathbf{1}&&\mathbf{6}&&\mathbf{15}&&\mathbf{20}&&\mathbf{15}&&\mathbf{6}&&\mathbf{1}& \\ \mathbf{1}&&\mathbf{7}&&\mathbf{21}&&\mathbf{35}&&\mathbf{35}&&\mathbf{21}&&\mathbf{7}&&\mathbf{1} \end{array}$$

Fig.4: Binomial Coefficients forming the Pascal Triangle.

Coefficients of terms of commuting quantities like **m** and **n** form the usual and well-known Pascal Triangle (see fig. 4).

But coefficients of powers of anti-commuting quantities like the orthogonal legs **a** and **b** of a right-angled triangle show a completely different structure:

$$\begin{aligned} \mathbf{c}^0 &= 1 \\ \mathbf{c}^1 &= 1\,\mathbf{a} + 1\,\mathbf{b} \\ \mathbf{c}^2 &= 1\,\mathbf{a}^2 + 0\,\mathbf{a}\,\mathbf{b} + 1\,\mathbf{b}^2 \qquad \{49\} \\ \mathbf{c}^3 &= 1\,\mathbf{a}^3 + 1\,\mathbf{a}^2\,\mathbf{b} + 1\,\mathbf{a}\,\mathbf{b}^2 + 1\,\mathbf{b}^3 \\ \mathbf{c}^4 &= 1\,\mathbf{a}^4 + 0\,\mathbf{a}^3\,\mathbf{b} + 2\,\mathbf{a}^2\,\mathbf{b}^2 + 0\,\mathbf{a}\,\mathbf{b}^3 + 1\,\mathbf{b}^4 \\ \mathbf{c}^5 &= 1\,\mathbf{a}^5 + 1\,\mathbf{a}^4\,\mathbf{b} + 2\,\mathbf{a}^3\,\mathbf{b}^2 + 2\,\mathbf{a}^2\,\mathbf{b}^3 + 1\,\mathbf{a}\,\mathbf{b}^4 + 1\,\mathbf{b}^5 \end{aligned}$$

They form the Pauli Pascal Triangle [10], [11] (see fig, 5).

And there is a clear, inevitable and absolutely irrefutable connection between the symmetry of two different quantities and the Pascal pattern of their coefficients, which will emerge.

- If coefficients form a Pascal triangle, the basic building blocks must inevitably have been quantities which commute.
- And if coefficients form a Pauli Pascal triangle, the basic buildings blocks must inevitably have been quantities which anti-commute.

You should tell this your students!

Now please have a look at the coefficients of different powers of a complex number z = x + i y:

$$z^n = (x + i\,y)^n \qquad \{50\}$$

Of course the coefficients form a Pascal Triangle (with alternating pairs of negative and positive diagonal lines originating from powers of the imaginary unit i^n). This clearly shows that the basic building blocks x and i y commute.

Now we compute powers of higher orders of products of imaginary numbers z = x + iy and complex conjugates z* = x – i y.

This time no Pascal Triangle is formed. Using complex conjugated building blocks inevitably results in

$$\begin{array}{ccccccccccccccc} &&&&&&&\mathbf{1}&&&&&&& \\ &&&&&&\mathbf{1}&&\mathbf{1}&&&&&& \\ &&&&&\mathbf{1}&&\mathbf{0}&&\mathbf{1}&&&&& \\ &&&&\mathbf{1}&&\mathbf{1}&&\mathbf{1}&&\mathbf{1}&&&& \\ &&&\mathbf{1}&&\mathbf{0}&&\mathbf{2}&&\mathbf{0}&&\mathbf{1}&&& \\ &&\mathbf{1}&&\mathbf{1}&&\mathbf{2}&&\mathbf{2}&&\mathbf{1}&&\mathbf{1}&& \\ &\mathbf{1}&&\mathbf{0}&&\mathbf{3}&&\mathbf{0}&&\mathbf{3}&&\mathbf{0}&&\mathbf{1}& \\ \mathbf{1}&&\mathbf{1}&&\mathbf{3}&&\mathbf{3}&&\mathbf{3}&&\mathbf{3}&&\mathbf{1}&&\mathbf{1} \end{array}$$

Fig.5: Coefficients of the Pauli Pascal Triangle.

a pattern which can clearly be identified with the Pauli Pascal Triangle of fig. 5:

$$\begin{aligned} z^0 &= 1 \\ z &= 1\,x + 1\,iy \\ z^*z &= 1\,x^2 + 0\,ixy + 1\,y^2 \qquad \{51\} \\ zz^*z &= 1\,x^3 + 1\,ix^2y + 1\,xy^2 + 1\,iy^3 \\ z^*zz^*z &= 1\,x^4 + 0\,ix^3y + 2\,x^2y^2 + 0\,ixy^3 + 1\,y^4 \\ zz^*zz^*z &= 1\,x^5 + 1\,ix^4y + 2\,x^3y^2 + 2\,ix^2y^3 + 1\,xy^4 + 1\,iy^5 \end{aligned}$$

And as the coefficients form the Pauli Pascal Triangle, the underlying symmetry clearly is identical to the symmetry of anti-commuting basic building blocks.

Using complex conjugation means: Using a structure which is formed of commuting building blocks (x and ± iy) to model an anti-commuting situation. Using complex conjugation thus means cheating: You do not see the anti-commuting building blocks, but they are there. They are mimicked by additional minus signs which have been fraudulently inserted via complex conjugation.

With complex conjugation we are modeling anti-commutative structures, which we hide behind illegitimate commutativity.

Complex conjugation is a dirty mathematical trick. Please, do not use this trick. Please be open and honest and use directly anti-commutating building blocks if you intend to model anti-commutative structures.

9. Literature

[1] Derbyshire, John (2006): Unknown Quantity. A Real and Imaginary History of Algebra. Joseph Henry Press, Washington, DC.

[2] Horn, Martin Erik (2019): Cheating with Complex Numbers. Der Selbstbetrug mit den komplexen Zahlen. Preprint at Url: www.vixra.org/abs/1911.0023 (01.11.2019).

[3] Grassmann, Hermann (1844): Die Wissenschaft der extensiven Grösse oder die Ausdehnungslehre, eine neue mathematische Disciplin. Erster Theil, die lineale Ausdehnungslehre enthaltend. Verlag von Otto Wigand, Leipzig.

[4] Rota, Gian-Carlo (1997). Indiscrete Thoughts. Birkhäuser, Boston, Basel, Berlin.

[5] Hestenes, David (2003): Oersted Medal Lecture 2002: Reforming the Mathematical Language of Physics. In: American Journal of Physics, Vol. 71, No. 2, pp. 104 – 121.

[6] Horn, Martin Erik (2020): Generalizing the Pythagorean and Euclidean Theorems. Die GDM hat Ihren Namen nicht verdient. German powerpoint slides of the GDM Online Conference 2020 with a short English summary. Url: www.vixra.org/abs/2003.0643 (29.03.2020).

[7] Snygg, John (1997): Clifford Algebra – A Computational Tool for Physicists. Oxford University Press, New York, Oxford.

[8] Gull, Stephan; Lasenby, Anthony; Doran, Chris (1993): Imaginary Numbers are not Real – The Geometric Algebra of Spacetime. In: Foundations of Physics, Vol. 23, No. 9, pp. 1175 – 1201.

[9] Parra Serra, Josep Manel (2009): Clifford Algebra and the Didactics of Mathematics. In: Advances of Applied Clifford Algebras, Vol. 19, No. 3/4, pp. 819 – 834.

[10] Horn, Martin Erik (2007): Die didaktische Relevanz des Pauli-Pascal-Dreiecks. In: Dietmar Höttecke (Ed.): Naturwissenschaftlicher Unterricht im Internationalen Vergleich. Proceedings of the Annual Meeting of the Society of Chemistry and Physics Education in Bern – Tagungsband zur Jahrestagung der Gesellschaft für Didaktik der Chemie und Physik in Bern, GDCP Band 27. LIT-Verlag, Berlin, Münster, pp. 557 – 559.
Horn, Martin Erik (2006): The Didactical Relevance of the Pauli Pascal Triangle. Preprint, arXiv:physics/0611277 (28.11.2006)

Original version of this paper

The original German version of this paper has been submitted to the Proceedings of the online conference 2020 of the Society of Mathematics Education (GDM – Gesellschaft für Didaktik der Mathematik) at Julius Maximilian University of Würzburg. It will be published as:

[11] Horn, Martin Erik (2020): Äquivalenzumformungen in der Geometrie am Beispiel der Satzgruppe des Pythagoras. In: Beiträge zum Mathematikunterricht – BzMU 2020. WTM-Verlag für wissenschaftliche Texte und Medien, Münster.